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ADIABATIC INVARIANTS FOR LINEAR HAMILTONIAN SYSTEMS

Anthony LEUNG
and
Kenneth MEYER *
University of Cincinnati

RESUME

On considère un système d'équations de la forme $\epsilon\dot{u} = \mathrm{A}(t)\,u$ pour lequel la matrice $\mathrm{A}(t)$ de dimension $2n$ est réelle et hamiltonienne ; quel que soit t dans l'intervalle $[-\infty, +\infty]$, les valeurs propres de $\mathrm{A}(t)$ sont supposées distinctes et imaginaires ; enfin, pour tout $j \geqslant 1$, $d^j\,\mathrm{A}/dt^j$ appartient à l'espace fonctionnel $\mathrm{L}_1(-\infty, +\infty)$.

Dans ces conditions, on peut choisir les vecteurs propres $d_1(t), \ldots, d_2(t)$ de $\mathrm{A}(t)$ de façon lisse, c'est-à-dire de manière que $d_i = d_{i+n}$ pour $i = 1, 2, \ldots, n$ et que la suite $(d_1(t), \ldots, d_{2n}(t))$ constitue une base symplectique de C^n en tout point t. Moyennant quoi, les n fonctions $\mathrm{I}_i(t\,, u) = \langle d_i(t)\,, u\rangle\langle d_{i+n}(t)\,, u\rangle$ $(1 \leqslant i \leqslant n)$ sont indépendantes et forment un système en involution. Les fonctions $\mathrm{I}_1, \ldots, \mathrm{I}_n$ constituent des invariants adiabatiques du système $\epsilon\dot{u} = \mathrm{A}(t)\,u$ au sens que voici. Soient $(t_0\,, u_0)$ des constantes et $\phi(t\,,\epsilon)$ la solution du système $\epsilon\dot{u} = \mathrm{A}(t)\,u$ qui satisfait la condition initiale $u(t_0) = u_0$; posons $\mathfrak{I}_i(t\,,\epsilon) = \mathrm{I}(t\,,\phi\,(t\,,\epsilon))$. Il en résulte que

$$\mathfrak{I}_i(\infty\,,\epsilon) - \mathfrak{I}_i(-\infty\,,\epsilon) = 0(\epsilon^n)$$

lorsque ϵ tend vers 0^+ par valeurs positives, et cela pour tout $n = 1, 2, \ldots$. Ce résultat est la généralisation naturelle d'une théorème de Littlewoods qui caractérise les invariants adiabatiques d'un système linéaire à un degré de liberté.

INTRODUCTION

This paper is a shortened version of [5]. In the classical literature a conservative dynamical system of n degrees of freedom was considered solved when n independent integrals in involution were found. One need only look at the chapters in Whittaker [9] titled "The soluble problems of particle dynamics" and "The soluble problems of rigid dynamics" to see

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the importance of n integrals in involution. Almost every example is analyzed by such integrals.

In systems which vary slowly with time, integrals must be replaced by quantities which also vary slowly with time, i.e., with adiabatic invariants. Of course the knowledge of n independent adiabatic invariants in involution for a dynamical system does not imply that the system is "solved" as it does in the conservative case. However, a great deal of mathematical and physical information can be obtained from adiabatic invariants [3], [4], [1].

In order to illustrate our theorem consider the system

$$\dot{u} = \mathrm{A}u \quad ; \quad \cdot = \frac{d}{dt} \tag{1.1}$$

where u is a $2n$ dimensional column vector and A is a constant $2n \times 2n$ real Hamiltonian matrix with distinct pure imaginary eigenvalues $\lambda_1, \ldots, \lambda_{2n}$. Let the eigenvalues be ordered so that $\lambda_{s+n} = -\lambda_s = \bar{\lambda}_s$ for $s = 1, \ldots, n$. If $c_1, \ldots, c_{2n}$ are row eigenvectors of A corresponding to $\lambda_1, \ldots, \lambda_{2n}$ (i.e., $c_s\mathrm{A} = \lambda_s c_s$) which satisfy the reality condition $c_{s+n} = \bar{c}_s$, $s = 1, \ldots, n$ then the n real functions $\mathrm{I}_s(u) = (c_s u)\,(c_{s+n} u) = |c_s u|^2$, $s = 1, \ldots, n$ form a set of n independent integrals in involution for 1.1).

If the matrix A were now allowed to vary slowly with t, then one would expect that there would exist n functions close to $\mathrm{I}_1, \ldots, \mathrm{I}_n$ which also vary slowly with t. This is the general content of our result.

