SUBHARMONICS IN HAMILTONIAN SYSTEMS

by

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I. Introduction:

In this note we shall discuss the existence and nature of subharmonic periodi solutions of an ordinary differential equation without damping. Except for the example the results discussed here have been given in [5] and [6] in greater detail. In order explain our problem consider the equation

$$\text{1)} \qquad \ddot{x} + x + \delta x^3 = \epsilon p(\omega t) \quad \text{or} \qquad \dot{x} = y \; , \; \dot{y} = -x - \delta x^3 + \epsilon p(\omega t)$$

where all the variables and parameters are real scalors and p is a smooth 2π periodic function. We note that 1) is a special case of Duffing's equation with no damping and therefore can be considered as a $2\pi/\omega$ periodic Hamiltonian system of one of freedom.

We shall be interested in the existence of $k2\pi/\omega$ periodic solutions of 1) (subharmonics) when ω is near the integer $k, k \geq 4$, and ϵ is small. Particular att will be placed on the dependence of these solutions on the parameter ω.

When $\epsilon = 0$ the origin, $x = y = 0$, can be considered as a $2\pi/\omega$ periodic solu of 1) with characteristic multipliers $\exp(\pm 2\pi i/\omega)$. Since the characteristic multipliers are not $+1$ for $\omega > 1$ the usual implicit function theorem argument yields existence of an $\epsilon_o(\omega)$ such that for each ϵ, $0 \leq \epsilon \leq \epsilon_o(\omega)$, the equation 1) has a $2\pi/\omega$ periodic solution $\varphi(t,\omega,\epsilon)$ with $\varphi \to 0$ as $\epsilon \to 0$. Moreover this periodic solut is locally unique in the sense that there exists a neighborhood of φ that contains no other $2\pi/\omega$ periodic solutions for fixed ω and ϵ. We shall call φ the harmonic solution of 1).

If $\epsilon = 0$ and $\omega = k$ then the characteristic multipliers of φ are k^{th}

roots of unity and so one can not conclude from the first approximation whether or not there are periodic solutions of period $k2\pi/\omega$ (subharmonics of order k) near φ when ω is near k. It is the existence and nature of such solutions that we wish to discuss here.

In particular we discuss two cases under which one can conclude the existence of subharmonic solutions. In the first case (lemma 2) a great deal of computation is necessary but very precise information is given about the nature of the subharmonics. The lemma 2 describes the generic case if one considers the class of all such differential equations depending on a parameter. See [5] for a precise definition of "generic" and for a complete discussion of generic bifurcation. We shall not prove lemma 2 here but state that the proof is based on the usual small parameter methods of bifurcation theory (Poincare's continuation method).

In the second case (lemma 3) very little computation is necessary to establish the existence of the subharmonics but the precise number and nature of these subharmonics is not give. This lemma is applied to 1) to establish the existence of subharmonics of all orders k, $k \geq 5$. This lemma is established by using a modification of a fixed point theorem of Birkhoff [1]. It was first used in its present form by J. Palmore and the author in [6] to establish the existence of a new class of periodic solutions in the restricted three body problem.

The history of subharmonics is extensive. An early reference is [7] where subharmonics are called periodic solutions of the second type. More resent references are [2] and [4] where further bibliographic information can be found. Subharmonics for Duffing's equation are discussed in [3] and [8].

II. Birkhoff's Normal Form:

In order to discuss subharmonics of 1) we shall investigate the period map P defined by 1). Let $x(t,x_o,y_o)$, $y(t,x_o,y_o)$ be the solution of 1) such that $x(o,x_o,y_o)=x_o$ and $y(o,x_o,y_o)=y_o$ then P is defined by $P(x_o,y_o) = (x(2\pi/\omega,x_o,y_o), y(2\pi/\omega,x_o,y_o))$. It is easy to verify that P defines a global area preserving diffeomorphism of the plane into itself which depends smoothly on the parameters $\delta, \epsilon,$ and ω. The map P is area preserving since there is no damping term in 1). We also note that a fixed point of P corresponds to a periodic solution of 1) of period $2\pi/\omega$ and a fixed point of P^k (a periodic point of period k) corresponds to a periodic solution of 1) of period $k2\pi/\omega$. We shall discuss the particular period map P in section V and discuss the general case here and in the next two sections. For the present we shall discuss a mapping depending on only one parameter (essentially $\eta = \omega - k$) and suppress the other parameters.

