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Geometric Averaging of Hamiltonian Systems: Periodic Solutions, Stability, and KAM Tori*

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To the memory of André Deprit and Andrée Deprit-Bartholomé

Abstract. We investigate the dynamics of various problems defined by Hamiltonian systems of two and three degrees of freedom that have in common that they can be reduced by an axial symmetry. Specifically, the systems are either invariant under rotation about the vertical axis or can be made approximately axially symmetric after an averaging process and the corresponding truncation of higher-order terms. Once the systems are reduced we study the existence and stability of relative equilibria on the reduced spaces which are unbounded two- or four-dimensional symplectic manifolds with singular points. We establish the connections between the existence and stability of relative equilibria and the existence and stability of families of periodic solutions of the full problem. We also discuss the existence of KAM tori surrounding the periodic solutions.

Key words. averaging, normalization, reduced space, angular momentum, periodic solutions, KAM tori, degenerate KAM theories, polar and polar-nodal coordinates, planar and spatial restricted N-body problems, radiation pressure, rotating double material segment, Hénon's isochrone, spring pendulum

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1. Introduction.

We will discuss a common thread that runs through several applications of Hamiltonian mechanics. Most of these applications are small perturbations of the Hamiltonian

$$\mathcal{G} = \mathcal{G}(x, y) = x_2 y_1 - x_1 y_2, \tag{1.1}$$

where $x \in \mathbb{R}^2$ stands for the coordinates and $y \in \mathbb{R}^2$ for the momentum conjugate to x. The reader will recognize that $\mathcal{G}$ appears in a Hamiltonian as an angular momentum, a Coriolis term, a magnetic term, a spin term, or even as the Hamiltonian of two harmonic oscillators.

Our analysis looks at the flow on the reduced (orbit) space defined by the Hamiltonian $\mathcal{G}$ in the same spirit as [65]. In particular, we prove some results on the reduced system, extracting qualitative conclusions about the full system. We are interested in establishing the existence of periodic solutions, their stability, and the persistence of KAM tori.

More specifically, we deal with several examples that include different cases of the planar and spatially restricted N-body problems, as well as other examples that can be cast into a

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special class of the restricted three-body problems. To all of these examples we apply high-order normalization in order to make them approximately axially symmetric. We also apply the theory to two cases that can be reduced without applying averaging, namely, the Hénon's isochrone and the spring pendulum. We obtain new periodic solutions and invariant tori for all the problems considered.

Background material on Hamiltonian systems, averaging, reduction of symmetric Hamiltonians, invariant theory, the restricted three-body problem, KAM theory, normal forms, stability, etc., can be found in [2, 4, 7, 12, 49, 61].

The key point of this paper is the combination of different techniques such as averaging, normal forms, reduction theory, or degenerate KAM theorems to deal with the analysis of Hamiltonian systems which are either axially symmetric or approximated by axially symmetric systems. This analysis is performed in several steps. We start with a certain Hamiltonian that is invariant under rotation about the vertical axis or that can be approximated by an axially symmetric Hamiltonian after applying averaging theory. We apply singular reduction to get the reduced Hamiltonian written in terms of the invariants associated with the axial reduction. This Hamiltonian is studied (its equilibria, stability character, and possible bifurcations) by means of standard techniques. Some features about the flow of the full system related to the existence of periodic solutions and invariant tori are established using Reeb's theorems [56] and KAM theory [4, 61].

In order to prove the existence of KAM tori in the examples tackled in the paper, we have applied different versions of KAM theorems that are valid for degenerate systems. Indeed, these tori appear around the relative equilibria of center type; thus we need to use KAM theory after reducing a symmetric system, discussing the relative equilibria, and analyzing their stability. Recently there has been considerable progress on degenerate KAM theorems; see, for example, the papers by Féjoz [25, 26], Chierchia and Pinzari [10], and the references therein. However, the proof of the existence of KAM tori in the comet case of the spatially restricted three-body problem that we show in section 4 has been made possible thanks to a recent result by Han, Li, and Yi [29], valid for very highly degenerate Hamiltonian systems.