In order to be precise we must make some definitions. A function $f : (-\infty, \infty) \to \mathrm{R}$ or C will be *gentle* if $\dfrac{d^s f}{dt^s} \in \mathrm{L}_1(-\infty, \infty)$ for $s = 0, 1, 2, \ldots$. If f or even $\dfrac{df}{dt}$ is gentle then $\lim\limits_{t \to \pm\infty} \dfrac{d^s f}{dt^s}$, $s = 0, 1, 2, \ldots$ exists and so we may consider f and all its derivatives as defined and continuous on $[-\infty, \infty]$. The assumption that a system varies slowly with t is expressed by considering a system of the form

$$\epsilon \dot{u} = \mathrm{A}(t)\, u \tag{1.2}$$

where A is a $2n \times 2n$ real matrix such that each entry of $\dfrac{d\mathrm{A}}{dt}$ is gentle and ϵ is a small positive parameter. This assumption is more easily understood when one uses the parameter $\tau = \epsilon^{-1} t$ so that 1.2) becomes $\dfrac{du}{d\tau} = \mathrm{A}(\epsilon\tau)\, u$.

Furthermore the system 1.2) is assumed to be Hamiltonian. Thus the matrix $\mathrm{S}(t) = -\mathrm{JA}(t)$ is symmetric where J is the usual $2n \times 2n$ matrix of Hamiltonian mechanics given by $J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$. The system 1.2) is then written in the Hamiltonian form

$$\dot{u} = J \frac{\partial H}{\partial u} \tag{1.3}$$

where

$$H = \frac{1}{2\epsilon} u^T S(t) u. \tag{1.4}$$

Let $\psi(t, t_0, u_0, \epsilon)$ be the solution of (1.2) which satisfies

$$\psi(t_0, t_0, u_0, \epsilon) = u_0.$$

A function $I(u, t)$ will be an *adiabatic invariant* of 1.2) if

$$\mathcal{I}(\infty, t_0, u_0, \epsilon) - \mathcal{I}(-\infty, t_0, u_0, \epsilon) = 0(\epsilon^s) \quad \text{as} \quad \epsilon \to 0^+$$

for all $s = 0, 1, 2, \ldots$ where $\mathcal{I}(t, t_0, u_0, \epsilon) = I(\psi(t, t_0, u_0, \epsilon), t)$.

Let $I_1, \ldots, I_\ell$ be ℓ functions of $(u, t) \in R^{2n+1}$. The Poisson bracket of I_r and I_s, $\{I_r, I_s\}$ is defined by

$$\{I_r, I_s\} = \frac{\partial I_r^T}{\partial u} J \frac{\partial I_s}{\partial u}. \tag{1.5}$$

The set of functions $I_1, \ldots, I_\ell$ are said to be in involution if $\{I_r, I_s\} \equiv 0$ for $1 \leqslant s, r \leqslant \ell$. The set of functions $I_1, \ldots, I_\ell$ are said to be independant if $\frac{\partial I_1}{\partial u}, \ldots, \frac{\partial I_\ell}{\partial u}$ are independent vectors for all (u, t) except for a subset of R^{2n+1} with no interior.

We can now state our main result.

Theorem 1. *Let the eigen values of* $A(t)$ *be distinct and pure imaginary for each* $t \in [-\infty, \infty]$. *Then the system* (1.2) *admits n independant adiabatic invariants in involution.*

In fact the adiabatic invariants are constructed as follows. Let

$$\lambda_1(t), \ldots, \lambda_{2n}(t)$$

be the eigenvalues of $A(t)$ with the order such that

$$\lambda_{n+s}(t) = -\lambda_s(t) = \overline{\lambda_s}(t) \quad \text{for} \quad s = 1, \ldots, n.$$

Let $c_1(t), \ldots, c_n(t)$ be the eigenvectors of $A(t)$ of unit length corresponding to $\lambda_1(t), \ldots, \lambda_n(t)$. The c_s are determined uniquely up to $\pm c_s$

and it will be shown that the entries of $\frac{dc_s}{dt}$ are gentle. Indeed any smooth choice of eigenvectors with $\frac{dc_s}{dt}$ gentle will work. Let $c_{n+s} = \overline{c}_s$ for $s = 1, \ldots, n$ so that $c_1, \ldots, c_{2n}$ are a full set of eigenvectors of A. Here as before we take the c_s to be row eigenvectors so $c_s(t)\, \mathrm{A}(t) = \lambda_s(t)\, c_s(t)$. Then the n adiabatic invariants of theorem 1 are

$$\begin{aligned} \mathrm{I}_s(u\,,\,t) &= |c_s(t)\, \mathrm{J} c^{\mathrm{T}}_{s+n}(t)|^{-1}\, (c_s(t)\, u)\, (c_{s+n}(t)\, u) \\ &= |c_s(t)\, \mathrm{J} c^{\mathrm{T}}_{s+n}(t)|^{-1}\, |c_s(t)\, u|^2 \end{aligned} \tag{1.6}$$

for $s = 1, \ldots; n$.

