Tori in Resonance

by

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Introduction: This paper gives three examples of ordinary differential equations which depend on one or more parameters and which admit invariant tori for some values of the parameters. These examples illustrate how invariant tori evolve as the parameters are changed; in particular how they disappear, bifurcate and lose smoothness. The equations presented are choosen to be as simple as possible in order to clearly show the interesting phenomenon without unnecessary details. However, the theory of normal forms and unfoldings was used to select typical examples, but no attempt will be made to define precisely the universe of discourse where these examples are generic. The unfolding of invariant tori would consist of a multitude of cases not all of which are that interesting.

Each example contains the small parameter ε which is introduced simply in order to use small parameter methods in the analysis. The parameter ε should be considered simply as a disposable small parameter and not the parameter of interest. The reason for introducing a small parameter is so that examples can be analysized which would otherwise be highly degenerate. These examples are choosen from the class of equations where the usual functional analysis method either fail or are difficult to apply. I consider these examples as counter-examples to several folk conjectures about the evolution of invariant tori.

The invariant tori given in the examples are simply the union of the stable (or unstable) manifolds of a finite number of periodic orbits. This representation of an invariant torus is easier to handle in the situations where there is a loss of smoothness or a bifurcation. Thus in all the examples given below there is an extrainment of the frequencies or the frequencies are in resonance (thus the title of the paper).

I hope that this presentation will help dispel the widely held belief that the evolution of an invariant torus is not much more complicated than the evolution of a periodic solution. The common feeling seems to be that what is complicated is the flow on the invariant torus and not what happens to the torus itself. This belief comes in part from the lack of simple examples and precise theorems. One of the strongest supports of the thesis that the evolution of an invariant torus is complex and interesting is the computer studies found in Aronson, Chory, Hall and McGehee [1]. This paper considers the equivalent problem of tracing an invariant circle of a map as two parameters a varied. This paper shows that the evolution of an invariant circle for a map is rich in phenomena and wrothy of study.

The first example illustrates what happens when two invariant tori almost undergo a saddle-node type bifurcation. Resonance terms are so choosen that only part of the invariant tori undergo a saddle-node bifurcation before the rest of them does. Thus as a parameters is varied two invariant tori come together and touch along one periodic orbit for a particular value of the parameter. Changing the parameter more destroy the structure of the tori; but two periodic solutions (one from each torus) presist for a short range of the parameters. Finally, the remaining periodic solutions undergo a saddle-node bifurcation and all remnants of the invariant tori are gone. Thus there is an interval in the parameter range between the values where there are two tori and where the invariant sets have completely disappeared.

The second example illustrates what happens when an invariant torus almost undergoes a Hopf bifurcation. Like the first example, part of the invariant torus undergoes a Hopf bifurcation before the rest does. This bifurcation gives raise to a range of parameter values where there is an invariant set which is a type of "pinched manifold". The invariant set in this parameter range can be visualized as a three torus which is pinced along one circle.

The third example illustrates the loss of smoothness that an invariant torus can experience when one of the periodic orbits in it becomes a sprial. This example models one of the phenomena discussed in Aronson et. al [1].

I would like to thank Professor George Sell for his though provoking comments on this subject. Indeed, he suggested the phenomena which is illustrated by example 2.

Example 1. A Saddle-Node Bifurcation

This example considers perturbations of a two dimensional autonomous system which depends on a parameter μ in such a way that for $\mu < \mu_0$ the system has two limit cycles (one stable and one unstable), a semi-stable limit cycle for $\mu = \mu_0$, and no limit cycle for $\mu > \mu_0$. The simplist model for the unperturbed system is

$$(1) \qquad \begin{aligned} \dot{\rho} &= (\mu - \mu_0) - (\rho - \rho_0)^2 \\ \dot{\theta} &= \omega \end{aligned}$$

where ω is a non-zero constant (the frequence). In the above (ρ,θ) are polar coordinates in an annular region around the circle $\rho = \rho_0$. One readily sees that this equation admits two limit cycles, $\rho = \rho_0 \pm \sqrt{(\mu - \mu_0)}$ for $\mu > \mu_0$; one limit cycle $\rho = \rho_0$ for $\mu = \mu_0$; and no limit cycle for $\mu < \mu_0$. The periods of these limit cycles are $2\pi/\omega$.

Now add a small non-autonomous, but T periodic, perturbation to equation (1). If we artifically consider equation (1) as a T-periodic system, then the limit cycles discussed above become invariant tori in space-time when one identifies the time coordinate modulo T. It is natural to suspect that even after the perturbation is added there is a μ^* close to μ_0 such that for $\mu > \mu^*$ there are two invariant tori, for $\mu = \mu^*$ there is one tori and for $\mu < \mu^*$ there are none. The example given below shows that this conjecture is false at least in the resonance case. Chenciner [2] has shown the conjecture is true in the case when ω is badly approximated by rationals. Thus we are left with the usual gap in differential equations, what happens in the case when the frequencies are well approximated by rationals?

In the resonance case, when $\omega = p/q$, (p and q relative prime integers) the general theory of normal forms predicts that a non-trivial perturbation term should be a 2π-periodic function of $(q\theta - pt)$. That is one can eliminate any other type of perturbation term by a normalization transformation. Instead of considering the most general case, consider only the perturbation given by

$$\rho = (\mu - \mu_0) - (\rho - \rho_0)^2 + \varepsilon^2\alpha \cos(q\theta - pt)$$

(2)

$$\theta = p/q + \varepsilon\beta \sin(q\theta - pt) .$$

In the above α and β are positive constants and the system is $2\pi/p$ periodic. We wish to study these equations when ρ is near ρ_0 and μ is near μ_0 so make the scaling transformation $\varepsilon r = \rho - \rho_0$, $\varepsilon^2 \nu = \mu - \mu_0$. The equations (2) become then

$$r = \varepsilon\{\nu - r^2 + \alpha \cos(q\theta - pt)\}$$

(3)

$$\theta = p/q + \varepsilon\beta \sin(q\theta - pt) .$$

In the above equation only $q\theta - pt$ appears and so it is natural to introduce a new angle ψ equal to this combination but this would mix the time variable and the spacial variables. In order to keep the geometry straight introduce a new angular variable τ defined modulo 2π, agument the equation (3) with the equation $\tau = 1$ and replace t by τ on the right hand side of (3). The variables (r,θ,τ) are now variables in $R^1 \times S^1 \times S^1 = R^1 \times T^2$. Since p and q are relative prime integers there are integers a and b such that $ap + bq = 1$. Make the change of variables

$$\psi = q\theta - p\tau \tag{4}$$
$$\sigma = a\theta + b\tau$$

so that the equations become

$$r = \varepsilon\{\nu - r^2 + \alpha \cos \psi\}$$
$$\psi = \varepsilon\beta \sin \psi \tag{5}$$
$$\sigma = 1/q + \varepsilon a\alpha \sin \psi .$$

Since the coefficients q, p, a, b in (4) are integers and the determinant of the coefficients in (4) is $+1$ the transformation (4) and its inverse preserve the integer lattice Z^2 in R^2. Thus (4) represents a valid change of variables on T^2 or both ψ and σ are angular coordinates defined mod 2π.

Since the first two equations in (5) do not depend on σ they can be analyzed separately. Also the first two equations in (5) are autonomous so the classical phase plane analysis method can be used.