In section 2 we study the integral manifolds, the reduced space defined by the Hamiltonian $\mathcal{G}$, and the invariants associated with the corresponding symmetry. The treatment here is geometric in nature.

The purpose of section 3 is to introduce the planar restricted three-body and N-body problems and investigate the comet case when the infinitesimal is far from the primaries. The comet problem is seen to be a small perturbation of the Hamiltonian $\mathcal{G}$ by appropriate scaling. Specifically for the planar restricted N-body problems we establish the existence of orbitally stable near-circular periodic solutions of large radii surrounded by KAM 2-tori. The results extend to any (small) perturbation of the planar Kepler problem in rotating coordinates. Most of these results are not particularly new, but we think the approach is.

In section 4 several three-degrees-of-freedom Hamiltonians are studied, which are again small perturbations of the Hamiltonian $\mathcal{G}$. In particular we partially generalize the results of section 3 to the spatial circular restricted three-body and restricted N-body problems, concluding the existence of elliptic near-circular periodic solutions of large radii. As in the planar case, the results extend to any small perturbation of the spatial Kepler problem in rotating coordinates. However, concerning the existence of KAM tori enclosing the periodic solutions the situation is more degenerate than in the planar case, and we need to use a theorem of Han, Li, and Yi. Thus, we need to specify the perturbation in order to make the computations. Explicitly, we deal with the spatial comet case of the restricted three-body problem, the radiation pressure problem, and the rotating double material segment problem, showing the existence of KAM 3-tori in all the cases. Here not only is the approach new, but also many of the results.

We deal with two more examples in section 5, where we do not need to apply averaging theory to obtain the periodic solutions and invariant tori. More specifically we treat the isochrone problem of Hénon, which was introduced to model the motion of the stars in our galaxy, and the spring pendulum, a system that analyzes the motion of a particle attached to a spring under a constant gravitation field. Both examples can be reduced by axial symmetry and therefore are treated by the methods of our paper. Thus, for the two problems we get periodic solutions and analyze their stability. We also obtain KAM 3-tori for the spring pendulum.

The paper is closed by section 6, devoted to the conclusions.

All the symbolic manipulations involved in the computations of Lie transformations, normal forms, changes of coordinates, Taylor expansions, and so on have been made with the software Mathematica, Version 8, on a MacBook Pro. The Mathematica function `TeXForm[]` has been used to transfer all the expressions into LaTeX, avoiding therefore possible mistakes in the translation of the formulae.

For self-completeness of our presentation we summarize the results of paper [65] in relation to the particular contexts to which they are applied.

We start with (M, Ω), a symplectic manifold of dimension $2n$, and $\mathcal{H}_0 : M \to \mathbb{R}$, a smooth Hamiltonian which defines a Hamiltonian vector field $Y_0 = (d\mathcal{H}_0)^{\#}$ with symplectic flow ϕ_0^t. Let $\mathbb{I} \subset \mathbb{R}$ be an interval such that each $h \in \mathbb{I}$ is a regular value of $\mathcal{H}_0$ and $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a connected circle bundle over a base space (i.e., the reduced space) $B(h)$ with projection $\pi : \mathcal{N}_0(h) \to B(h)$. Assume that all the solutions of Y_0 in $\mathcal{N}_0(h)$ are periodic. Then the following results hold:

(i) The base space $B(h)$ inherits a symplectic structure ω from (M, Ω); i.e., $(B(h), \omega)$ is a symplectic manifold.

