30. <u>Homoclinic points of area preserving maps.</u> K. Meyer.

Let M be a 2-dimensional symplectic manifold s.t. any simple closed curve separates M into two regions, one of finite area, e.g. M is the plane. Given a diffeomorphism f, a point $p \in M$ is <u>homoclinic</u> to a hyperbolic fixed point q of f if $p \in W^s(q) \cap W^u(q)$ and $p \neq q$. p is non-degenerate if $W^s(q)$ and $W^u(q)$ meet transversally at p. Let $\mathcal{F}$ be the set of sympletic (or area preserving) diffeomorphisms of M with the compact open C^1 topology and let $\mathcal{H} \subset \mathcal{F}$ be those diffeomorphisms that have a point homoclinic to a hyperbolic fixed point.

<u>Theorem.</u> $\mathcal{H}$ is open in $\mathcal{F}$.

<u>Idea of proof.</u> If p is a non-degenerate homoclinic point of $f \in \mathcal{H}$ the transversal intersection survives C^1 perturbations of f. If the intersection is non-transversal then g near f might have a transversal homoclinic point near p so $g \in \mathcal{H}$ or else $W_g{}^s(q)$, $W_g{}^u(q)$ are as in figure 1a. But in this latter case the bounded region in figure 1b

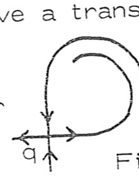

Fig 1a.

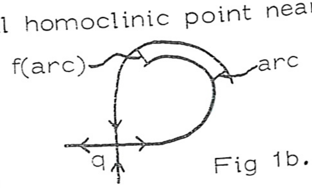

Fig 1b.

enclosed by $W^s(q)$, $W^u(q)$ and a small arc is mapped inside itself which contradicts the area preserving property of f.

Corollary. Let $H(t,x,\varepsilon) = \varepsilon H_1(t,x)$, $x \in R^2$, be a C^2 time-dependent Hamiltonian function on the plane depending on a further small parameter ε. Assume H_1 is periodic with period T and let H_0 be the averaged function $H_0(x) = (1/T)\int_0^T H_1(s,x)ds$. Let H_0 have a non-degenerate saddle point q and assume the level surface $\{x; H_0(x) = H_0(q)\}$ contains a simple closed curve C through q. Let $\varphi_\varepsilon : R^2 \to R^2$ be the period map, i.e. the time T map of the flow given by the Hamiltonian $H(t,x,\varepsilon)$. Then there exists $\varepsilon_0 > 0$ s.t. $\forall \varepsilon$, $0 < |\varepsilon| < \varepsilon_0$, φ_ε has a homoclinic point.

Example. The equation $\ddot{v} + v + \varepsilon\{2\alpha v + 4\beta v^3 + \gamma \cos t\} = 0$ comes from a Hamiltonian which after a change of coordinates has the form $H = \varepsilon H_1$,

$$H_1(t,x) = \alpha(x_1 \cos t + x_2 \sin t)^2 + \beta(x_1 \cos t + x_2 \sin t)^4 + \gamma(x_1 \cos t + x_2 \sin t) \cos t.$$

$$T = 2\pi \text{ and } H_0(x) = \alpha(x_1^2 + x_2^2)/2 + 3\beta(x_1^2 + x_2^2)^2/8 + \gamma x_1/2.$$

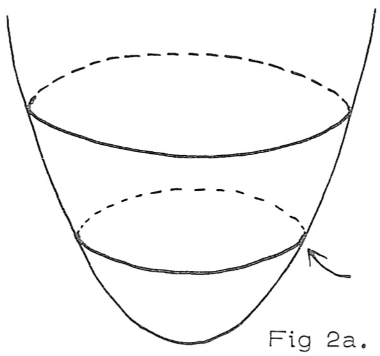

Fig 2a.

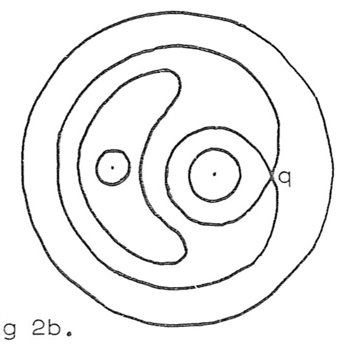

Fig 2b.

