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# ON THE EXISTENCE OF LYAPUNOV FUNCTIONS FOR THE PROBLEM OF LUR'E*

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**Introduction.** This paper is an extension of the work of Yacubovich and Kalman on the existence of Lyapunov functions for the problem of Lur'e. The primary result of this paper is the removal of the unnecessary hypothesis of complete controllability and complete observability from the theorem of Kalman. These hypotheses have been used either explicitly or implicitly by many authors working in this field. Indeed, the change of coordinates introduced by Lur'e, the so-called Lur'e transformations, can be made only if the system is completely controllable.

The first section contains a summary of elementary results and definitions from linear algebra and control theory.

The proofs of these preliminaries are elementary and can be found in [1], [2], and [3].

The second section contains the extensions of the lemma of Kalman-Yacubovich. The proof of the first lemma follows very closely the proof as given by Kalman in [2].

The third section contains a few applications of the lemmas developed in the second section.

**1. Preliminaries.** Let $A$ be a real $n \times n$ matrix and $b$, $c$ two real $n$-vectors (column). Let $E^n$ be Euclidean $n$-space. Denote by $A(z)$ the characteristic matrix of $A$, that is, $A(z) = zI - A$, where $I$ is the identity matrix and $z$ is a scalar complex variable and let $A(z)^{-1} = \{A(z)\}^{-1}$. Let $'$ denote the transpose, $^*$ the conjugate transpose and $|\quad|$ the determinant. Thus $|A(z)|$ is the characteristic polynomial of $A$. The subspaces of $E^n$ generated by the vectors $b$, $Ab$, $\cdots$ will be denoted by $[A, b]$. The orthogonal complement of $[A, b]$ in $E^n$ will be denoted by $[A, b]^0$. Let the dimension of $[A, b]$ be $p$.

LEMMA A. *In general,*

$$\begin{aligned}
[A, b]^0 &= \{x \in E^n : x'A^k b = 0, k = 0, 1, 2, \cdots\} \\
&= \{x \in E^n : x'(\exp At)b \equiv 0 \textit{ for all } t \in (-\infty, \infty)\} \\
&= \{x \in E^n : x'A(z)^{-1}b \equiv 0 \textit{ for any set of } z \textit{ having finite limit point}\},
\end{aligned}$$

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*and if all the characteristic roots of $A$ have negative real parts then*

$$[A, b]^0 = \{x \in E^n : \operatorname{Re} x'A(i\omega)^{-1}b \equiv 0 \text{ for all real } \omega\}.$$

One says the pair $(A, b)$ is *completely controllable* provided $[A, b] = E^n$ and the pair $(A, c')$ is *completely observable* if $(A', c)$ is completely controllable.

LEMMA B. *There exists a basis for $E^n$ such that*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix},$$

*where $A_1$, $A_2$, and $A_3$ are $p \times p$, $p \times (n - p)$ and $(n - p) \times (n - p)$ matrices respectively, $b_1$ is a $p$-vector and $(A_1, b_1)$ is completely controllable.*

LEMMA C. *Let $(A, b)$ be completely controllable and let $\tilde{g}(z) = g_1 + g_2 z + \cdots + g_n z^{n-1}$ be any polynomial with real coefficients of degree less than $n$. Then there exists a real $n$-vector $g$ such that $g'A(z)^{-1}b = \tilde{g}(z)\{|A(z)|\}^{-1}$.*

LEMMA D. *Let $(A, b)$ be completely controllable and $k$ any real $n$-vector. Let $k'A(z)^{-1}b = p(z)\{|A(z)|\}^{-1}$. Then the degree of the greatest common divisor of $p(z)$ and $|A(z)|$ is equal to the dimension of $[A', k]^0$.*

A rational function $f(z)$ is said to be a *positive real function* provided $\operatorname{Re} f(z_0) \geqq 0$ whenever $z_0$ is not a pole of $f(z)$ and $\operatorname{Re} z_0 \geqq 0$.

**2. The main lemmas.** The extension of the Kalman-Yacubovich lemma will require several steps. The first lemma is a slight extension of the lemma as given by Kalman [2] and the proof of this lemma follows very closely his proof. We obtain the additional information that $B$ is positive definite and that $(A, q')$ is completely observable.