In order to compare our theorem with similar results in the literature consider the equation

$$\epsilon^2 \ddot{\xi} + \phi^2(t)\, \xi = 0. \tag{1.7}$$

where ϕ is a positive function of t such that $\frac{d\phi}{dt}$ is gentle and $\phi(\infty) > 0$, $\phi(-\infty) > 0$. The equation (1.7) can be written as a system in the form (1.2) by introducing $\epsilon\dot{\xi} = \eta$. In this case the matrix A turns out to be $\begin{pmatrix} 0 & 1 \\ -\phi^2 & 0 \end{pmatrix}$ which satisfies the hypothesis of our theorem. The eigenvalues of A are $\pm\, i\phi(t)$ and the eigenvectors are $(\pm\, i\phi\,,\, 1)$. The quantity $|c_s(t)\, \mathrm{J} c^{\mathrm{T}}_{s+n}(t)|$ in this case is 2ϕ and so the adiabatic invariant is

$$\mathrm{I} = \frac{1}{2\phi}\,(i\phi\xi + \eta)\,(-\,i\phi\xi + \eta) = \frac{1}{2\phi}\,(\phi^2\xi^2 + \eta^2).$$

This complete result was first obtained by Littlewood [6] and the reader is referred to this paper for a discussion of earlier partial results. Recently Wasow [7] has given an eloquent proof of Littlewood's theorem by fully describing the form of the fundamental matrix solution of equation (1.7). Wasow has even obtained the precise asymptotic order of the adiabatic invariant under further mild assymptions on ϕ by using turning point theory [8]. Indeed the present authors were first stimulated by the results of Professor Wasow and wish to thank him for several enlightening conversations on the subject of adiabatic invariants.

2. OUTLINE OF THE PROOF

Here we shall give a brief outline of the proof of the main theorem. For details see [5].

Lemma 1. Let $\lambda_1(t), \ldots, \lambda_{2n}(t)$ be the eigenvalues of $A(t)$ and

$$c_1(t), \ldots, c_{2n}(t)$$

the corresponding smooth row eigenvectors of unit length. Then $\dot{\lambda}_s(t)$ and the entries of $\dot{c}_s(t)$ are gentle for $s = 1, \ldots, 2n$.

Proof : The proof of this lemma is elementary.

With the information from lemma I we shall construct a symplectic change of variables. Let the ordering be such that $\lambda_{s+n}(t) = -\lambda_s(t) = \overline{\lambda}_s(t)$ and $c_{s+n}(t) = \overline{c}_s(t)$ for $s = 1, \ldots, n$.

Since A is Hamiltonian $AJ + JA^T = 0$ so

$$\lambda_r c_r J c_s^T = c_r A J c_s^T = -c_r J A^T c_s^T = -\lambda_s c_r J c_s^T.$$

Thus $c_r J c_s^T = 0$ unless $\lambda_r + \lambda_s = 0$ or $|r - s| = n$. Since $c_r J c_s^T = 0$ for all s is impossible we have that $c_r J c_s^T \neq 0$ when $|r - s| = n$. Now let $1 \leqslant r \leqslant n$.

$$c_r J c_{r+n}^T = c_r J \overline{c}_r^T = \overline{c_r J \overline{c}_r^T} = \overline{c_r J^T c_r^{-T}} = -\overline{c_r J c_{r+n}^T}$$

and so $c_r J c_{r+n}^T$ is pure imaginary. By interchanging λ_r, λ_{r+n} and c_r, c_{r+n} if necessary we may assume $c_r J c_{r+n}^T = ai$ with $a > 0$. Now define

$$d_r(t) = |c_r(t)\, J c_{r+n}(t)^T|^{1/2}\, c_r(t) \quad \text{for} \quad r = 1, \ldots, n$$

and $d_{r+n}(t) = \overline{d}_r(t)$ for $r = 1, \ldots, n$. Thus we have $d_r J d_s^T = 0$ for $|r - s| \neq n$ and $d_r J d_{r+n} = +i$.

Remark : Note that the adiabatic invariants defined in the introduction are just $I_r(t, u) = (d_r u)(d_{r+n} u)$.