For $\alpha > 0$, $\beta < 0$ and $\gamma = 0$ H_0 looks like a paraboloid as in figure 2a and when $\gamma \neq 0$ this surface is modified by pushing inward and upward a little at one point as indicated by the arrow and at the same time tilting the paraboloid. Under an extra condition on α and β the level surfaces of H_0 are as in figure 2b. There is one hyperbolic saddle point q and there are two simple closed curves through q in its level surface. The corollary applies and there are two (possibly degenerate) homoclinic points of φ_ε for small non-zero ε.

Another example, proofs and diagrams are in [1].

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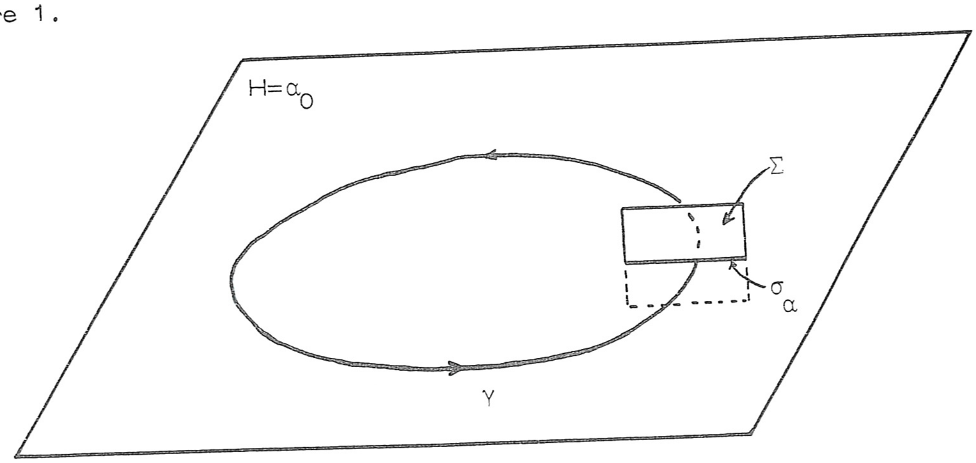
31. Generic Bifurcations in Hamiltonian Systems. Kenneth R. Meyer.

Introduction : The literature on the bifurcation of periodic solutions of Hamiltonian systems is found in celestial mechanics, engineering, mathematics and physics journals. Thus there has been considerable duplication of effort. Most of the papers are concerned with the existence and bifurcation of periodic solutions for particular equations, however a culling of the literature yields several generic phenomena. Even though the authors of these articles do not state their results in the language of Baire category theory they place conditions on the equations which are obviously generic in the proper setting. Since transversality theory is fully developed it is an easy step to translate these results into the modern framework. I would like to give a short account of some of these generic phenomena. The first part of this account gives some improvements and extensions of my previous work and the second part gives a survey of the literature on bifurcations near resonance equilibria.

Notation and Background : Let M be a smooth $(2n+2)$-dimensional manifold with sympletic form Ω. Ω defines an isomorphism $\flat : T_pM \to T_p{}^*M$, $v_p \mapsto \Omega_p(v_p, \cdot)$ with inverse $\#: T_p{}^*M \to T_pM$. If $H: M \to R$ is smooth then $X = (dH)^{\#}$ is a Hamiltonian vector field on M. Let $p \in M$ be such that the solution γ of X through p is periodic, Σ a $(2n+1)$-dimensional local cross section to X at p and $\sigma_\alpha = \Sigma \cap \{q \in M; H(q) = \alpha\}$. Thus if Σ is small enough, σ_α is a 2n-dimensional cross section to X in the level surface $H = \alpha$ (see figure 1). Let $\alpha_0 = H(p)$.

Figure 1.



Let φ_α be the first return map (Poincaré map) for the flow generated by X on the local cross section σ_α. Thus the study of the nature and bifurcation of periodic solutions of X near γ is reduced to the study of the nature and bifurcation of the fixed points of φ_α near p. $\{\varphi_\alpha\}$ can be considered as a one parameter family of local diffeomorphisms. Some basic facts are summerised in

Theorem 1. (1) $\Omega|\sigma_\alpha = \omega_\alpha$ is a sympletic structure.