LEMMA 1. *Let $A$ be an $n \times n$ real matrix all of whose characteristic roots have negative real parts, let $\tau$ be a nonnegative real number and let $b$, $k$ be two real $n$-vectors. Assume $(A, b)$ is completely controllable. If the function*

$$T(z) = \tau + 2k'A(z)^{-1}b \tag{1.1}$$

*is a positive real function then there exist two $n \times n$ real symmetric matrices $B$ and $D$ and a real $n$-vector $q$ such that*

(a) $A'B + BA = -qq' - D$,
(b) $Bb - k = \sqrt{\tau}\, q$,
(c) *$(A, q')$ is completely observable,*
(d) *$B$ is positive definite and $D$ is positive semidefinite,*
(e) *if $i\omega_0$, $\omega_0$ real, is a zero of $-q'A(z)^{-1}b + \sqrt{\tau}$, then it is a zero of $b'A(-z)^{-1}DA(z)^{-1}b$, and*
(f) *all the zeros of $-q'A(z)^{-1}b + \sqrt{\tau}$ are in the closed left halfplane.*

*Proof.* Let $m(z) = A(z)^{-1}b$ and $\psi(z) = |A(z)|$. Since $T(z)$ is positive real,

$$0 \leqq \tau + m(i\omega)^* k + k' m(i\omega) = \frac{\eta(i\omega)}{\psi(i\omega)\psi(-i\omega)}. \tag{1.2}$$

Clearly $\eta(z)$ is an even polynomial with real coefficients and hence its zeros are symmetric about both the real axis and the imaginary axis. Since Re $\eta(i\omega) \geqq 0$ for all real $\omega$, the zeros of $\eta(z)$ on the imaginary axis are of even multiplicity. Thus $\eta(z) = \theta(z)\theta(-z)$, where $\theta(z)$ is a real polynomial all of whose zeros have nonpositive real parts.

Let $\theta(z) = \theta_1(z)\theta_2(z)$, where all the zeros of $\theta_1(z)$ have negative real parts, all the zeros of $\theta_2(z)$ are pure imaginary, and the leading coefficient of $\theta_2(z)$ is one. Let $\epsilon_0$ be the greatest lower bound of $\theta_1(i\omega)\theta_1(-i\omega)$ taken over all real $\omega$. Since $\theta_1(z)$ has no pure imaginary zeros, $\epsilon_0 > 0$. Let $\alpha$ be a real positive number such that $\alpha^2 \leqq \epsilon_0$ and $\alpha^2 \neq \theta_1(\lambda_i)\theta_1(-\lambda_i), i = 1, \cdots, n$, where $\lambda_i$ is a zero of $\psi(z)$. If $\theta_1(z)$ is a constant, take $\alpha = 0$. Consider $\Gamma(z) = \theta_2(z)\theta_2(-z)[\theta_1(z)\theta_1(-z) - \alpha^2]$. By the definition of $\alpha$ and $\Gamma$ it follows that (i) $\Gamma(i\omega) \geqq 0$ for all real $\omega$, and (ii) the greatest common divisor of $\Gamma(z)$ and $\psi(z)\psi(-z)$ is one.

Since $\Gamma(z)$ is an even polynomial and Re $\Gamma(i\omega) \geqq 0$ for all real $\omega$, there exists a polynomial $\nu(z)$ with real coefficients all of whose zeros have nonpositive real parts such that $\Gamma(z) = \nu(z)\nu(-z)$. Define the vector $g$ such that $g' A(i\omega)^{-1} b = \alpha\theta_2(z)\{\psi(z)\}^{-1}$. Thus

$$\begin{aligned} 0 &\leqq \tau + m(i\omega)^* k + k' m(i\omega) - m(i\omega)^* g g' m(i\omega) \\ &= \frac{\Gamma(i\omega)}{\psi(i\omega)\psi(-i\omega)} \\ &= \frac{\nu(i\omega)\nu(-i\omega)}{\psi(i\omega)\psi(-i\omega)}. \end{aligned} \tag{1.3}$$

The formal degree of $\nu(z)$ is $n$ and its leading coefficient is $\sqrt{\tau}$ and so

$$\frac{\nu(z)}{\psi(z)} = -\frac{\mu(z)}{\psi(z)} + \sqrt{\tau},$$

where $\mu$ is real and of degree less than or equal to $n - 1$. The vector $q$ is then defined by $\mu(z)\{\psi(z)\}^{-1} = q' m(z)$. By construction, $\mu(z)$ and $\psi(z)$ have greatest common divisor one and so $(A, q')$ is completely observable. Thus property (c) holds. Define $D = gg'$; since by construction the pure imaginary zeros of $g' m(z)$ and $-q' m(z) + \sqrt{\tau}$ are the same, property (e) holds.