Let $P(t)$ be the $2n \times 2n$ matrix whose r^{th} row is d_r. Then from the above $P(t)\, JP(t)^T = iJ$ and

$$P(t)\, A(t)\, P^{-1}(t) = A_0(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_{2n}(t)).$$

Note that by lemma 1 the matrix $\dot{P}$ has gentle entries. It will be important in the argument that follows to keep track of the fact that the equations are real. Let Q be the $2n \times 2n$ matrix defined by $Q = \begin{pmatrix} O_n & I_n \\ I_n & O_n \end{pmatrix}$. Now by construction $\overline{P} = QP$ and so $Q\overline{A}_0 Q = A_0$.

We are now ready to make the change of variables $x = P(t)\, u$ in equation (1.2) to get

$$\epsilon \dot{x} = (A_0(t) + \epsilon A_1(t))\, x \tag{2.1}$$

where

$$A_0(t) = P(t) A(t) P(t)^{-1} \tag{2.2}$$

$$A_1(t) = \dot{P}(t) P^{-1}(t). \tag{2.3}$$

Since $PJP^T = iJ$ and $|\det P(t)| = 1$, $A_1(t)$ is gentle. A_0 and A_1 satisfy the reality condition $QA_sQ = \overline{A}_s$, $s = 0, 1$. The change of variables $x = P(t)u$ is a symplectic change of variables with multiplier i and so equation (2.1) is Hamiltonian.

The new Hamiltonian is of the form

$$\epsilon^{-1}\{H_0(x, t) + \epsilon H_1(x, t)\} \tag{2.4}$$

where $H_0(x, t) = \frac{1}{2} x^T S_0(t) x$, $H_1(x, t) = \frac{1}{2} x^T S_1(t) x$. Here, both the matrices $S_0 = -JA_0$ and $S_1 = -JPP^{-1}$ are symmetric, because $PJP^T = iJ$. Since $QJ = -JQ$ one sees that S_0 and S_1 satisfy the reality condition $QS_sQ = -\overline{S}_s$, $s = 0,1$.

Remark. In the new coordinates the adiabatic invariants of our theorem now take the simple form $I_s(x, t) = x_s x_{n+s}$. Thus it is clear that they are independant and in involution.

The above lemma shows how to diagonalize the equations of motion to first order. Now we proceed to formally diagonalize the equations to all orders. In order to do this we shall deal with a class of functions which we shall now define. A function $K(x, t)$ will be called a GR function if $K(x, t) = x^T S(t) x$ where $S(t)$ is a $2n \times 2n$ symmetric matrix with gentle entries which satisfies the reality condition $\overline{S} = -QSQ$.

Lemma 2 : Consider a Hamiltonian of the form

$$\epsilon^{-1} H_*(x, t, \epsilon) = \epsilon^{-1} \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} H_j(x, t) \tag{2.5}$$

where $H_0(x, t) = \sum_{s=1}^{n} \lambda_s(t) x_s x_{s+n}$, H_j is a GR function for $j \geqslant 1$ and the above series is a formal expansion in ϵ. Then there exists a formal linear symplectic change of variables

$$x = \sum_{j=0}^{\infty} \left(\frac{\epsilon^j}{j!}\right) \Phi^j(t)\, y \tag{2.6}$$

which transforms (2.5) and hence (2.4) to the Hamiltonian

$$\epsilon^{-1} \mathrm{K}(y, t, \epsilon) = \epsilon^{-1} \sum_{j=0}^{\infty} \left(\frac{\epsilon^j}{j!}\right) \mathrm{K}^j (y, t) \tag{2.7}$$

where $\mathrm{K}^0 = \mathrm{H}_0$ and for $j \leqslant 1$

$$\mathrm{K}^j (y, t) = \sum_{s=1}^{n} a^j(t)\, y_s\, y_{s+n}, \tag{2.8}$$

and K^j are GR functions. In (2.6), $\Phi^0(y) = \mathrm{I}$ the identity and $\Phi^j(t), j \geqslant 1$, is a $2n \times 2n$ matrix with gentle entries.

The proof of this lemma is carried out with the aid of the Lie transform formulas found in [2]. Now all that remains are some standard estimates.