(2) φ_α is a symplectomorphism.

(3) The characteristic multipliers of γ are $1, 1, \lambda_1, \ldots, \lambda_{2n}$ where $\lambda_1, \ldots, \lambda_{2n}$ are the eigenvalues of $D\varphi_{\alpha_0}(p)$.

(4) If $\lambda_1, \ldots, \lambda_{2n} \neq 1$ then γ lies in a smooth one parameter family of periodic solutions of X and the parameter may be taken to be the value of H locally.

In view of the above only symplectomorphisms which depend on one or more parameters will be considered henceforth. Since the types of bifurcations considered here are basically local in nature one can always use local coordinates to reduce the problem to the study of a symplectomorphism of R^{2n} where R^{2n} is given the usual symplectic structure defined by the matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The notation used here is basically that found in Abraham [1]. Further references for this material are Meyer [9] and Poincaré [14].

Connection with Singularity Theory : In [9] a generating function suggested by Poincaré was used to study the fixed points of a symplectomorphism by applying standard singularity theory. Poincaré's generating function seems somewhat artificial and so a slightly different generating function suggested by the work of Weinstein [20] will be used here (also see [16]). By the implicit function theorem no new fixed points will occur under small perturbations near a fixed point where the linearised map does not take the eigenvalue 1. Thus one should first investigate what happens generically when a symplectomorphism has a fixed point where the linearised map does take the eigenvalue 1. With this in mind we shall construct the special generating function.

Consider the fractional linear transformation Φ of $\mathbb{C}$ given by $\Phi : z \mapsto w = (1+z)(1-z)^{-1}$ with inverse given by $\Phi^{-1} : w \mapsto z = (w-1)(w+1)^{-1}$. Clearly Φ maps $0 \mapsto 1$, $i \mapsto i$ and $\infty \mapsto -1$ and so carries the imaginary axis onto the unit circle with the left half plane going to the interior. In the usual way Φ

may be extended to matrices.

Lemma 2. (a) Φ maps the set of $k \times k$ matrices with no eigenvalue $+1$ bijectively onto the set of $k \times k$ matrices with no eigenvalue -1.

(b) The eigenvalues of $\Phi(A)$ are $\Phi(\lambda_1),\ldots,\Phi(\lambda_k)$ where $\lambda_1,\ldots,\lambda_k$ are the eigenvalues of A.

(c) If A is a $2n \times 2n$ Hamiltonian matrix with $\Phi(A)$ defined then $\Phi(A)$ is symplectic and if B is a $2n \times 2n$ symplectic matrix with $\Phi^{-1}(B)$ defined then $\Phi^{-1}(B)$ is Hamiltonian.

Proof. Parts (a) and (b) are well known. Let A be a Hamiltonian matrix i.e. $A^TJ + JA = 0$ and $B = (I+A)(I-A)^{-1}$. Then

$$\begin{aligned} B^TJB &= (I-A^T)^{-1}(I+A^T)J(I+A)(I-A)^{-1} \\ &= (I-A^T)^{-1}(J+A^TJ)(I+A)(I-A)^{-1} \\ &= (I-A^T)^{-1}J(I-A)(I+A)(I-A)^{-1} \\ &= (I-A^T)^{-1}J(I+A) = (I-A^T)^{-1}(J+JA) \\ &= (I-A^T)^{-1}(I-A^T)J = J \end{aligned}$$

and so B is symplectic. The second part of (c) is similar.