These equations have critical points at $\psi = 0$, $r = \pm\sqrt{\nu + \alpha}$ and $\psi = \pi$, $r = \pm\sqrt{\nu - \alpha}$. Linearizing about these critical points gives that the critical points are of the following types:

$\psi = 0 \, , \, r = +\sqrt{\nu + \alpha}$ is a saddle

$\psi = 0 \, , \, r = -\sqrt{\nu + \alpha}$ is a source

$\psi = \pi \, , \, r = +\sqrt{\nu - \alpha}$ is a sink

$\psi = \pi \, , \, r = -\sqrt{\nu - \alpha}$ is a saddle.

In the above ν is assumed large enough that the square roots are real and non-zero. The critical points along $\psi = 0$ undergo a saddle-node bifurcation when $\nu = -\alpha$, whereas the critical points along $\psi = \pi$ undergo a saddle-node bifurcation when $\nu = +\alpha$. Also note that the rays $\psi = 0$ and π are invariant and that the flow along these rays are as shown in figure 1a) when $\nu > +\alpha$. When referring to the figures 1a)b)c)d) recall that $r = 0$ is not the origin in the plane but the circle $\rho = \rho_0$.

When $\nu > +\alpha$ all four critical points exist, r is decreasing on the circle $r = \pm 2\sqrt{\nu + \alpha}$, and r is increasing along the circle $r = 0$. Consider the half of the unstable manifold of the saddle point at $\psi = 0$, $r = +\sqrt{\nu + \alpha}$ which lies in the upper half plane. It is trapped in the semi-annular ring $0 \leq r \leq 2\sqrt{\nu + \alpha}$, $0 \leq \psi \leq \pi$. There are no critical points in the interior of this ring and ψ is increasing. Thus this unstable manifold must approach the sink at $\psi = \pi$, $r = +\sqrt{\nu - \alpha}$ as $t \to \infty$. The same is true for the other half of the unstable manifold of the critical point at $\psi = 0$, $r = +\sqrt{\nu + \alpha}$. Thus the first two equations in (5) have an invariant circle for $\nu > +\alpha$ which consists of the saddle point at $\psi = 0$, $r = +\sqrt{\nu + \alpha}$ and its unstable manifold plus the sink at $\psi = \pi$, $r = +\sqrt{\nu + \alpha}$. A similar analysis (by reversing time) shows that when $\nu > +\alpha$ there is another invariant circle consisting of the saddle $\psi = \pi$, $r = -\sqrt{\nu - \alpha}$ and its stable manifold plus the source at $\psi = 0$, $r = \sqrt{\nu + \alpha}$. See figure 1a). There invariant curves are smooth except possible at the source and sink where they may have a kink.