By the remark above it is enough to prove that $\mathrm{I}_s(x) = x_s x_{s+n}$ is an adiabatic invariant for the system (2.1) for $s = 1, \ldots, n$. Let m be a positive integer and consider the truncated change of variables

$$x = \left\{ \sum_{j=0}^{m} \left(\frac{\epsilon^j}{j!}\right) \Phi^j(t) \right\} v \tag{2.9}$$

(c.f. equation 2.6). Then equation 2.1) becomes

$$\epsilon \dot{v} = \left\{ \sum_{j=0}^{m} \left(\frac{\epsilon^j}{j!}\right) \mathrm{A}^j(t) \right\} v + \epsilon^{m+1} \mathrm{D}(t, \epsilon)\, v \tag{2.10}$$

where $\mathrm{A}^j = \mathrm{diag.}\ (a_1^j, \ldots, a_n^j, -a_1^j, \ldots, -a_n^j)$, a_s^j are gentle and pure imaginary and $\mathrm{D}(t, \epsilon)$ is a $2n \times 2n$ matrix with gentle entries such that $\int_{-\infty}^{\infty} \|\mathrm{D}(t, \epsilon)\|\, dt \leqslant \mathrm{B}$ for $0 \leqslant \epsilon \leqslant \epsilon_0$, ϵ_0 a sufficiently small number.

Now consider 'the truncated system

$$\epsilon \dot{w} = \left\{ \sum_{j=0}^{m} \left(\frac{\epsilon^j}{j!}\right) \mathrm{A}^j(t) \right\} w. \tag{2.11}$$

Because all the matrices A^j are diagonal with pure imaginary entries the fundamental matrix solution of (2.11) which is the identity at $t = 0$ is uniformly bounded for $t \in [-\infty, \infty]$ and $\epsilon \in (0, 1]$. Moreover the functions $w_s w_{s+n}$, $s = 1, \ldots, n$ are integrals for 2.11).

Let x_0 be fixed and $v_0(\epsilon)$ be computed from

$$x_0 = \left\{ \sum_{j=0}^{m} \left(\frac{\epsilon^j}{j!}\right) \Phi^j(0) \right\} v_0(\epsilon) \quad \text{so} \quad v_0(0) = x_0 .$$

Let $x(t) = x(t, \epsilon, x_0)$ be the solution of (2.1) satisfying $x(0) = x_0$ and let

$$v(t) = v(t, \epsilon, v_0(\epsilon)) \quad \text{and} \quad w(t, \epsilon, v_0(\epsilon))$$

be the solutions of (2.10) and (2.11) respectively which are equal to $v_0(\epsilon)$ when $t = 0$.

Thus $x(t)$ is carried into $v(t)$ by (2.9) and since $\Phi^0 = I$, $\Phi^j(\pm\infty) = 0$ for $j \geqslant 1$ we have $x(\pm\infty) = v(\pm\infty)$. Represent v in (2.10) by the variation of parameters formula, with $\epsilon^{m+1} D(t, \epsilon) v$ as the inhomogeneous term. Then, by a standard Gronwall inequality estimate we have

$$v(t) - w(t) = 0(\epsilon^m),$$

uniformly for $t \in [-\infty, \infty]$. Thus

$$\begin{aligned} &x_s(\infty)\, x_{s+n}(\infty) - x_s(-\infty)\, x_{s+n}(-\infty) = \\ &v_s(\infty)\, v_{s+n}(\infty) - v_s(-\infty)\, v_{s+n}(-\infty) = \\ &w_s(\infty)\, w_{s+n}(\infty) - w_s(-\infty)\, w_{s+n}(-\infty) + 0(\epsilon^m) = 0(\epsilon^m). \end{aligned} \tag{2.12}$$

Since m is arbitrary we have shown that

$$x_s(\infty)\, x_{s+n}(\infty) - x_s(-\infty)\, x_{s+n}(-\infty)$$

is asymptotic to zero as $\epsilon \to 0^+$ and so $I_s = x_s x_{s+n}$ is an adiabatic invariant for (2.1).

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DISCUSSION

Pr. Voros – What similar statements can be made for the adiabatic invariants of an analogous system where $A(t)$ is now taken to be a periodic function of t ?

Pr. **Meyer** – I do not know if the periodic case has been considered separately in the literature. However, I can comment on what our procedure would yield for periodic systems. If $A(t)$ were T-periodic instead of having a gentle derivative then all of the changes of variables discussed in section 2 would be T-periodic also. Thus the system could be diagonalized by a formal, T-periodic, symplectic change of variables. The estimates of this section would all be carried out in the same way with say 0 replacing $-\infty$ and T replacing $+\infty$. Only at the last step would a difficulty occur. Since $\Phi^j(\pm\infty) = 0$ for all $j > 0$ we were able to conclude that $x(\pm\infty) = v(\pm\infty)$. Thus the first equality in (2.12) would no longer hold. However, the second and third would still hold. This would mean that an adiabatic invariant could be constructed order by order but would depend on ϵ in the periodic case.