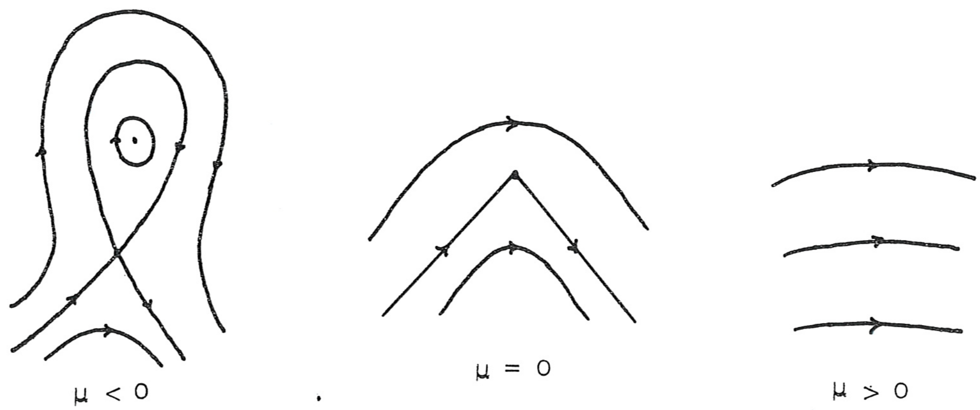
Now we shall extend Φ to nonlinear maps. Let V be an open neighbourhood of 0 in R^{2n} and $H:V \to R$ a smooth map such that $\text{grad}H(0) = 0$ and the matrix $J\partial^2H/\partial x^2(0)$ has no eigenvalue equal to $+1$. Then $g = J\,\text{grad}\,H$ is a Hamiltonian vector field on V with 0 as a critical point. By assumption $id - g$ is locally invertable near 0. Let $\Phi(g) = f = (id+g)\circ(id-g)^{-1}$ and so f is defined in a neighbourhood of the origin in R^{2n} and by part (c) of the lemma above f is a symplectomorphism since $g = J\,\text{grad}\,H$ is a Hamiltonian vector field. Also note that the critical points of g give rise to fixed points of f. For if $g(x) = 0$ and x is close enough to zero then x is the unique solution of $y - g(y) = x$. So $(id+g)\circ(id-g)^{-1}(x) = (id+g)(x) = x$. Thus (locally) there is an association which sends the function H to the symplectomorphism f in such a way that the critical points of H are exactly the fixed points of f. Clearly the inverse is defined up to an additive constant.

If H depends smoothly on parameters then of course so does f and so the bifurcation of fixed points of f is reduced to the bifurcation of the critical points of H. But the bifurcation of critical points has been extensively studied.

When $n = 1$ a symplectomorphism is just an area preserving diffeomorphism of a planar region. In this case by part (b) of lemma 2 a nondegenerate saddle point (nondegenerate maximum or minimum) of H corresponds to a hyperbolic (an elliptic) fixed point of f.

If there are no parameters then generically a function has only nondegenerate saddles, maxima and minima and so generically an area preserving mapping has only hyperbolic and elliptic fixed points. With one parameter a new type of critical point can occur namely $H = x^2 + y^3 + \mu y$. (Warning : In general a simple singularity can be brought to polynomial form by a change of variables but it may not be symplectic!) The corresponding f will have an elliptic and a hyperbolic fixed point for $\mu < 0$ which approach each other as $\mu \to 0-$. At $\mu = 0$, f will have a degenerate fixed point and for $\mu > 0$ no fixed point (see figure 2). For 2 parameters a new type of singularity is given by $H = \pm x^2 + y^4 + \mu y^2 + \nu y$. One can easily analyse the critical points of this simple polynomial.

Figure 2.



Higher Order Bifurcations : In general periodic points bifurcate from a fixed point when the eigenvalues of the linearised map are pth roots of unity. This type of bifurcation was completely investigated in [9] for area preserving mappings depending on one parameter. However the analysis found in [9] and [10] was clumsy in the case when the map has a fixed point where the linearised map has eigenvalues -1 (the transition points). F. Takens suggested the following lemma as a means of simplifying the arguments.

Lemma 3. Let $\varphi: V \to R^2$, V an open neighbourhood of $0 \in R^2$, be an area preserving mapping with the origin as a fixed point. Assume that $D\varphi(0)$ is similar to $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Then there exists a symplectic coordinate system (x,y) at $0 \in R^2$ such that $\varphi:(x,y) \mapsto (X,Y)$ where

$$X = -x + y + \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 + X_5$$

$$Y = -y + ax^3 + bx^2 y + cxy^2 + dy^3 + Y_5$$

and $X_5, Y_5 = o(\{x^2 + y^2\}^{5/2})$.