(ii) If ε is a small parameter, the Hamiltonian $\mathcal{H}_1 : M \to \mathbb{R}$ is smooth, such that $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^{\#}$, $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$, and ϕ_ε^t is the flow defined by Y_ε. Let the average of $\mathcal{H}_1$ be

$$\bar{\mathcal{H}} = \frac{1}{T}\int_0^T \mathcal{H}_1(\phi_0^t)dt,$$

which is a smooth function on $B(h)$, and let $\bar{\phi}^t$ be the flow on $B(h)$ defined by $\bar{Y} = d\bar{\mathcal{H}}^{\#}$. If $\bar{\mathcal{H}}$ has a nondegenerate critical point at $\pi(p) = \bar{p} \in B(h)$ with $p \in \mathcal{N}_0$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for ε small with $p(0) = p$, $T(0) = T$, $p(\varepsilon) \in \mathcal{N}_\varepsilon$, and the solution of Y_ε through $p(\varepsilon)$ is $T(\varepsilon)$-periodic.

(iii) If p and $\bar{p}$ are as in the previous item and if the characteristic exponents of $\bar{Y}(\bar{p})$ (that is, the eigenvalues of the matrix $A = J\partial^2\bar{\mathcal{H}}/\partial y^2(\bar{p})$, where J denotes the standard skew-symmetric matrix) are $\lambda_1, \lambda_2, \dots, \lambda_{2n-2}$, then the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are 1, 1, $1 + \varepsilon\lambda_1 T + O(\varepsilon^2)$, $1 + \varepsilon\lambda_2 T + O(\varepsilon^2)$,

$\ldots, 1 + \varepsilon\lambda_{2n-2}T + O(\varepsilon^2)$,

(iv) If one or more of the characteristic exponents λ_j is real or has nonzero real part, then the periodic solution through $p(\varepsilon)$ is unstable. If the matrix A is the coefficient matrix of a parametrically stable system, then the periodic solution through $p(\varepsilon)$ is elliptic.

(v) If p and $\bar{p}$ are as before, suppose there are symplectic action-angle variables $(I_1, \ldots, I_{n-1}, \varphi_1, \ldots, \varphi_{n-1})$ at $\bar{p}$ in $B(h)$ such that

$$\bar{\mathcal{H}} = \sum_{k=1}^{n-1} \omega_k I_k + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} C_{kj} I_k I_j + \mathcal{H}^{\#}, \tag{1.2}$$

where the ω_k are nonzero, $C_{kj} = C_{jk}$, and $\mathcal{H}^{\#}(I_1, \ldots, I_{n-1}, \varphi_1, \ldots, \varphi_{n-1})$ is at least cubic in $I_1, \ldots, I_{n-1}$. Assume that $\det C_{kj} \neq 0$. That is, assume the system has been put into Birkhoff normal form and that the "twist" condition is satisfied, and assume that the period varies with $\mathcal{H}_0$ in a nontrivial way. Then near the periodic solutions given above there are invariant KAM tori of dimension n. In particular, when $n = 2$ the periodic solution is orbitally stable.

Points (i)–(v) correspond respectively to Theorems 2.1 and 2.2, Corollaries 2.2 and 2.3, and Theorem 2.5 of subsection 2.3 of [65]. Proofs and details appear in [65]. We do not need the compactness assumption on $\mathcal{N}_0(h)$ and $B(h)$, as the results (i)–(v) are local in nature; compare with [65].

2. The Hamiltonian $\mathcal{G}$.

Since this paper considers various systems that are perturbations of $\mathcal{G}$, it is appropriate that we spend some time looking at this Hamiltonian.

One way that (1.1) arises is by the introduction of rotating symplectic coordinates, so let us return to fixed coordinates (q, p) by

$$x = e^{J(t-t_0)} q, \qquad y = e^{J(t-t_0)} p, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The Hamiltonian $\mathcal{G}(x, y) = -x^T J y$ becomes $\mathcal{G}(q, p) \equiv 0$; thus q and p are constants. The transformation is orthogonal, and so $r_1 = \|x\| = \|q\|$, $r_2 = \|y\| = \|p\|$ are constants of the motion. Without trepidation we introduce nonsymplectic polar coordinates

$$x_1 = r_1 \cos\theta_1, \quad x_2 = r_1 \sin\theta_1, \quad y_1 = r_2 \cos\theta_2, \quad y_2 = r_2 \sin\theta_2.$$