Now define

$$B = \int_0^\infty e^{A't}\{qq' + D\} e^{At}\, dt,$$

and so, $A'B + BA = -qq' - D$. Since $(A, q')$ is completely observable, $B$ is positive definite. From (1.3) it follows that

$$\begin{aligned} m^*(i\omega)k + k'm(i\omega) \\ &= m^*(i\omega)Dm(i\omega) + (-q'm(i\omega) + \sqrt{\tau})(-m^*(i\omega)q + \sqrt{\tau}) - \tau \\ &= m^*(i\omega)\{qq' + D\}m(i\omega) - \sqrt{\tau}(q'm(i\omega) + m^*(i\omega)q) \\ &= b'Bm(i\omega) + m^*(i\omega)Bb - \sqrt{\tau}(q'm(i\omega) + m^*(i\omega)q) \end{aligned}$$

and hence Re $\{Bb - k - \sqrt{\tau}q\}'m(i\omega) = 0$ and so $Bb - k = \sqrt{\tau}q$.

The next step is the removal of the assumption that $(A, b)$ be completely controllable. This is done with the following lemma.

LEMMA 2. *Let $A$ be a real $n \times n$ matrix all of whose characteristic roots have negative real parts; let $\tau$ be a real nonnegative number and let $b$, $k$ be two real $n$-vectors. If*

$$T(z) = \tau + 2k'A(z)^{-1}b$$

*is a positive real function then there exist two $n \times n$ real symmetric matrices $B$, $D$ and a real $n$-vector $q$ such that*

(a) $A'B + BA = -qq' - D$,
(b) $Bb - k = \sqrt{\tau}q$,
(c) *$D$ is positive semidefinite and $B$ is positive definite,*
(d) $\{x \in E^n : x'Dx = 0\} \cap [A', q]^0 = \{0\}$,
(e) $q \notin [A, b]^0$, *and*
(f) *if $i\omega$, $\omega$ real, is a zero of $-q'A(z)^{-1}b + \sqrt{\tau}$, then it is a zero of $b'A(-z)^{-1}DA(z)^{-1}b$.*

*Proof.* Choose a coordinate system for $E^n$ such that

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \qquad k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

where $A_1$, $A_2$, $A_3$ are $p \times p$, $p \times (n - p)$, $(n - p) \times (n - p)$ matrices, respectively, $b_1$, $k_1$ are $p$-vectors, $k_2$ is an $(n - p)$-vector, and such that $(A_1, b_1)$ is completely controllable. Clearly if $A$ has all characteristic roots with negative real parts then so do $A_1$ and $A_3$. If we partition $B$, $D$ and $q$ in the same way, i.e.,

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2' & B_3 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_3 \end{pmatrix}, \qquad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

we find that we must solve the following set of matrix equations:

(1) $A_1'B_1 + B_1A_1 = -q_1q_1' - D_1$,
(2) $A_2'B_1 + A_3'B_2' + B_2'A_1 = -q_2q_1'$,

(3) $A_2'B_2 + A_3'B_3 + B_2'A_2 + B_3A_3 = -q_2q_2' - D_3$,
(4) $B_1b_1 - k_1 = \sqrt{\tau}q_1$,
(5) $B_2'b_1 - k_2 = \sqrt{\tau}q_2$.

By hypothesis, $\tau + 2k_1'A_1(z)^{-1}b_1$ is a positive real function and so by Lemma 1 there exists a solution to (1) and (4) and by Lemma 1(c) the condition of Lemma 2(e) is satisfied. Also by Lemma 1(e) the condition of Lemma 2(f) is satisfied. Now let us consider (2) and (5). Since $B_1$ and $q_1$ are known by Lemma 1 these two equations have only $B_2'$ and $q_2$ as unknowns. We can solve (2) for $B_2'$ in terms of $q_2$ by the formula

$$B_2' = \int_0^\infty e^{A_3't}\{q_2\, q_1' - A_2'\, B_1\}\, e^{A_1t}\, dt,$$

and then substitute this into (5) to obtain

$$Rq_2 = \left\{\int_0^\infty e^{A_3't}\, q_1 e^{A_1t} b\, dt - \sqrt{\tau} I\right\} q_2 = k + \int_0^\infty e^{A_3't} A_2' B_1\, e^{A_1t}\, b\, dt.$$