The proof of this lemma proceeds as the proof of Birkhoff's normalisation theorem. Using this lemma the complicated conditions on the higher order terms in the transition case can be replaced by $a \neq 0$. The analysis proceeds exactly as in [9] but the computations are simpler.

Periodic Solutions near Equilibrium Points : Let M be a 2n-dimensional symplectic manifold, $H:M \to R$ a Hamiltonian, $X = (dH)^{\#}$ and $p \in M$ an equilibrium point of X i.e. $X_p = (dH)^{\#}(p) = 0$. The eigenvalues of $D(dH)^{\#}(p)$-- the Hessian of X at p -- are called the characteristic exponents of X at p. The characteristic exponents appear in negative pairs [1] and thus may be ordered $\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n$. It is generic (codimension zero) that $\lambda_1, \ldots, \lambda_n$ are independent over the integers [8]. If $\lambda_1, \ldots, \lambda_n$ are independent over the integers then a classical theorem of Liapunov states that for each pair of pure imaginary exponents the flow admits a local invariant surface containing p which is filled with periodic orbits.

Codimension 1 bifurcations of these Liapunov families have been considered in the celestial mechanics literature for a system of two degrees of freedom. Most of this literature is devoted to a study of the periodic solutions near the Lagrange triangular libration points in the restricted three body problem. However a careful selection will yield an almost complete list of codimension 1 phenomena.

Again the problem is local and so one may assume that $M = R^4$ and $p = 0 \in R^4$. Let H depend smoothly on a single parameter μ. Then X is given by

$$\dot{x} = A(\mu)x + f(x,\mu) = J \operatorname{grad}_x H(x,\mu)$$

where $f(0,\mu) = 0$, $D_1 f(0,\mu) = 0$. Let the eigenvalues of A be $\lambda_1(\mu), \lambda_2(\mu), -\lambda_1(\mu), -\lambda_2(\mu)$. In this case it can happen generically that there are non-zero integers p and q such that $p\lambda_1 + q\lambda_2 = p\lambda_1(0) + q\lambda_2(0) = 0$ and λ_1, λ_2 are pure imaginary.

<u>Case I : $p = q = 1$.</u> In this case the eigenvalues of A generically are of the form $\pm i\omega(\mu) \pm \beta(\mu)\sqrt{\mu}$ where $\omega(\mu)$, $\beta(\mu)$ are real and $\omega(0) \neq 0$, $\beta(0) \neq 0$. Thus for $\mu < 0$ there are two families of periodic solutions given by Liapunov's theorem and for $\mu > 0$ there are no periodic solutions near zero by the stable manifold theorem. Generically one of two things can happen. Either 1) for $\mu < 0$ and μ small the two Liapunov families are globally connected and as a unit tend to zero as μ tends to zero or 2) the two families exist even for $\mu = 0$ and as μ receeds from zero through positive values the two families detach as a unit and the whole family receeds from the origin. (See [12], [3] and [4], also see figures 3a and b.)

<u>Case II : $p = 1$, $q = 2,3$.</u> This case has not been completely explored but when $q = 2$ the system has only one family of periodic solutions when $\mu = 0$. (See [2], [17].)

<u>Case III : $p = 1$, $q = 4$.</u> In this case the two Liapunov families exist even when $\mu = 0$. The unfolding of this critical point is quite surprising since a whole family of periodic solutions receed from the origin as μ receeds from 0 through positive values (see [13] and [6]). I believe $p = 1$, $q > 4$ is similar to $p = 1$, $q = 4$. (See figure 3c.)

<u>Case IV : $p \geqslant 2$, $q \geqslant 2$ and (p,q) relatively prime.</u> In this case there exist additional periodic solutions of much longer period (approximate period equal to $q2\pi/i\lambda_2 = p2\pi/i\lambda_1$). There are two main subcases :

A) For $\mu \leqslant 0$ there are no periodic solutions of period near $q2\pi/i\lambda_2$. For $\mu > 0$ two families of periodic solutions -- one elliptic and one hyperbolic -- receed from the origin as μ receeds from zero. These two families for $\mu > 0$ connect the two families given by Liapunov's classical theorem.