Let $\psi : N \times (-\tau, \tau) \to R^2$ be a smooth map where N is an open neighborhood of ψ_η the origin in R^2 such that for each $\eta \in (-\tau, \tau)$ the map $\psi\eta = \psi(\cdot, \eta) : N \to R^2$ is area preserving and $\psi_\eta(o) = o$. Since ψ_η is area preserving the Jacobian determinant of ψ_η at the origin is 1. Thus the eigen values λ_1, and λ_2 of the Jacobian matrix at the origin are either real and $\lambda_1 = (\lambda_2)^{-1}$ or complex and $\lambda_1 = \lambda_2 = \bar{\lambda}_2^{-1}$. In the first case we shall call the origin a <u>hyperbolic</u> fixed point of ψ and in the second use we shall call the origin an elliptic fixed point of ψ.

Henceforth we shall assume that for $\eta = o$ the eigen values of the Jacobian matrix of ψ_η at o are $\exp(\pm 2\pi \ell i/k)$ where ℓ and k are relatively prime integers $o < \ell < k$. Thus the origin is an <u>elliptic</u> fixed point for ψ and the eigen values of the Jacobian matrix are k^{th} roots of unity for $\eta = o$.

For area preserving diffeomorphisms there exists a special theory of coordinate transformations and canonial forms. These are discussed in detail in [1]. In particular a minor modification of the normalization proceed [1] pages 71 - 74 yields the following:

Lemma 1. <u>There exists an area preserving change of variables which is well defined in</u> $N' \subset N$ <u>for all</u> $\eta \in (-\tau_1, \tau_1)$, $o < \tau_1 < \tau$, <u>such that the map</u> ψ_η <u>in the new coordinates</u> <u>maps</u> $(q,p) \to (Q,P)$ <u>where</u>

$$q = r^{1/2} \cos\theta, \quad p = r^{1/2} \sin\theta$$

$$Q = R^{1/2} \cos\Theta, \quad P = r^{1/2} \sin\Theta$$

$$\Theta = \theta + 2\pi\ell/k + \alpha\eta + \Sigma_1^m \beta^j r^j + \gamma(\cos k\,\theta)\, r^{(k-2)/2} + \Theta_1$$

$$R = r + 2\gamma\,(\sin k\,\theta)\, r^{k/\theta} + R_1$$

$$\Theta = O(r^{(k-1)/2}) \quad \text{as} \quad r \to o$$
$$R = O(r^{(k+1)/2}) \quad \text{as} \quad r \to o$$
$$m = [(k-2)/2]$$

Θ_1 <u>and</u> R_1 <u>are smooth functions of</u> θ , $\rho = r^{1/2}$ <u>and</u> η <u>and</u> 2π <u>periodic in</u> θ . <u>Also</u> α , β_j <u>and</u> γ <u>are smooth functions of</u> η .

At first sight the formulas in the above lemma are quite formidable. The following remarks will clarify the lemma.

1. Note that there really are two changes of variables in this lemma. First a change of variables to rectangular variables q,p and then a change of variables to polar variables r,θ . Also note that r is the square of the usual polar radius ρ . This choice of r is made so that the area preserving nature of the problem is kept.

2. Θ_1 and R_1 are simply error terms that are higher order in r than any other indicated term.

3. The linearized equation is in these coordinates just $\Theta = \theta + 2\pi \ell/k + \alpha\eta$ $R = r$. The eigen values of this linear map are just $\exp \pm (2\pi\ell/k + \alpha\eta$. Thus if $a(o) \neq o$ then the eigen values of the linearized equation move along the unit circle and cross the k^{th} root of unity at $\eta = o$.