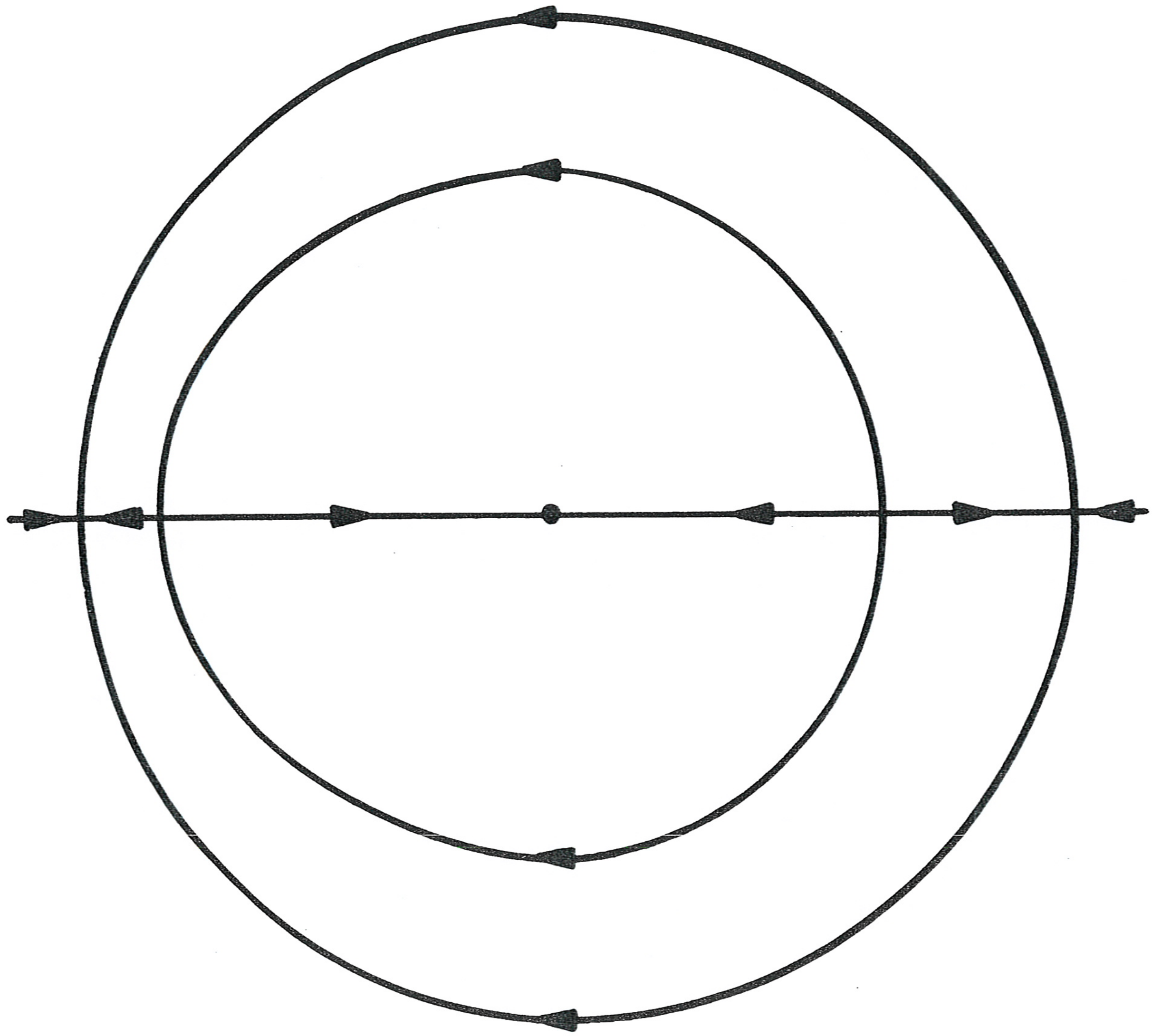

Figure 1a

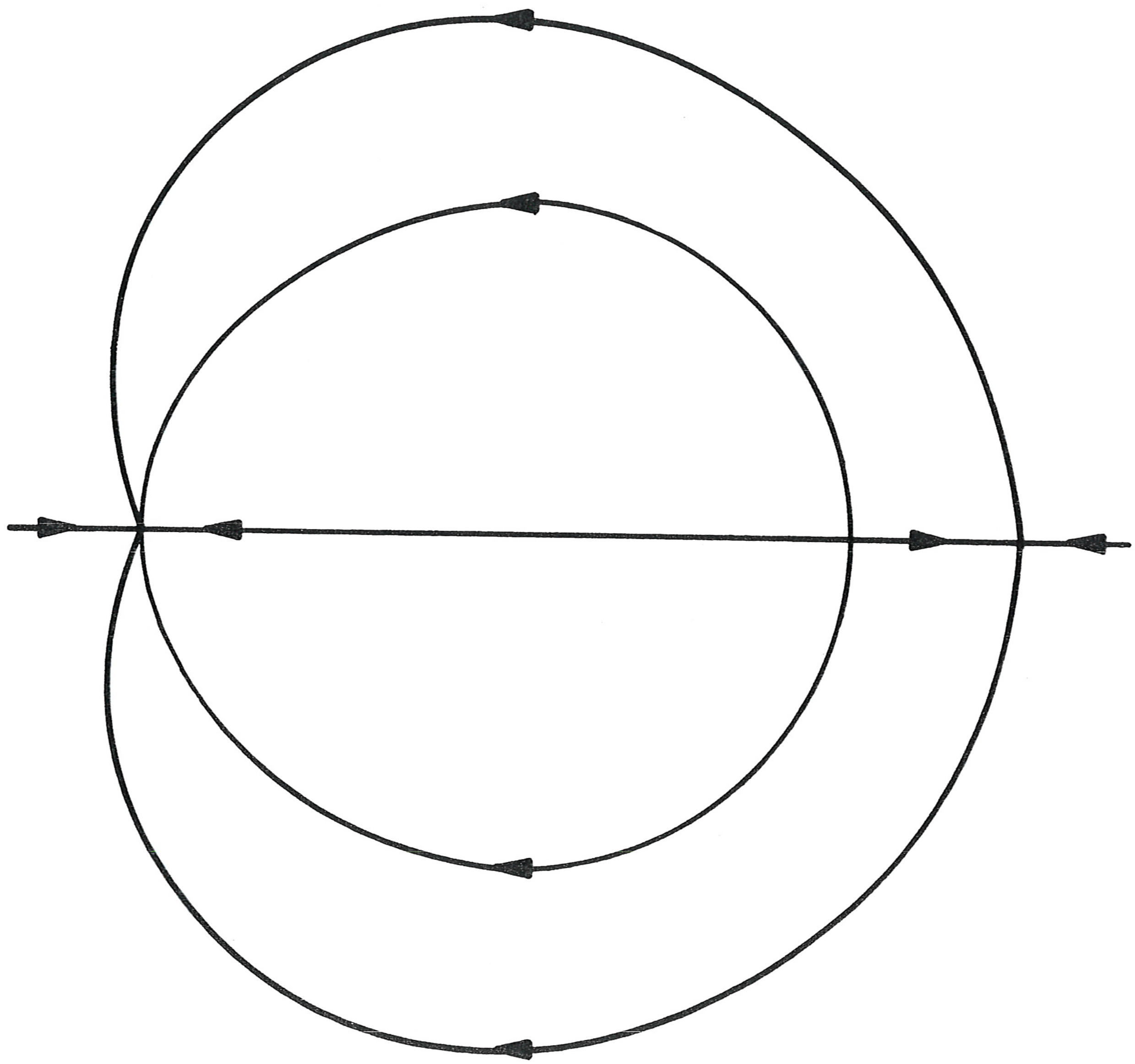

Figure 1b

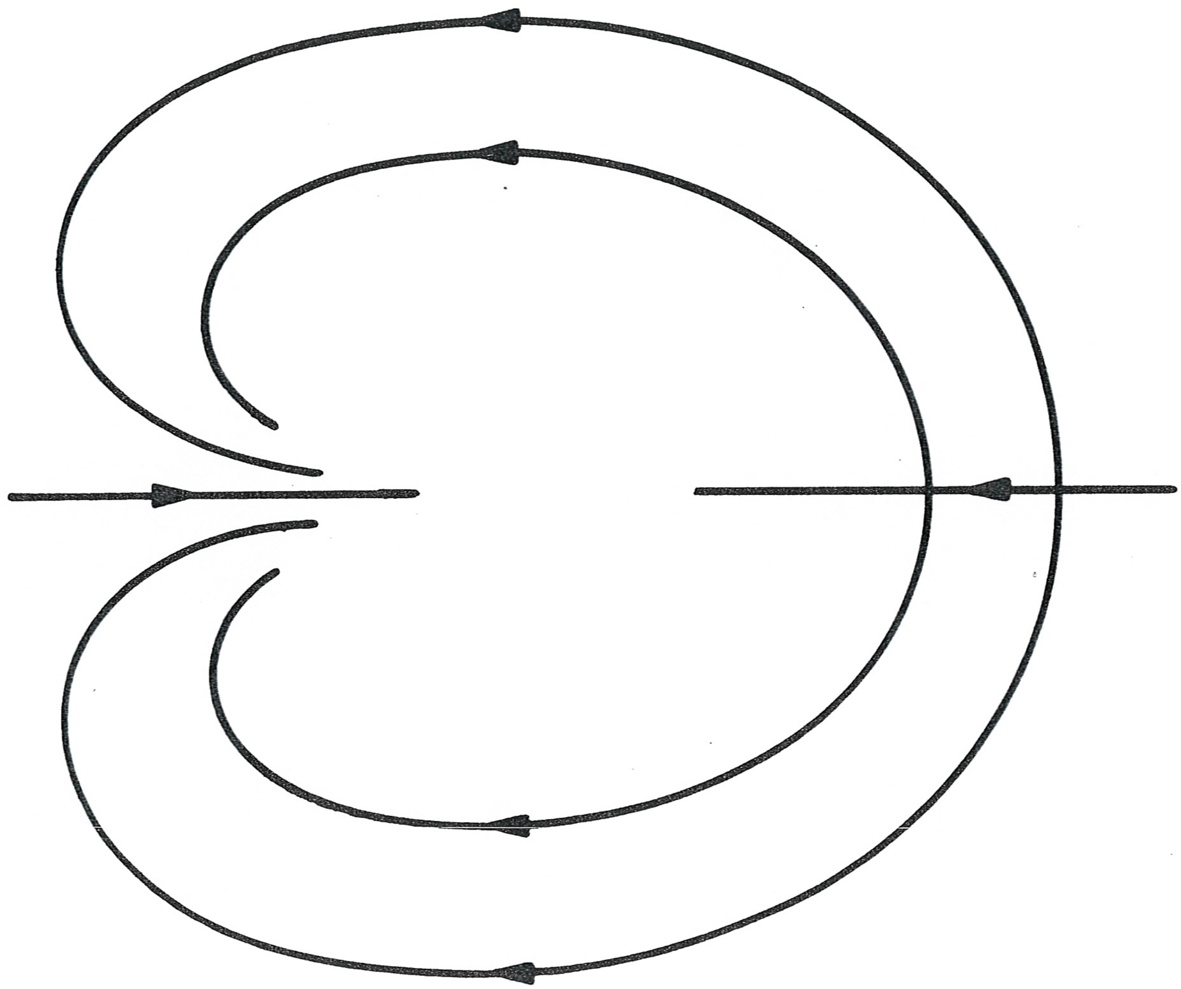

Figure 1c

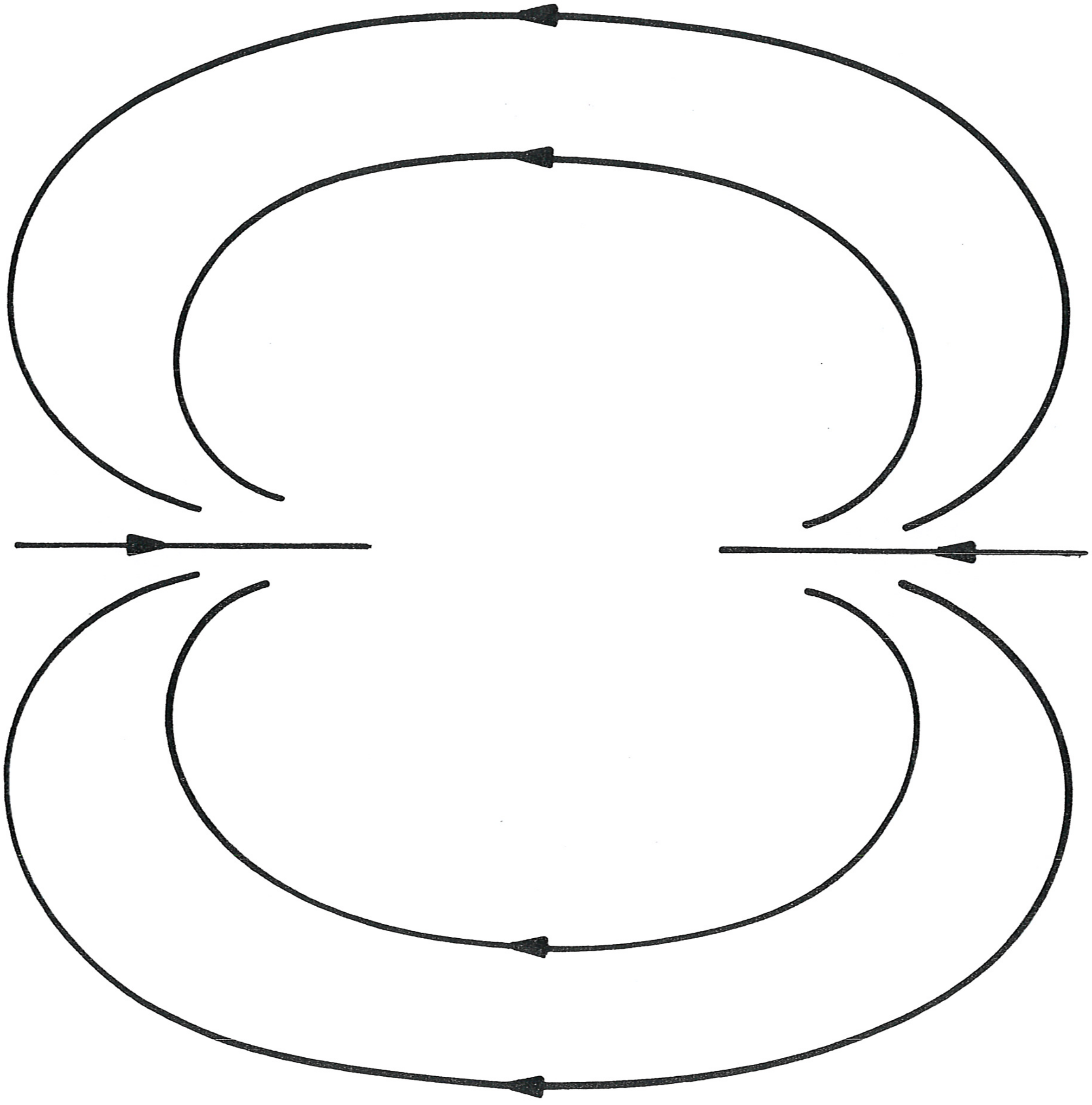

Figure 1d

The same type of analysis can be carried out when $\nu = +\alpha$ to show that these first two equations in 5a) have an invariant set which consists of two circles which meet at $\psi = \pi$, $r = 0$ as shown in figure 1b). A similar analysis yields figure 1c) when $\alpha > \nu > -\alpha$ and figure 1d) when $\nu < -\alpha$.

I claim that these figures 1a)b)c)d) also represents a picture of the section map for the full set of equation in (5) where the surface of section is taken as $\sigma = 0$. Let $r(t,r_0,\psi_0), \psi(t,r_0,\psi_0)$, $\sigma(t,r_0,\psi_0)$ be the solution of equations (5) which pass through $r = r_0$, $\psi = \psi_0$, $\sigma = 0$ when $t = 0$. Clearly $\sigma(t,r_0,\psi_0) = t/q + O(\varepsilon)$ so we may apply the implicit function theorem to the equation $\sigma(t,r_0,\psi_0) = 2\pi$ to yield the existence of a smooth function $T(r_0,\psi_0) = 2\pi q + O(\varepsilon)$ which is the first return time to the $\sigma = 0$ section. The section map is then the map $(r_0,\psi_0) \to (r(T(r_0,\psi_0),r_0,\psi_0),\psi(T(r_0,\psi_0),r_0,\psi_0))$. Thus the section map is obtained by following the flow of the first two equations in (5) by a time $T = 2\pi q + O(\varepsilon)$.

Going back to the full three dimensional problem one sees that the equation (5) and hence equations (3) and (2) have two invariant tori when $\mu > +\alpha$ which contain two periodic solutions per torus. As μ approaches $+\alpha$ from above these tori aproach each other alone one circle and at $\mu = +\alpha$ the equations admit an invariant set which consists of two tori which have a circle in common. When $+\alpha > \mu > -\alpha$ the equations still have two periodic solutions but there unstable manifolds no longer form tori. As $\mu \to -\alpha$ from above the two remaining periodic solutions undergo a saddle-node bifurcation and disappear.