B) For $\mu < 0$ there are two families -- one elliptic and one hyperbolic -- which bifurcate from one orbit of one of the two Liapunov families. As μ tends to zero, $\mu < 0$, the orbit from which these new orbits bifurcate tends to the origin. For $\mu = 0$ these two families exist and are connected to the origin. For $\mu > 0$ these two families bifurcate from a periodic orbit on the other Liapunov family. (See [6], [11], [13], [15], [17], [18] and figure 3d.)

Figure 3 gives a brief indication of the bifurcations described above. In each figure the y-axis may be taken as the value of H and the x-axis as

Figure 3a

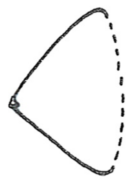


Figure 3b

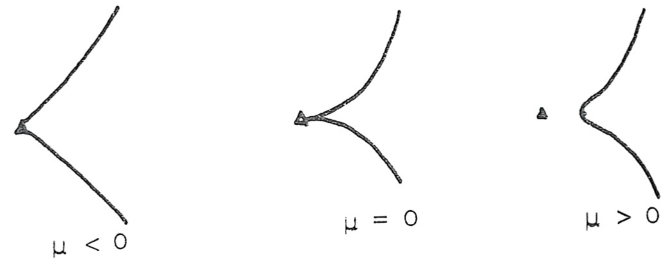


Figure 3c

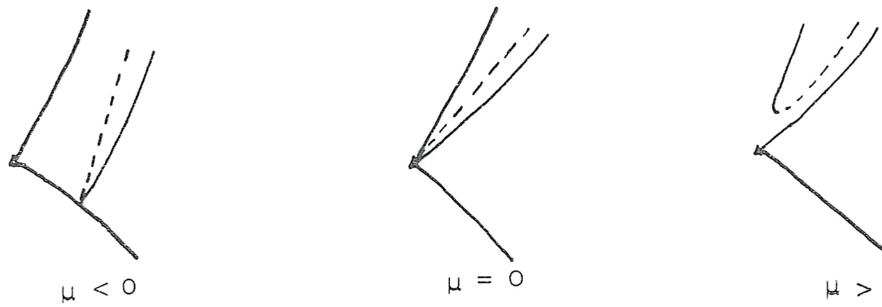
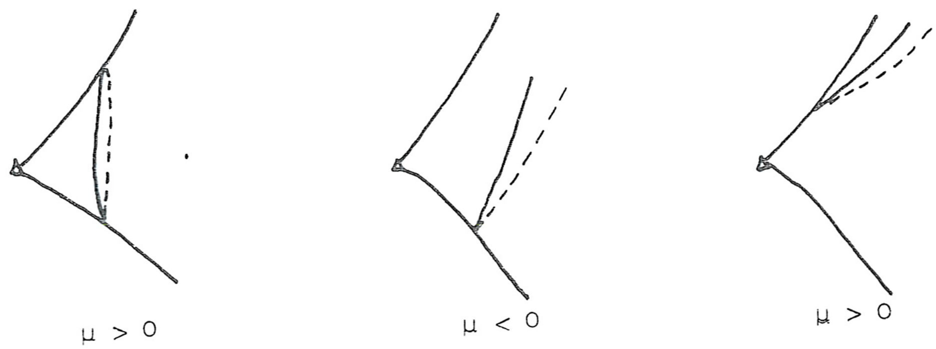


Figure 3d



as the other spatial coordinates. The dashed lines represent families of hyperbolic periodic orbits and the solid lines represent families of elliptic periodic orbits. A point on one of these lines represents a periodic solution. The delta represents the equilibrium point.

<u>Some MSc. problems :</u> 1. Make a complete list of codimension 2 bifurcations of periodic points of an area preserving mapping and compare the results with those found by numerical computations in the restricted problem

(see [4] and [5]).

2. Consider the bifurcations of symmetric periodic orbits in a Hamiltonian system of two degrees of freedom which is invariant under Z_2 action (codimension 0 only). Compare the results with those found in [4], [5] and [7].

3. Complete the list of codimension 1 bifurcations near resonance equilibria.

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