4. Observe that if $k=3$ the term containing γ dominates the term $\beta_1 r$ in the expression for Θ . If $k=4$ this term containing γ may or may not dominate the term $\beta_1 r$. For this reason we shall discuss only the case when $k \geq 5$. The case when $k=3$ and 4 are discussed in [5] .

5. In the usual Birkhoff normalization as discussed in [1] and other places the eigin values of the linear part are assumed not to be k^{th} roots of unity. In this case the term containing γ does not occur. Thus the term containing γ arrises from the fact that we are investigating the case when the characteristic multipliers are k^{th} roots of unity.

6. The second approximation to the map for $\eta = o$ is $\Theta = \theta + 2\pi\ell/k + \beta_1 r$, $R = r$. Thus to this approximation circles are carried into circles and if $\beta_1 \neq o$ the amount of rotation on each circle varies with the radius. Thus the assumption that $\beta_1(o) \neq o$ is the usual twist assumption found in the theory of small divisors and the invariant

curve theorem of Moser et al.

III The Generic Case:

In general one would not expect all the coefficients in the normal form of lemma 1 to be zero. In [5] we define precisely the mean of "in the general case" or "the generic case" by use of Baire category theory and show that indeed $\alpha(o) \neq o$, $\beta_1(o) \neq o$ and $\gamma(o) \neq o$ in general.

Under the assumptions that these coefficients are not zero it is an easy task to analyze the map for the existence of periodic points of period k. One simply uses the usual tricks of small parameter perturbation theory and the implicit function theorem. We shall not analyze this case here but refer the reader to lemma 1.18 of [5] for a proof of lemma 2 given below.

Let ψ be as in section II and chose coordinates as indicated in lemma 1.

Lemma 2: Let $\alpha(o) \neq o$, $\beta_1(o) = o$, $\gamma(o) \neq o$ and $k \geq 5$. Then there exists an $\eta_o \geq 0$. and smooth functions z^i and w^i, $i = 1,2,\ldots, k$, from (o,η_o) or $(-\eta_o, o)$ into N' such that $\overset{k}{\underset{1}{U}} z^i(\eta)$ is the orbit of an elliptic periodic point of least period k for ψ and $\overset{k}{\underset{1}{U}} w^i(\eta)$ is the orbit of a hyperbolic periodic point of least period k for ψ for each $\eta \in (o,\eta_o)$ (or $(\eta \in (-\eta_o, o))$. Moreover $z^i(\eta)$ and $w^i(\eta)$ tend to zero as η tends to zero.

We note that if the above map is the section map about the harmonic solution of a differential equation then there would exists two subharmonic periodic solutions of order k, one elliptic and one hyperbolic. These subharmonics bifurcate from the harmonic as the parameter passes through the value where the harmonic solution has multipliers that are k^{th} roots of unity.

Remark: Similar conclusions hold if $k = 4$ with $|\beta| > |\gamma|$ but $k = 4$ with $|\beta| < |\gamma|$ and $k = 3$ are considerably different. See remark 4 of section II and [5].

IV The Computable Case:

In applications it is difficult to verify the hypothesis $\gamma(o) \neq o$ of the previous lemma. In applications the period map is not precisely known and will not be in the normalized form of lemma 1. Finding the change of variables described in lemma 1 is a difficult computations problem. Thus we would like to dispense with the condition $\gamma(o) \neq o$. In several interesting examples one can compute the coefficients $\alpha(o)$ and $\beta_1(o)$ and find they are not zero. (cf. sections V and [6]) .

With only knowledge that $\alpha(o) \neq o$ and $\beta_1(o) \neq o$ we can use an ingenious idea of Birkhoff to give the existence of periodic points that bifurcate from the fixed point at the origin.

Let ψ be as in section II and choose coordinates as described in lemma 1.