Remark: The figures 1a)b)c)d) do not represent the period map for equations (2) or (3). The surface $\sigma \equiv 0 \bmod 2\pi$ corresponds to $a\theta + b\tau \equiv 0 \bmod 2\pi$ in the original variables. The picture of the period map can be obtained from these pictures by cutting anyone of these figures along the ray $\psi = 0$ and then linearly contracting the cut plane in the angular direction until it fits in a wedge of angular width $2\pi/q$. Now fill out the figure by duplicating the drawing in the wedge in each of the q wedges $j2\pi \leq \theta < (j+1)2\pi$, $j = 0,\ldots,q-1$.

Example 2: A Hopf bifurcation.

This example was suggested by Professor George Sell and should be considered as an example which proves the necessity of the non-resonance condition in his paper [3]. The example is a perturbation of an autonomous three dimensional system which depends on a parameter μ . The unperturbed autonomous system has a limit cycle for all values of the parameter μ , but for $\mu = 0$ it undergoes a Hopf bifurcation, i.e. an invariant two torus bifurcates from the limit cycle as the limit cycle changes stability. The unperturbed equation is

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega \\ \dot{\phi} &= \lambda . \end{aligned} \tag{1}$$

Here (r,θ,ϕ) are coordinates for a solid torus, i.e. (r,θ) are polar coordinates in R^2 and ϕ is a coordinate on S^1 (see figure 2c)). (A completely different interpretation of this example can be given when (r,ϕ) are the polar coordinates in R^2 and θ is a coordinate in S^1 .) Both θ and ϕ are angular variables defined modulo 2π and $r \geq 0$. The equations (1) admit a limit cycle $r = 0$ for all values of μ which is asymptotically stable when $\mu < 0$ and unstable when $\mu > 0$. For $\mu > 0$ this limit cycle is encircled by a stable invariant torus $r = \sqrt{\mu}$, θ and ϕ arbitrary and the flow on this invariant torus is the linear flow $\dot{\theta} = \omega$, $\dot{\phi} = \lambda$.