The angles θ_1, θ_2 are not well defined since the orientation of the qp-frame of reference is not uniquely defined (e^{-Jt_0} is arbitrary), but the angle between q and p is a constant. Thus, the systems whose Hamiltonian is $\mathcal{G}$ have three geometric integrals of motion,

$$r_1, \quad r_2, \quad \theta_1 - \theta_2.$$

We return to rectangular coordinates by being mindful of the d'Alembert character [32, 49] and defining

$$\begin{aligned} a_1 &= r_1^2 = x_1^2 + x_2^2, & a_2 &= r_2^2 = y_1^2 + y_2^2, \\ a_3 &= r_1 r_2 \cos(\theta_1 - \theta_2) = x_1 y_1 + x_2 y_2, & a_4 &= r_1 r_2 \sin(\theta_1 - \theta_2) = x_2 y_1 - x_1 y_2 = \mathcal{G}. \end{aligned} \tag{2.1}$$

The familiar trigonometry identity $\cos^2\phi + \sin^2\phi = 1$ yields

$$a_3^2 + a_4^2 = a_1 a_2, \qquad a_1 \geq 0, \quad a_2 \geq 0. \tag{2.2}$$

Specifying a_1, a_2, a_3, and a_4 subject to the constraints (2.2) uniquely specifies an orbit because the constraints $a_1 \geq 0$, $a_2 \geq 0$ allow one to solve for $r_1 \geq 0$ and $r_2 \geq 0$, and the constraint $a_3^2 + a_4^2 = a_1 a_2$ allows one to uniquely solve for $\cos(\theta_1 - \theta_2)$ and $\sin(\theta_1 - \theta_2)$ and hence the angle $\theta_1 - \theta_2$.

Since a_1, a_2, and a_3 are integrals for the system whose Hamiltonian is $\mathcal{G} = a_4$ we have

$$\{a_1, a_4\} = \{a_2, a_4\} = \{a_3, a_4\} = 0, \tag{2.3}$$

and a direct computation gives

$$\{a_1, a_2\} = 4a_3, \;\; \{a_3, a_2\} = 2a_2, \;\; \{a_3, a_1\} = -2a_1. \tag{2.4}$$

By making the linear symplectic change of coordinates

$$\begin{aligned} x_1 &= \tfrac{1}{\sqrt{2}}(u_1 - v_2), \qquad & y_1 &= \tfrac{1}{\sqrt{2}}(u_2 + v_1), \\ x_2 &= \tfrac{1}{\sqrt{2}}(u_2 - v_1), \qquad & y_2 &= \tfrac{1}{\sqrt{2}}(u_1 + v_2), \end{aligned}$$

the Hamiltonian becomes

$$\mathcal{G} = \mathcal{G}(u, v) = \frac{1}{2}(-u_1^2 + u_2^2 - v_1^2 + v_2^2), \tag{2.5}$$

thus manifesting that the Hamiltonian (1.1) is equivalent to the quadratic part of a Hamiltonian system with semisimple $1 : -1$ resonance. The bifurcations related to Hamiltonian systems enjoying this resonance are dealt with in [36, 39].

On the integral manifold where $\mathcal{G} = \gamma > 0$ we have

$$u_2^2 + v_2^2 - 2\gamma = u_1^2 + v_1^2 = \rho^2.$$

So above each point P in the u_2v_2-plane outside the (blue) circle of radius $\sqrt{2\gamma}$ there is a circle of radius ρ in the u_1v_1-plane, and above each point p on the blue circle there is a point (the origin) in the u_1v_1-plane; see Figure 1(a). Thus above the (green) dashed ray in the u_2v_2-plane through p and P there lies the whole u_1v_1-plane. Letting the ray rotate all the way around in the u_2v_2-plane, we get a solid torus, $S^1 \times \mathbb{R}^2$. When $\mathcal{G} = \gamma < 0$ the picture is the same with the subscripts reversed. Thus, the energy surface $\mathcal{G} = \gamma \neq 0$ is a three-dimensional hyperboloid that is homeomorphic to a solid torus.