Lemma 3: Let $\alpha(o) \neq o$, $\beta_1(o) \neq o$ and $k \geq 5$. Then there exists an $\eta_o > o$ such that for each $\eta \in (o, \eta_o)$ if $\alpha(o)\ \beta_1(o) < o$ (resp. $\eta \in (-\eta_o, o)$ if $\alpha(o)\ \beta_1(o) > o$) the map ψ has ℓ , $\infty \geq \ell \geq 2k$, periodic points of least period k . Moreover as $\eta \to o$ these periodic points tend to the origin.

Remark: Note that this statement is only an existence statement for each $\eta \in (o, \eta_o)$ or $\eta \in (-\eta_o, o)$. In particular ℓ may change with η . Also one can not assert any continuity properties of these periodic points in the parameter η except that they tend to the origin as $\eta \to o$.

Proof in outline: Let $\alpha(o) = \alpha$ and $\beta_1(o) = \beta$ and assume for simplicity that $\alpha\beta < o$. Make the change of scale $r \to \eta r$ and compute the k^{th} iterate of $\psi_\eta^{\ k} : (\theta, r) \to (\Theta^k, R^k)$

$$\Theta^k = \theta + k\dot{\eta}(\alpha + \beta r(+ O\ (\eta^{3/2}) \qquad R^k = r + O\ (\eta^{5/2})$$

Since $\beta \neq o$ we can apply the implicit function theorem to assert the existence of a function $\rho(\theta, \eta)$ such that ρ is 2π periodic in θ , $\rho(\theta, o) = \alpha\beta^{-1}$ and hence positive for all θ if η is sufficiently small, and such that

$$\Theta^k(\theta\ ,\ \rho(\theta,\eta)\ ,\ \eta) - \theta \equiv o\ .$$

Thus the circle $S_\eta : r = \rho(\theta, \eta)$ is a curve of zero rotation for the map $\psi_\eta^{\ k}$. Since $\psi_\eta^{\ k}$ is area preserving and leaves the origin fixed the image of the circle S_η under $\psi_\eta^{\ k}$ must intersect itself, i.e. $S_\eta \cap \psi_\eta^{\ k}(S_\eta) \neq \phi$. For η sufficiently small bo S_η and $\psi_\eta^{\ k}(S_\eta)$ are smooth curves that meet a ray from the origin in only one point.

Thus if $x \in S_\eta \cap \psi_\eta^k(S_\eta)$ the angular coordinate of x does not change and since it is on both S_η and $\psi_\eta^k(S_\eta)$ its radial coordinate can not change. Thus x is a fixed point of ψ_η^k. Provided η is small enough one can show that x is not a fixed point of ψ_η^j, $o < j < k$ and so x, $\psi_\eta^1(x)$, ..., $\psi_\eta^{k-1}(x)$ are distinct fixed points of ψ_η^k or equivalently the orbit of x under ψ_η. By the same argument as found in [1] pp 215 - 18 one can show that either there are an infinite number of periodic points or at least one has index $+1$ and at least one has index -1 and so there are at least $2k$ fixed points.

V <u>Applications</u>:

We shall now proceed to apply lemma 3 to equation 1. First we note:

<u>For each integer</u> $k \geq 3$ <u>there exists a smooth function</u> $\omega_k(\epsilon)$ <u>defined for</u> ϵ <u>sufficiently small such that</u> $\omega_k(o) = k$ <u>and the characteristic multipliers of the harmonic solution</u> $\varphi(t, \omega_k(\epsilon), \epsilon)$ <u>are</u> $\exp(\pm 2\pi i/k)$.

We note that the characteristic multipliers are smooth functions of ϵ and ω provided the multipliers are not near ± 1. For $\epsilon = o$ the characteristic multipliers are $\exp(\pm 2\pi i/\omega)$ and so the above statement follows from the implicit function theorem.