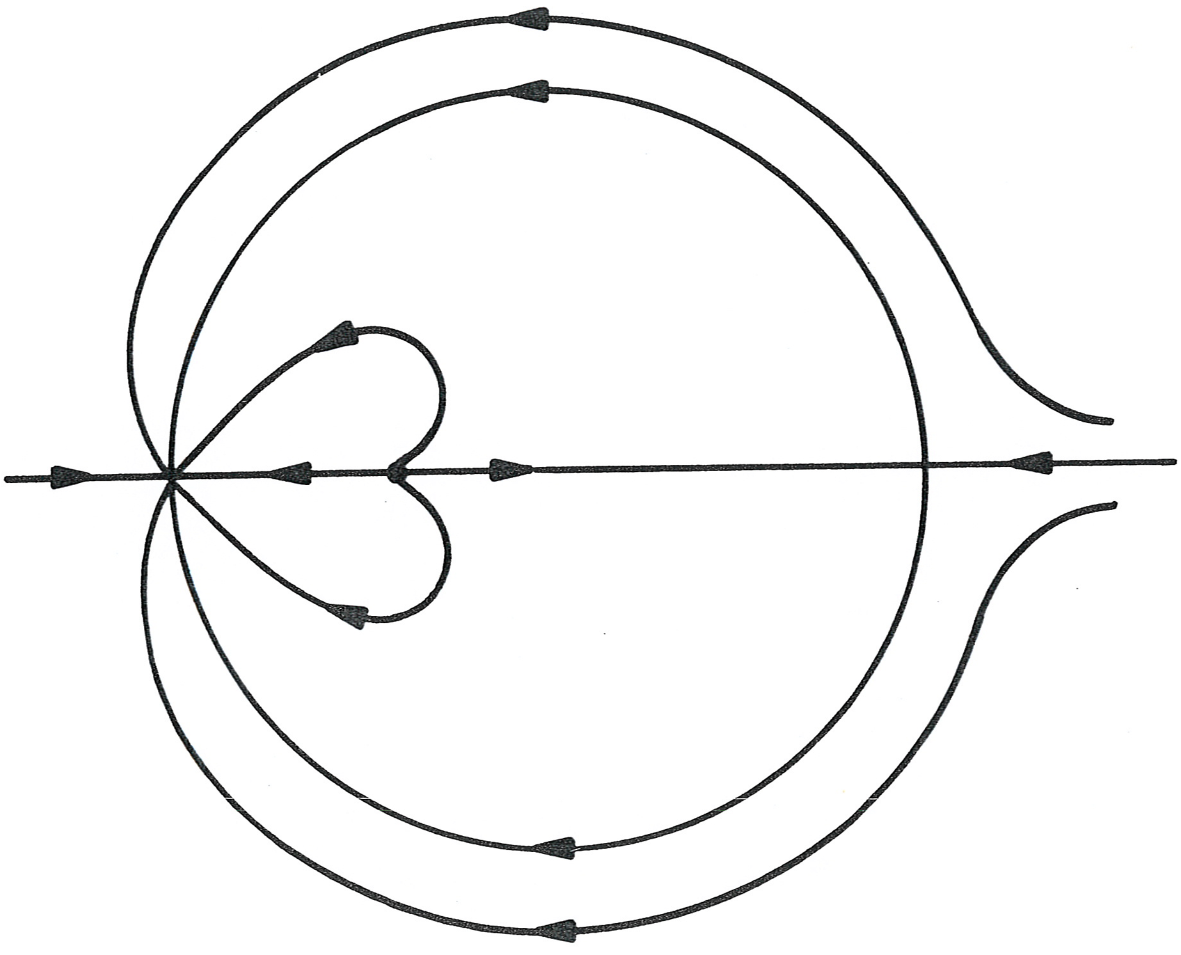

Figure 2a

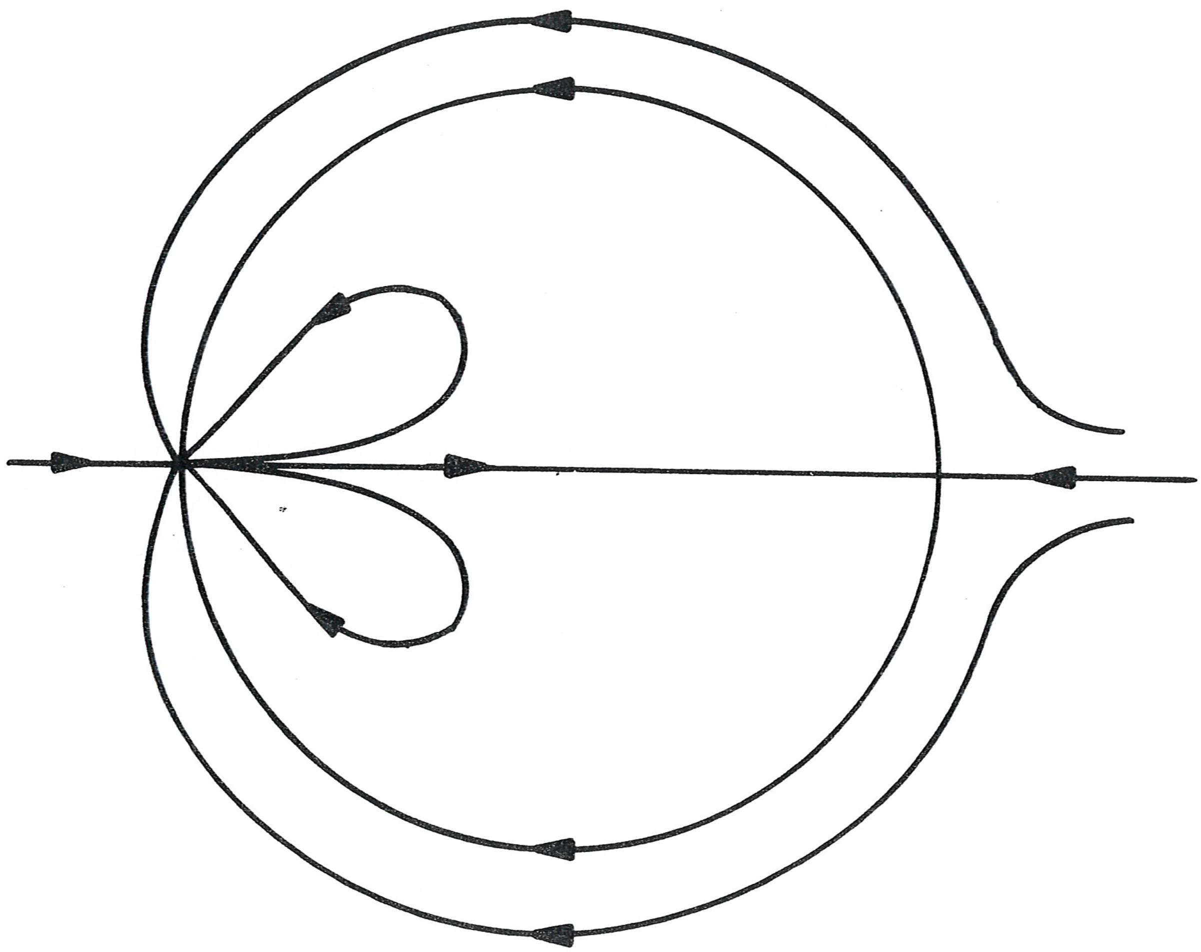

Figure 2b

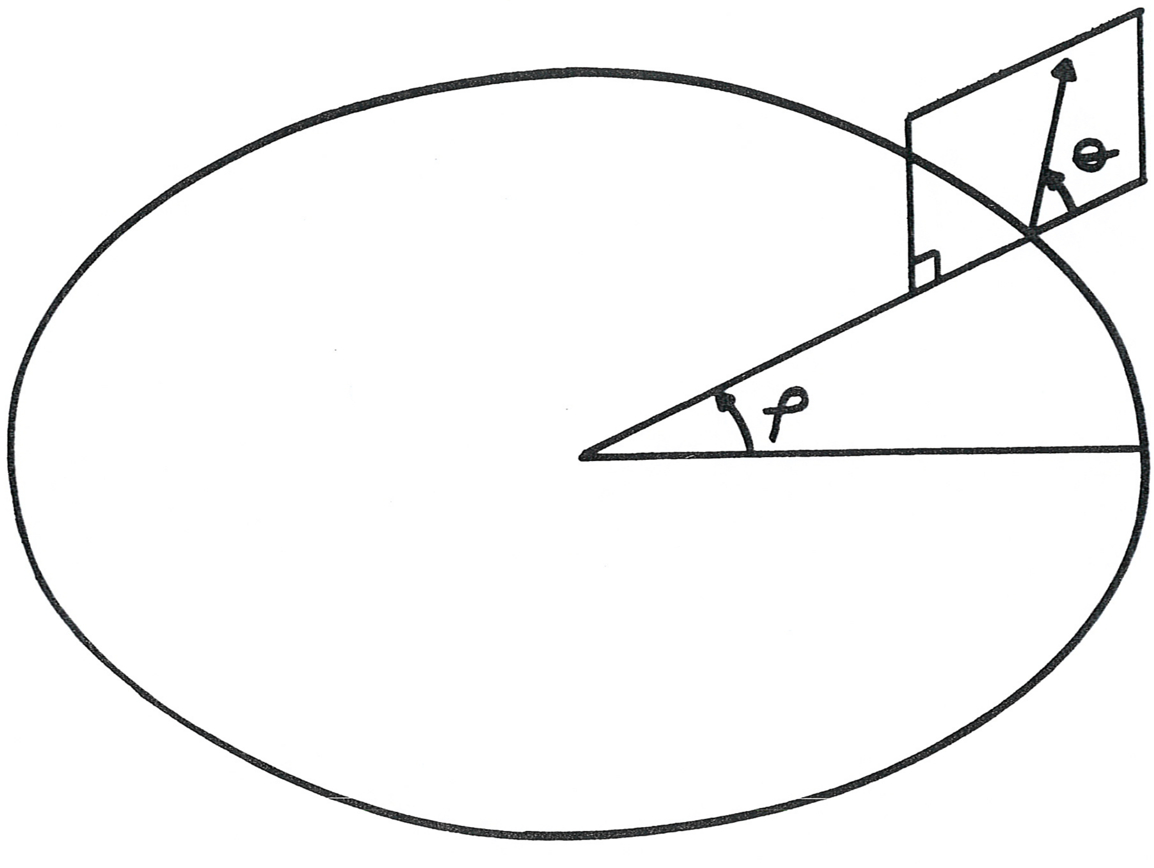

Figure 2c

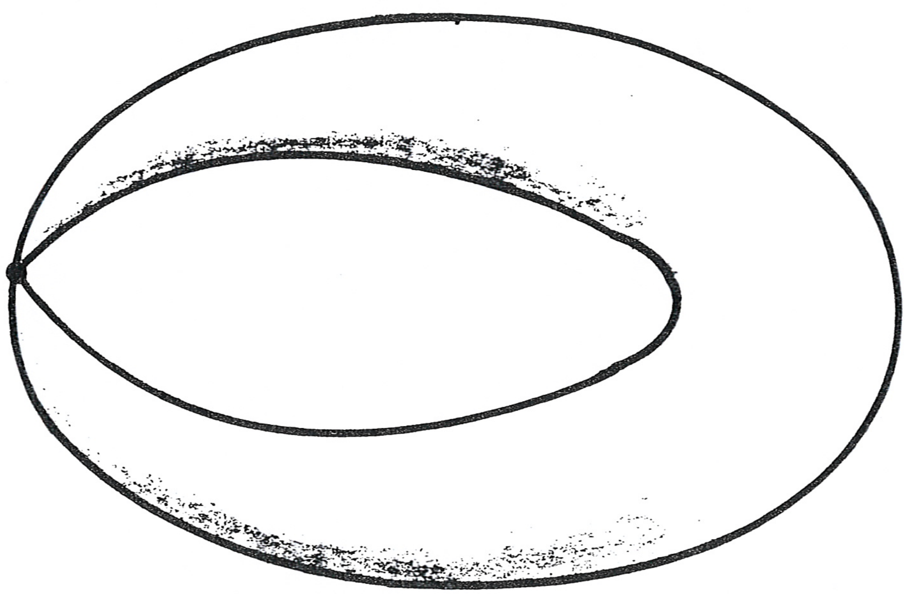

Figure 2d

Add to (1) a small non-autonomous, but periodic, perturbation which is in resonance with one of the frequencies, say λ . If equation (1) is artifically considered as a T-periodic system then the limit cycle becomes an invariant 2-torus and the invariant 2-torus becomes an invariant 3-torus in the space-time (r,θ,ϕ,t) where t is an angular variable modulo T . As in the previous example, the perturbation terms are choosen as the simplest non-trivial terms as predicted by the theory of normal forms.

Let the perturbed equations be

$$
(2)\qquad \begin{aligned}
\dot{r} &= \mu r - r^3 + \varepsilon^2 \cos (q\phi - pt)\\
\dot{\theta} &= \omega + \varepsilon^2 f(q\phi - pt)\\
\dot{\phi} &= p/q + \varepsilon^2 \sin (q\phi - pt)
\end{aligned}
$$

where f is an arbitrary, smooth, 2π-periodic function whose precise form is unimportant. In order to investigate what happens when μ and r are small scale by $r \to \varepsilon r$, $\mu \to \varepsilon^2\mu$. As before introduce τ and change coordinates by $\psi = q\phi - p\tau$, $\sigma = a\phi + b\tau$ where $qb + ap = 1$. The equations become

$$
(3)\qquad \begin{aligned}
\dot{r} &= \varepsilon^2\{\mu r - r^3 + r \cos \psi\}\\
\dot{\theta} &= \omega + \varepsilon^2 f(\psi)\\
\dot{\psi} &= \varepsilon^2 p \sin\psi\\
\dot{\sigma} &= 1/q + \varepsilon^2 a \sin \psi \,.
\end{aligned}
$$