Since the right hand side of the above is known, we can solve for $q_2$ provided the matrix in the braces, $R$, is nonsingular. There is no loss of generality in assuming that $A_3'$ is in triangular form and so $e^{A_3't}$ is in triangular form. A typical term from the diagonal of $R$ is then

$$\begin{aligned}\int_0^\infty e^{\lambda_i t} q_1\, e^{A_1t}\, b_1\, dt - \sqrt{\tau} &= q_1(-\lambda_i I - A_1)^{-1}\, b_1 - \sqrt{\tau} \\ &= q_1'\, A_1(-\lambda_1)^{-1}\, b_1 - \sqrt{\tau}.\end{aligned}$$

But this term is not zero since $-\lambda_i$ is in the open right halfplane and by Lemma 1(f), we know that the zeros of $q_1A_1(z)^{-1}b_1 - \sqrt{\tau}$ are in the closed left halfplane. Thus $R$ is nonsingular and $q_2$ and $B_2$ are determined.

Now choose $D_3$ to be any positive definite matrix. It is clear then that (5) has a solution and that (d) is satisfied.

Since $B$ satisfies $A'B + BA = -qq' - D$, it must be of the form

$$B = \int_0^\infty e^{A't} qq'\, e^{At}\, dt + \int_0^\infty e^{A't}\, De^{At}\, dt.$$

If $x_0$ is such that $x_0Bx_0 = 0$, then $x_0e^{A't}q \equiv 0$ and $x_0Dx_0 = 0$; and thus by (d), $x_0 = 0$. Hence $B$ is positive definite.

The converse of this lemma is true also. The proof of the converse as given in [2] does not depend on complete controllability and complete observability.

In some critical cases the following lemma is useful. This lemma is in essence due to Yacubovich [5] and was implicitly used by Meyer in [6].

LEMMA 3. *Let $A$ be a $2n \times 2n$ real matrix with simple distinct pure im-*

*aginary characteristic roots* $\pm i\omega_j$, $j = 1, \cdots, n$. *If the residues of* $k'A(z)^{-1}b$ *at each* $\pm i\omega_j$ *are positive then there exists a positive definite matrix* $B$ *such that*

$$A'B + BA = 0 \quad \text{and} \quad Bb - k = 0.$$

This lemma follows at once by making a change of coordinates so that $A$ is diagonal. In this coordinate system $B$ is chosen to be diagonal also.

Using the same procedure as used in the proof of Lemma 2 one can extend the lemma of [4, p. 115] as follows.

LEMMA 4. *Let* $A$ *be a real* $n \times n$ *matrix all of whose characteristic roots have negative real parts,* $\tau$ *be a nonnegative number and* $b$, $k$ *be any two real* $n$*-vectors. If*

$$\tau + 2 \operatorname{Re} k'A(i\omega)^{-1}b > 0$$

*for all real* $\omega$, *then there exist two real positive definite matrices* $B$ *and* $D$ *and a real* $n$*-vector* $q$ *such that*

(a) $A'B + BA = -qq' - D$,
(b) $Bb - k = \sqrt{\tau}q$.

This lemma is almost the same as the lemma given by Yacubovich [7].

**3. Applications.** The lemmas developed in §2 can be applied to many different systems that have been considered in the literature. Let us consider the so-called direct control system. The equations are

$$\begin{aligned} \dot{x} &= Ax - b\phi(\sigma), \\ \sigma &= c'x, \end{aligned} \tag{3.1}$$

where $A$ is a real $n \times n$ matrix, $b$, $x$ and $c$ are real $n$-vectors and $\phi(\sigma)$ is a continuous scalar function of the scalar $\sigma$ such that $\sigma\phi(\sigma) > 0$ for all $\sigma \neq 0$. The vector $x$ and the scalar $\sigma$ are functions of the real variable $t$, time, and $\dot{x}$ is the derivative of $x$ with respect to $t$. Let us assume also that through each point in $E^n$ there exists a unique trajectory of (3.1).