Now imagine the energy surface $\mathcal{G} = \gamma$ as $\gamma \to 0$. The inner (blue) circle tends to a point, and so the algebraic variety where $\mathcal{G} = 0$ is not a manifold but is homeomorphic to a solid torus with the blue circle identified with a point, i.e., $S^1 \times \mathbb{R}^2/(S^1 \times (0,0))$; see Figure 1(b).

From (2.5) the equations of motion in the uv-space are

$$\begin{aligned} \dot{u}_1 &= -v_1, \qquad & \dot{u}_2 &= v_2, \\ \dot{v}_1 &= u_1, \qquad & \dot{v}_2 &= -u_2, \end{aligned}$$

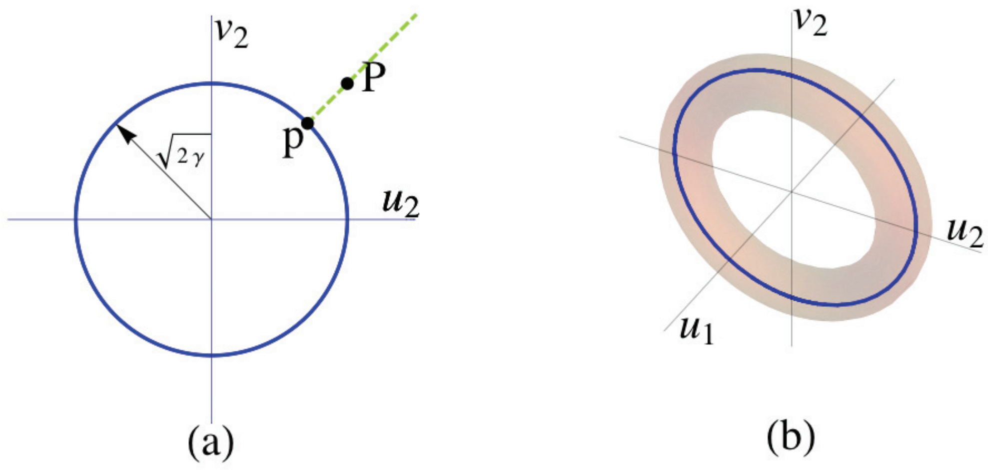

Figure 1. *The integral manifold.*

which are two harmonic oscillators of period 2π. Thus after t increases by 2π, both oscillators return to their initial value. So each orbit intersects the plane above the (green) dashed ray once and only once. Thus the reduced (orbit) space is homeomorphic to $\mathbb{R}^2$. When $\gamma = 0$ the reduced space is not smooth since it is a cone. Hence, a reduction process associated with the fact that $\mathcal{G}$ is converted into an integral of motion is regular provided that $\gamma \neq 0$, and singular when $\gamma = 0$. Regular reduction theory was first introduced in [45] (see also [44]), whereas singular reduction appeared for the first time in [3].

Looking at the reduced space from another perspective, the invariants a_1, a_2, a_3, a_4 subject to the constraints (2.2) uniquely specify an orbit. After fixing $\mathcal{G} = a_4 = \gamma$, the reduced space is

$$\mathcal{R}_\gamma = \{(a_1, a_2, a_3) : a_1 a_2 - a_3^2 = w^T S w = \gamma^2,\ a_1 \geq 0,\ a_2 \geq 0\}, \tag{2.6}$$

where

$$w = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

this is shown in Figure 2. It is a two-dimensional hyperboloid of revolution—revolving about the line $a_1 = a_2$. To put this quadratic surface into standard form, rotate the a_1, a_2 axes by 45° and scale a_3 by letting $a_1 = \frac{1}{\sqrt{2}}(\xi + \eta)$, $a_2 = \frac{1}{\sqrt{2}}(-\xi + \eta)$, $a_3 = \frac{1}{\sqrt{2}}\zeta$, so that $\eta^2 - \xi^2 - \zeta^2 = 2\gamma^2$, which is a two-dimensional hyperboloid of revolution that is homeomorphic to the plane when $\gamma \neq 0$ and is a cone when $\gamma = 0$. This reduced space and the invariants associated with the reduction appeared in [12] in the setting of singular reduction theory. See also [54] for a classification of reduced spaces related to Hamiltonian systems of two degrees of freedom.