For the moment ignore the θ and σ equations and call the remaining equations for r and ψ equations (3'). These equations have three critical points: $r = 0$; $r = \sqrt{\mu + 1}$, $\psi = 0$; and $r = \sqrt{\mu - 1}$, $\psi = \Pi$. When $\mu > -1$ the critical point at $r = \sqrt{\mu + 1}$, $\psi = 0$ is a saddle and when $\mu > 1$ the critical point at $r = \sqrt{\mu - 1}$, $\psi = \pi$ is a sink. When $\mu > +1$ all three critical points exist, r is decreasing on large circle and r is increasing on small circles. Thus the unstable manifold of the saddle must approach the sink at $r = \sqrt{\mu - 1}$, $\psi = \pi$ as shown in figure 2a).

As $\mu \to +1$ from above the sink approaches the critical point at the origin and at $\mu = 1$ only the saddle and the critical point at the origin presist. For all μ , r is decreasing on large circles and so for $1 \geq \mu > -1$ the unstable manifold of the saddle approaches the critical point at the origin as shown in figure 2b). As $\mu \to -1$ from above the saddle approaches the origin and for $\mu \leq -1$ there is only the critical point at the origin which is a sink.

Thus for $\mu > +1$ equation (3') have an invariant circle which is made up of the unstable manifold of the saddle and the sink and which is smooth except possible at the sink. Also for $\mu > +1$ there is a source at the origin. As $\mu \to +1$ from above the invariant circle approaches the critical point at the origin and at $\mu = +1$ the invariant circle attaches to the critical point at the origin. For $1 \geq \mu > -1$ there invariant circle is attached to the origin but as $\mu \to -1$ the invariant circle approaches the origin. For $\mu < -1$ the origin is a sink.