THEOREM 1. *If all the characteristic roots of* $A$ *have negative real parts and if there exist two nonnegative constants* $\alpha$ *and* $\beta$, $\alpha + \beta > 0$, *such that*

$$T(z) = (\alpha + \beta z)c'A(z)^{-1}b \tag{3.2}$$

*is a positive real function then all solutions of* (3.1) *are bounded, the trivial solution* $x = 0$ *is stable, and moreover if* $\alpha \neq 0$ *the trivial solution is asymptotically stable in the large.*

*If, in the case where* $\alpha = 0$, *all the characteristic roots of the matrix* $A - \mu bc'$ *have negative real parts for all* $\mu > 0$ *then the trivial solution* $x = 0$ *of* (3.1) *is asymptotically stable.*

*Proof.* Using the relation $zI = A(z) + A$ we obtain

$$T(z) = \beta c'b + 2 \operatorname{Re} \left(\frac{\alpha c + \beta A'c}{2}\right)' A(z)^{-1} b,$$

and thus by Lemma 2 there exist a real $n$-vector $q$ and two positive symmetric matrices $B$ and $D$ such that

$$A'B + BA = -qq' - D, \quad Bb - \left(\frac{\alpha c + \beta A'c}{2}\right) = \sqrt{\beta c'b}\, q,$$

and moreover $B$ is definite. Thus

$$V = x'Bx + \beta \int_0^\sigma \phi(\sigma)\, d\sigma \tag{3.3}$$

is a positive definite function and tends to $\infty$ as $|x| \to \infty$. The derivative $\dot{V}$ of $V$ along the trajectories of (3.1) is given by

$$\begin{aligned} -\dot{V} &= -x'(A'B + BA)\, x + 2\left(Bb - \frac{\alpha}{2} c - \frac{\beta}{2} A'c\right)' x\phi(\sigma) \\ &\qquad + \beta c'b\phi(\sigma)^2 + \alpha\sigma\phi(\sigma) \\ &= x'Dx + (\sqrt{\tau}\, \phi(\sigma) + q'x)^2 + \alpha\sigma\phi(\sigma). \end{aligned} \tag{3.4}$$

Note that $\alpha\sigma\phi(\sigma)$ has been added and subtracted from $\dot{V}$ and that $\tau = \beta c'b$.

Clearly $-\dot{V}$ is also nonnegative and hence, by the well known theorems of Lyapunov theory all solutions are bounded and the origin is stable. In order to prove asymptotic stability we must show that no solution remains in the set where $-\dot{V} = 0$. Let $\alpha \neq 0$ and assume there exists a solution $x(t)$ of (3.1) such that $x(0) = x_0$ and $x(t)$ remains in the set where $-\dot{V} = 0$. But if $\dot{V} = 0$ then $\sigma = 0$, and thus, such a solution is a solution of $\dot{x} = Ax$. Hence $x(t) = (\exp At)x_0$. From the second term we obtain $q'(\exp At)x_0 \equiv 0$. Also, $x_0Dx_0 = 0$ and so by Lemma 2(d), $x_0 = 0$.

In general we cannot conclude more than stability in the case where $\alpha = 0$, but if the linear system $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$ then (3.1) is asymptotically stable in the large also. In order to rule out solutions that remain in the set where $-\dot{V} = 0$, we must be sure that there is no solution such that $\sqrt{\tau}\phi(\sigma(t)) = -q'x(t)$.

If $\tau \neq 0$ then a solution of (3.1) that remains in the set where $\dot{V} = 0$ must satisfy the linear equation $\dot{x} = \{A + \tau^{-1/2}bq'\}x$. By Lemma 2(e) there exists a nonnegative integer $m$ such that $q'b = q'Ab = \cdots = q'A^{m-1}b = 0$ and $q'A^m b \neq 0$. Hence if $\tau = 0$ there exists an $m$ such that a solution of (3.1) that remains in the set where $-\dot{V} = 0$ must satisfy $\dot{x} = \{A - (q'A^m b)^{-1} bq'A^{m+1}\}x$.