Now for fixed ϵ define $\eta = \omega - \omega_k(\epsilon)$. For $\epsilon = o$ the characteristic multipliers of the harmonic solution are $\exp(\pm 2\pi i/k)\,(\sum_{j=o}^{\infty} (-\eta|k)^j) = \exp(\pm 2\pi i/k \pm \eta\bar{\alpha}(\eta)i)$ where $\bar{\alpha}(o) = -2\pi/k^2$. Thus:

<u>For each integer</u> $k \geq 3$ <u>and</u> ϵ, η <u>sufficiently small the characteristic multipliers of the harmonic solution are</u> $\exp(\pm 2\pi i/k \pm \eta\, \alpha(\eta, \epsilon)i)$ <u>where</u> α <u>is a smooth function and</u> $\alpha(o,o) = -2\pi/k^2$.

For ϵ sufficiently small $\alpha(o, \epsilon) \neq o$ and so for fixed ϵ small the hypothesis $\alpha(o) \neq o$ of lemma 3 is verified. If we choose the origin as the initial value of the harmonic solution for $t = o$ then the linearized period map has eigen values $\exp(\pm 2\pi i/k \pm \eta\, \alpha(\eta, \epsilon)i)$.

Now we must proceed to show that the period map is a "twist map" i.e.

$\beta_1 \neq o$ for ϵ sufficiently small. First consider the case when $\epsilon = o$. When $\epsilon = o$ the equation 1) is derived from the Hamiltonian

$$H = \frac{1}{2}(u^2 + v^2) + \delta x^4/4 .$$

In this simple case the Birkhoff normalization procedure converges and so there exists a symplectic change of variables $(x,y) \to (u,v)$ such that

$$H(u,v) = H\,\frac{(u^2+v^2)}{2} =$$

$$\frac{u^2+v^2}{2} + 3\delta/8 \left(\frac{u^2+v^2}{2}\right)^2 + \ldots$$

Change variable by $u = (2r)^{1/2} \cos\varphi$, $v = (2r)^{1/2} \sin\varphi$ and so

$$H(\theta, r) = H(r) = r + (3\delta/4)\, r^2 + \ldots$$

The equations of motion are

$$\dot r = o \quad \text{and} \quad \dot\theta = 1 + (3\delta/4)\, r + \ldots$$

One easily computes the period map to be

$$r \to r \quad \text{and} \quad \varphi \to \varphi + 2\pi/\omega + (3\delta\pi/2\omega)\, r + \ldots$$

Thus for $\epsilon = o$ the twist coefficient is $(3\delta\pi/2\omega)$.

Since the coefficients in the normalization described in lemma 1 are continuous in ϵ we can conclude that $\beta \neq o$ for ϵ small provided $\delta \neq o$. Thus the hypothesis of lemma 3 are satisfied and we can conclude:

<u>For each integer</u> $k \geq 5$ <u>there exists an</u> $\eta_o > o$ <u>and an</u> $\epsilon_o > o$ <u>such that for fixed</u> ϵ, $o < \epsilon \leq \epsilon_o$ <u>equation 1) has at least two subharmonic solutions of order</u> k <u>for</u> $\omega^*(\epsilon) < \omega < \omega^*(\epsilon) + \eta_o$ <u>if</u> $\delta > o$ <u>or</u> $\omega^*(\epsilon) - \eta_o < \omega < \omega^*(\epsilon)$ <u>if</u> $\delta < o$. <u>Moreover these subharmonic approach the harmonic solution as</u> $\omega \to \omega^*(\epsilon)$.

If $k = 3$ we know that γ is zero when $\epsilon = o$ but this of course does not imply that γ will remain zero when $\epsilon \neq o$. Therefore one can not use the methods of lemma 3 to analyze this case. When $k = 4$ we know that γ is zero when $\epsilon = o$ and so is small when $\epsilon \neq o$ but small. By an easy extension of lemma 3 one can show that the same conclusions hold provided $|\gamma| < |\beta|$. Thus the above statement for equation 1) can be extended to the case when $k = 4$.

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