Now return to the full set of equations in (3). If ε is sufficiently small θ and σ are always increasing since $\omega > 0$, $1/q > 0$. Recall that (r,θ) are polar coordinates in R^2 while ψ and σ are angular varialbes in S^1 . Thus $r = 0$ corresponds to a 2 dimensional torus. Since $r = 0$ is invariant equations (3) always admit an invarint two torus. But for $\mu > +1$ the equations (3') also admit an additional invariant circle and so equations (3) admit an invariant three torus. As $\mu \to +1$ from above the invariant three torus approaches the two torus. For $1 \geq \mu > -1$ the invariant two torus remains but there is also an invariant set which is a 3 torus with one circle identified to a point. As $\mu \to -1$ from above this pinched 3 torus approaches the two torus and disappears. For $\mu < -1$ the two torus is a sink.

In order to picture the above bifurcation consider the $\sigma = 0$ section map. This map will have the circle $r = 0$ as invariant. For $-1 > \mu > 1$ this section map will have an invariant set which is a pinched two torus as shown in figures 2c) and 2d).

It is important to note that the equations (2) have the D'Alembert character with respect to r , θ and so represent analytic equations.

Example 3: Loss of Smoothness

Aronson, Chory, Hall and McGehee [1] made numerous computer studies of the evolution of invariant circles for a mapping of the plane. They discovered various ways in which an invariant curve can lose its smoothness and this explicit example illustrates one of the phenomena that they observed. In order for an invariant curve or torus to lose its smoothness it is necessary that the perturbation be of the same order of magnitude as the normal contraction to the invariant object. For this reason we shall not present the present example as a small perturbation of a very simple system, but simply give the equation as

$$(1) \qquad \begin{aligned} r &= \varepsilon\{-r - \mu \sin(q\theta - pt)\} \\ \theta &= p/q + (\varepsilon/q)\{r + \sin(q\theta - pt)\} . \end{aligned}$$

Here $r = 0$ corresponds to a circle of radius $\rho_0 > 0$ and so (r,θ) are coordinates in an annular region. As in the previous examples, introduce τ and let $\psi = q\theta - p\tau$, $\sigma = a\theta + b\tau$ where $qb + pa = 1$. The equations become then

$$(2) \qquad \begin{aligned} r &= \varepsilon\{-r - \mu \sin \psi\} \\ \psi &= \varepsilon\{r + \sin \psi\} \\ \sigma &= 1/q + (\varepsilon a/q)\{r + \sin \psi\} . \end{aligned}$$

Consider the first two equations in (2) and call them equations (2'). For small μ these equations have two critical points at $r = 0$, $\psi = 0$ or π . The critical point at $\psi = 0$ is a saddle for all small μ whereas the critical point at $\psi = \pi$ is a stable mode for $\mu > 0$, a stable degenerate mode for $\mu = 0$ and a stable sprial for $\mu < 0$.

First consider the case when $\mu = 0$. Then the circle $r = 0$ is invariant for equations (2') and consists of the unstable manifold plus the sink. Since the local structure of a flow near a hyperbolic critical point in the plane is C^1 equivalent to the linearized flow, the degenerate node at $r = 0$, $\psi = \pi$ look (approximately) as shown in figure 3a).

For $\mu \neq 0$ but small, r is decreasing for $r > 0$ and increasing for $r < 0$. Thus the unstable manifold of the saddle point is trapped in an annular ring about $r = 0$ and so must tend to the sink at $r = 0$, $\psi = \pi$. Thus for small μ the equations (2') admit an invariant circle consisting of the unstable manifold of the saddle plus the sink.

When $\mu < 0$ the sink at $r = 0$, $\psi = \pi$ is a focus so even though the equations admit an invariant circle it is not even Lipschitzian. See figure 3b). When $\mu > 0$ the sink at $r = 0$, $\psi = \pi$ is a node. It is difficult to determine in which direction the unstable manifold approaches the sink. But since the direction of approach to a node always has a limit it is clear that the invariant circle is at least Lipschitzian. As μ increases the curve must become smoother and smoother. (see figure 3c).

As in the previous examples the figures 3a)b)c) represent the picture of the cross section of the $\sigma = 0$ section.