As we have seen, a solution that remains in the set where $-\dot{V} = 0$ is a solution of the linear constant coefficient differential equation. Let us assume that there exists a nontrivial solution $x(t)$ of (3.1) that remains in the set where $-\dot{V} = 0$. We can assume $\sigma(t) \not\equiv 0$ since if $\sigma \equiv 0$ we could repeat the previous argument. Since $x(t)$ is a solution of a linear equation and is bounded for all $t$, then $x(t)$ must be of the form

$$x(t) = \sum_{j=-N}^{N} v_j\{\exp i\omega_j t\},$$

where the $v_j$ are $n$-vectors such that $v_{-j} = \bar{v}_j$ and $\omega_j$ are real scalars such that $\omega_{-j} = -\omega_j$. Clearly $\phi(\sigma(t))$ must be of the form

$$\phi(\sigma(t)) = \sum_{j=-N}^{N} a_j\{\exp i\omega_j t\},$$

where the $a_j$ are scalars such that $a_j = -a_{-j}$. By substituting these forms into (3.1) one obtains

$$v_j = -a_j A(i\omega_j)^{-1}b.$$

Thus, by the well known formula from the theory of almost periodic functions,

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T \sigma(t)\phi(\sigma(t))\,dt = -\sum_{j=-N}^{N} |\, a_j\,|^2 c'A(i\omega_j)^{-1}\,b > 0.$$

We shall have a contradiction once we prove the following remark. *Let the characteristic roots of the matrix $A - \mu bc'$ have negative real parts for all $\mu > 0$. If $i\omega_j$, $\omega_j$ real, is a characteristic root of $A + \tau^{-1/2}bq'$ when $\tau \neq 0$ or of $A - (q'A^m b)^{-1}bq'A^{m+1}$ when $\tau = q'b = \cdots = q'A^{m-1}b = 0$ and $q'A^m b \neq 0$, then* $\operatorname{Im} c'A(i\omega_j)^{-1}b = 0$ *and* $c'A(i\omega_j)^{-1}b \geqq 0$.

We shall consider only the case where $\tau \neq 0$, since the other case is very similar. Since $\alpha = 0$ we may take $\beta = 1$. Then

$$qq' + D = -(A'B + BA) = A^*(i\omega_j)B + BA(i\omega_j),$$

and

$$|\, q'A(i\omega_j)^{-1}b\,|^2 + b'A^*(i\omega_j)^{-1}DA(i\omega_j)^{-1}b = 2\operatorname{Re} b'BA(i\omega_j)^{-1}b.$$

Now the characteristic polynomial of $A + \tau^{-1/2}bq'$ is

$$|\, A(z)\,|\{1 - \tau^{-1/2}q'A(z)^{-1}b\}$$

and so

$$\tau = \tau^{-1/2}q'A(i\omega_j)^{-1}b = b'BA(i\omega_j)^{-1}b - \tfrac{1}{2}c'AA(i\omega_j)^{-1}b.$$

Since $-\sqrt{\tau} + q'A(i\omega_j)^{-1}b = 0$ by Lemma 2(f), $b'A(i\omega_j)^{-1}DA(i\omega_j)^{-1}b = 0$.

Thus

$$\tau + 2 \operatorname{Re} c'AA(i\omega_j)^{-1}b = \operatorname{Re} i\omega_j c'A(i\omega_j)^{-1}b = 0$$

or

$$\operatorname{Im} c'A(i\omega_j)^{-1}b = 0.$$

Since the linear system $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$, the theorem of Nyquist gives $c'A(i\omega_j)^{-1}b \geqq 0$.

The above theorem can be modified several ways:

(i) If the matrix $A$ has some characteristic roots on the imaginary axis then Lemmas 2 and 3 can be used to prove asymptotic stability in a manner similar to that found in [5] and [6]. In particular, we have the following.

THEOREM 1$'$. *If $A$ has $2s$ simple, distinct, nonzero pure imaginary characteristic roots, the characteristic root zero of multiplicity $p$ where $p = 0, 1, 2$, and all other characteristic roots having negative real parts, then* (3.1) *is asymptotically stable in the large, provided:*

(1) *there exist two nonnegative constants $\alpha$ and $\beta$, $\alpha + \beta > 0$, such that $T(z) = (\alpha + z\beta)c'A(z)^{-1}b$ is a positive real function, and if $i\omega$, $\omega$ real, is a characteristic root of $A$, then the residue of $(\alpha + z\beta)c'A(z)^{-1}b$ at $i\omega$ is positive;*

(2) *if $p = 2$,* then $\lim_{z\to 0} z^2 c'A(z)^{-1}b \neq 0$,

(3) *when $\alpha = 0$, the characteristic roots of $A - \mu bc'$ have negative real parts for all $\mu > 0$,*