We remark that all the points of $\mathcal{R}_0$ account for the rectilinear motions, as in this case the

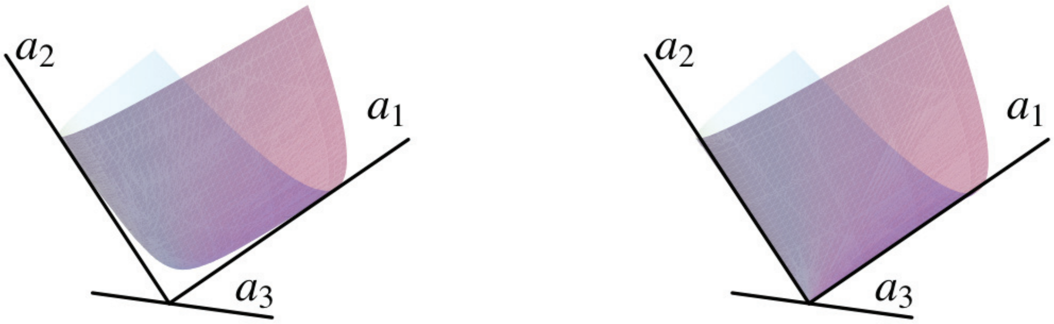

Figure 2. *The reduced space. On the left: $\gamma \neq 0$. On the right: $\gamma = 0$.*

angular momentum is zero. Additionally, the cone point $(0, 0, 0)$ reconstructs to the origin of $\mathbb{R}^4$, whereas for the rest of the points of $\mathcal{R}_\gamma$ (with γ having any value) a circle S^1 is attached.

The symmetry group of the reduced space defined by (2.6) is

$$\mathcal{S} = \{M \in Gl(3, \mathbb{R}^3) : (Mw)^T S(Mw) = w^T Sw, \forall w \in \mathbb{R}^3\} = \{M \in Gl(3, \mathbb{R}^3) : M^T SM = S\},$$

and its algebra is

$$\int = \{N \in gl(3, \mathbb{R}^3) : N^T S + SN = 0\}$$

with the three generators

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Lie brackets satisfy $[E, F] = 2D$, $[D, E] = E$, $[D, F] = -F$, which are the standard bracket relations for the generators D, E, F of $sl(2, \mathbb{R})$, the algebra of the special linear group $Sl(2, \mathbb{R})$; see [38].

Mapping $a_1 \to -2E$, $a_2 \to 2F$, $a_3 \to -2D$ sets up a Lie algebra isomorphism between the algebra generated by a_1, a_2, a_3, and $sl(2, \mathbb{R})$. That is, the brackets in (2.4) merely reflect the symmetry of the reduced space.

3. Two-degrees-of-freedom comet problems. The planar circular restricted three-body problem describes the motion of an infinitesimally small particle moving in the plane under the influence of the gravitational attraction of two finite particles that revolve around each other in a circular orbit with uniform velocity. The two finite particles, called the primaries, have mass $\mu > 0$ and $1 - \mu > 0$. Let $x \in \mathbb{R}^2$ be the coordinate of the infinitesimal particle in a uniformly rotating coordinate system, and $y \in \mathbb{R}^2$ the momentum conjugate to x. The rotating coordinate system is chosen so that the particle of mass μ is always at $(1 - \mu, 0)$