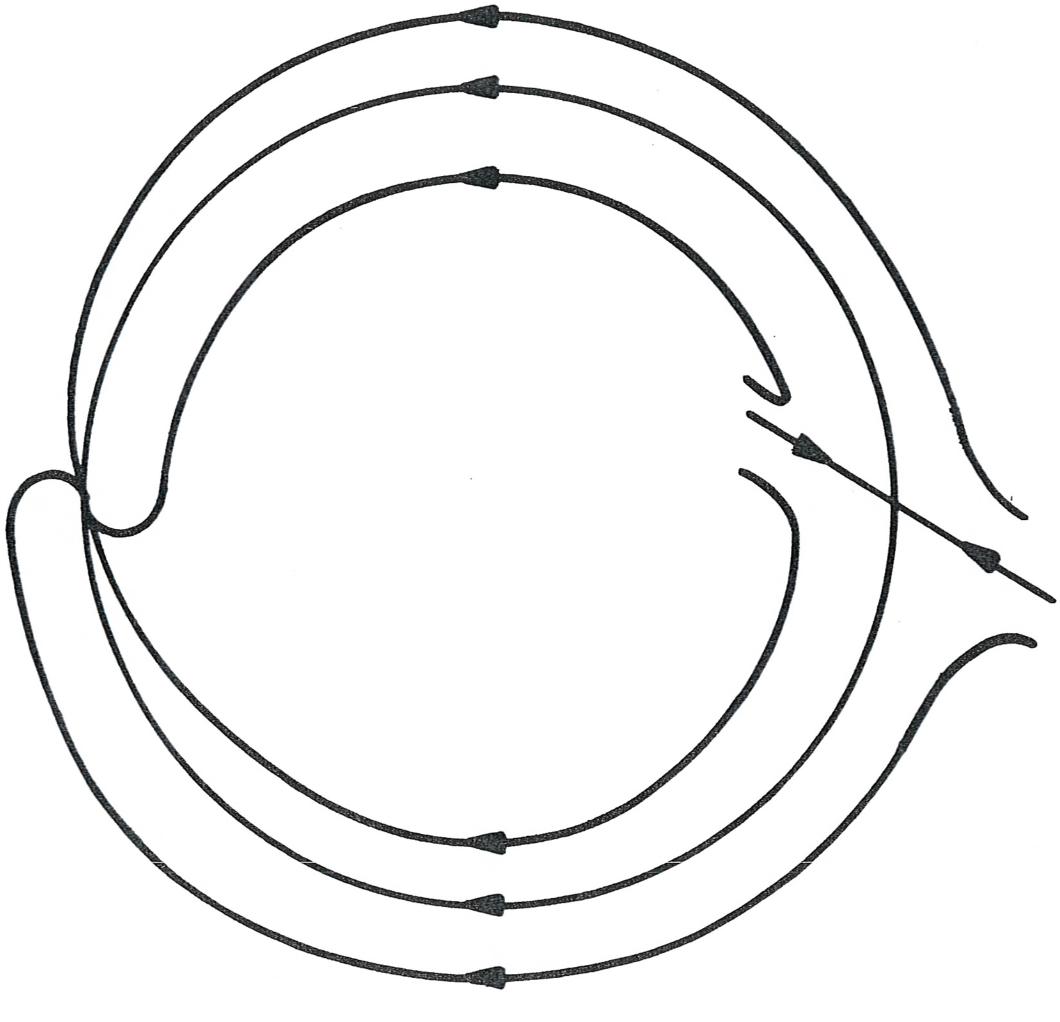

Figure 3a

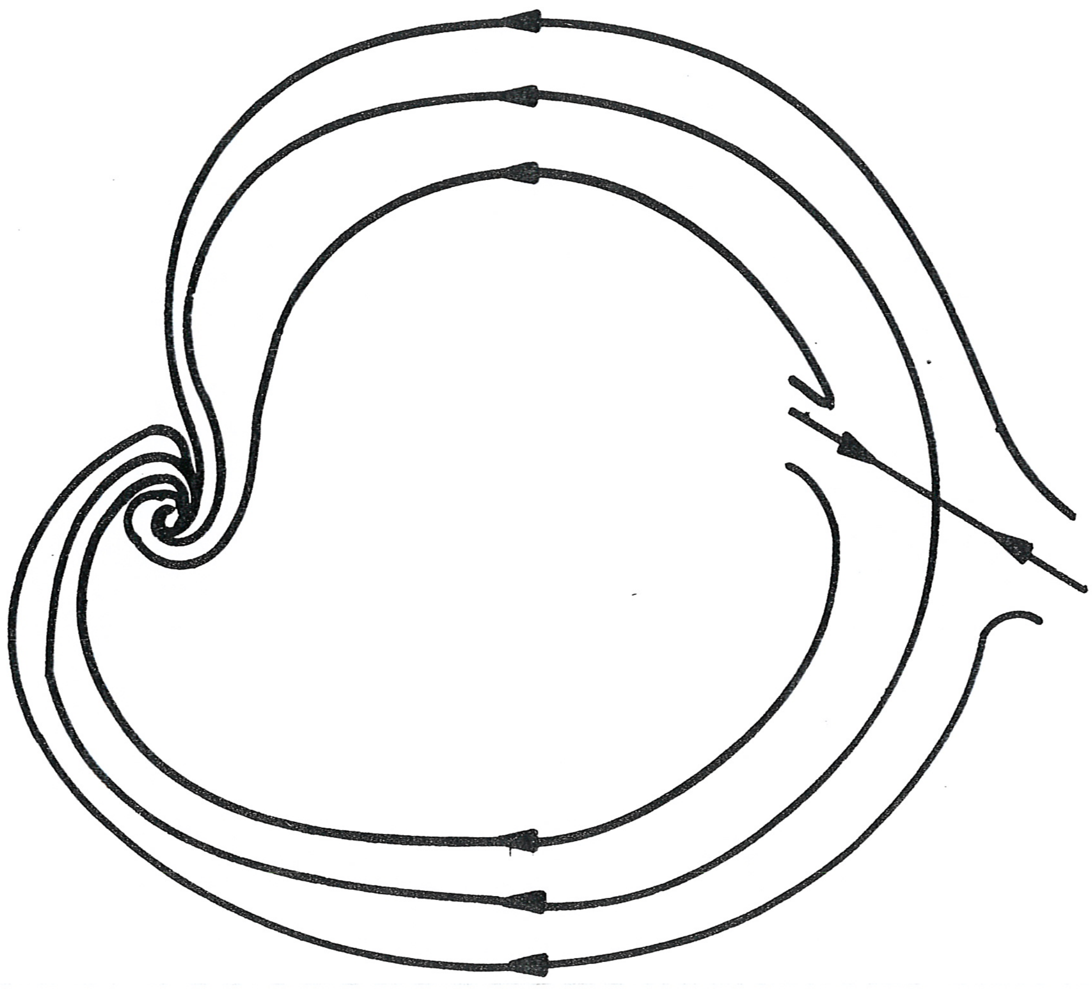

Figure 3b

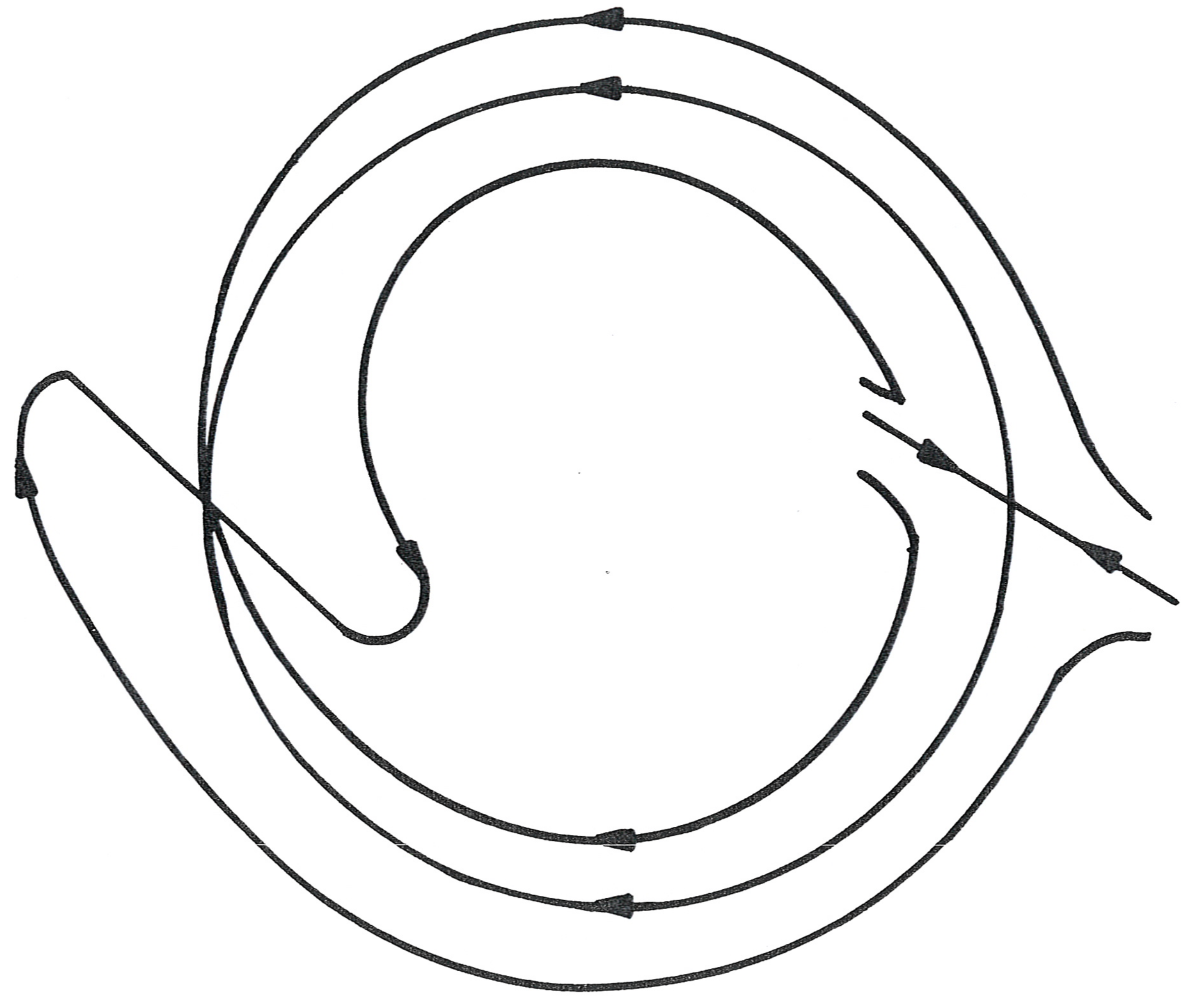

Figure 3c

References

[1] Aronson, D.G., Chory, M.A., Hall, G.R., McGehee, R.P., Bifurcation from an invariant circle for two-parameter families of maps of the plane, Comm. Math. Phy., 83, (1982) 303-354.

[2] Chenciner, A., Courbes fermees invariantes non normalement hyperboliques au voisinage d'une bifurcation de Hopf degeneree diffeomorphismes de(R^2,0) , preprint.

[3] Sell, G., Bifurcation of higher dimensional tori, Arch. Rat. Mech. Anal. 69 (1979), 199-230.