(4) *if $A$ is singular and $\alpha = 0$ then $\int_0^\sigma \phi(\tau)\, d\tau \to \infty$ as $|\sigma| \to \infty$.*

In order to prove this theorem one first changes coordinates such that the system (3.1) takes the form

$$\begin{aligned}
\dot{x}_1 &= A_1 x_1 - b_1\phi(\sigma),\\
\dot{x}_2 &= A_2 x_2 - b_2\phi(\sigma),\\
\dot{x}_3 &= A_3 x_3 - b_3\phi(\sigma),\\
\sigma &= c_1' x_1 + c_2' x_2 + c_3' x_3,
\end{aligned}$$

where $x_1$, $b_1$, $c_1$ are $r$-vectors, $x_2$, $b_2$, $c_2$ are $2s$-vectors and $A_1$, $A_2$ are $r \times r$, $2s \times 2s$ matrices, respectively. The vectors $x_3$, $c_3$ and $b_3$ are $p$-vectors and $A_3$ is a $p \times p$ matrix, where $p = 0, 1, 2$. The characteristic roots of $A_1$ all have negative real parts, the characteristic roots of $A_2$ are all simple nonzero pure imaginary numbers and the characteristic root of $A_3$ is zero. The matrix $A_3 = (0)$ if $p = 1$ and $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ if $p = 2$. Let

$$V = x_1'\, B_1\, x + x_2'\, B_2\, x_2 + x_3'\, B_3\, x_3 + \beta \int_0^\sigma \phi(\tau)\, d\tau,$$

where $B_1$ is given by Lemma 2 as in the above and $B_2$ is given by Lemma 3 and $B_3 = 0$ if $p = 0$, $B_3 = \alpha$ if $p = 1$, $B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if $p = 2$. Thus, $B_1$, $B_2$, and $B_3$ are $r \times r$, $2s \times 2s$ and $p \times p$ symmetric matrices, respectively, and $V$ is positive definite. One can proceed as before with only very minor changes in the argument.

(ii) If $\phi(\sigma)$ is restricted so that $0 < \sigma\phi(\sigma) < k\sigma^2$ for $\sigma \neq 0$, then instead of adding and subtracting $\alpha\sigma\phi(\sigma)$ from $-\dot{V}$ one can subtract $\alpha\phi(\sigma) \cdot (\sigma - k^{-1}\phi(\sigma))$. The proof carries over and the theorem remains the same except that $c'A(i\omega)^{-1}b$ is replaced by $c'A(i\omega)^{-1}b + k^{-1}$ (see [9]).

(iii) Let us make the change of variables $y(t) = e^{-\lambda t}x(t)$, where $x(t)$ is a solution of (3.1) and $\lambda$ is any real number such that $\lambda > \operatorname{Re} \lambda_i$, $i = 1, \cdots, n$, and $\lambda_i$, $i = 1, \cdots, n$, are the characteristic roots of $A$. Note that $\lambda$ may be positive or negative and the characteristic roots of $A$ may have positive or negative real parts. Then $y(t)$ satisfies the equation

$$\dot{y} = (A - \lambda I)y - be^{-\lambda t}\phi(e^{\lambda t}c'y). \tag{3.5}$$

Let $V = y'By$ and then the derivative of $V$ along the trajectories of (3.5) is

$$\begin{aligned} -\dot{V} = &-y'\{(A - \lambda I)'B + B(A - \lambda I)\}y \\ &+ 2\{Bb - \tfrac{1}{2}c\}'ye^{-\lambda t}\phi(e^{\lambda t}c'y) + c'ye^{-\lambda t}\phi(e^{\lambda t}c'y). \end{aligned}$$

As before there exists a $B$ such that $V$ is positive definite and $-\dot{V} \geqq 0$ for all $y$, provided

$$T_1(z) = c'(A - \lambda I)(z)^{-1}b = c'A(z + \lambda)^{-1}b$$

is a positive real function. Thus $y(t)$ is bounded and the bound depends only on $\| y_0 \|$.

THEOREM 2. *If $\lambda$ is as defined above and $T_1(z) = c'A(z + \lambda)^{-1}b$ is a positive real function, then there exists a nonnegative monotone scalar function $K(\cdot)$ such that $\| x(t) \| \leqq K(\| x_0 \|)e^{\lambda t}$, where $x(t)$ is the solution of* (3.1) *such that $x(0) = x_0$.*

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