and the particle of mass $1-\mu$ is at $(-\mu, 0)$. The Hamiltonian governing the motion of the infinitesimal particle in these coordinates is

$$\mathcal{H} = \frac{1}{2}\|y\|^2 - x^T J y - U, \tag{3.1}$$

where $x, y \in \mathbb{R}^2$ are conjugate, U is the self-potential

$$U = \frac{\mu}{d_1} + \frac{1-\mu}{d_2}, \tag{3.2}$$

and d_i is the distance from the infinitesimal body to the ith primary, or

$$d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2, \qquad d_2^2 = (x_1 + \mu)^2 + x_2^2. \tag{3.3}$$

The term $-x^T J y = \mathcal{G}$ in the Hamiltonian $\mathcal{H}$ reflects the fact that the coordinate system is not a Newtonian frame but a rotating system. It gives rise to the Coriolis force in the equations of motion.

A generalization is the planar restricted N-body problem with $N \geq 4$, defined as follows. Let $b_1, \ldots, b_{N-1} \in \mathbb{R}^2$ be a central configuration of the $(N-1)$-body problem, i.e.,

$$b_i = \sum_{j=1}^{N-1} \frac{m_j(b_j - b_i)}{\|b_j - b_i\|^3}, \quad i = 1, \ldots, N-1, \quad \text{and} \quad \sum_{j=1}^{N-1} m_j b_j = 0,$$

where $m_1, \ldots, m_{N-1}$ are the masses of the primaries normalized so that $m_1 + \cdots + m_{N-1} = 1$. The planar restricted N-body problem describes the motion of an infinitesimally small particle moving in the plane under the influence of the gravitational attraction of the primaries of mass m_i at b_i in a frame that rotates with uniform velocity. The Hamiltonian governing the motion of the infinitesimal particle in these coordinates is the same as (3.1) but with

$$U = \sum_{j=1}^{N-1} \frac{m_j}{\|b_j - x\|}. \tag{3.4}$$

See [46] for more details about this generalization.

In order to study the motion when the infinitesimal is far from the primaries, we introduce a small parameter ε. In the Hamiltonian (3.1), scale the variables by $x \to \varepsilon^{-2}x$, $y \to \varepsilon y$; this is symplectic with multiplier ε. This symplectic scaling procedure will be used throughout the paper many times; see [46, 48] for a discussion of scaling. This scaling, as with others, exploits the fact that the potential is homogeneous and gives yet another example for the general discussion found in [9]. The Hamiltonian becomes

$$\mathcal{H}_\varepsilon = -x^T J y + \varepsilon^3 \left(\frac{\|y\|^2}{2} - \frac{1}{\|x\|} \right) + O(\varepsilon^5). \tag{3.5}$$

We remark that when $N = 3$ the higher-order terms start at order $O(\varepsilon^7)$.

Now ε small means that the infinitesimal is near infinity, and (3.5) says that near infinity the Coriolis force dominates, and the next most important force looks like a Kepler problem with all the primaries at the origin. We point out that, in the context of the comet problem, we cannot allow the angular momentum to be zero because then the motion would be rectilinear, but the scaling is meaningful only if the infinitesimal is bounded away from the origin and infinity.

We write (3.5) in terms of the invariants (2.1) to get

$$\mathcal{H}_\varepsilon = a_4 + \varepsilon^3\left(\frac{a_2}{2} - \frac{1}{\sqrt{a_1}}\right) + O(\varepsilon^5).$$

To obtain the Hamiltonian on the reduced space $\mathcal{R}_\gamma$ where $a_4 = \gamma$, we drop this constant as well as the terms $O(\varepsilon^5)$ and divide by ε^3 to obtain

$$\bar{\mathcal{H}} = \frac{a_2}{2} - \frac{1}{\sqrt{a_1}}. \tag{3.6}$$

The phase portrait of the flow on the reduced space can be obtained by intersecting the level surfaces of (3.6) with the constraint surface (2.2), as illustrated in Figure 3.

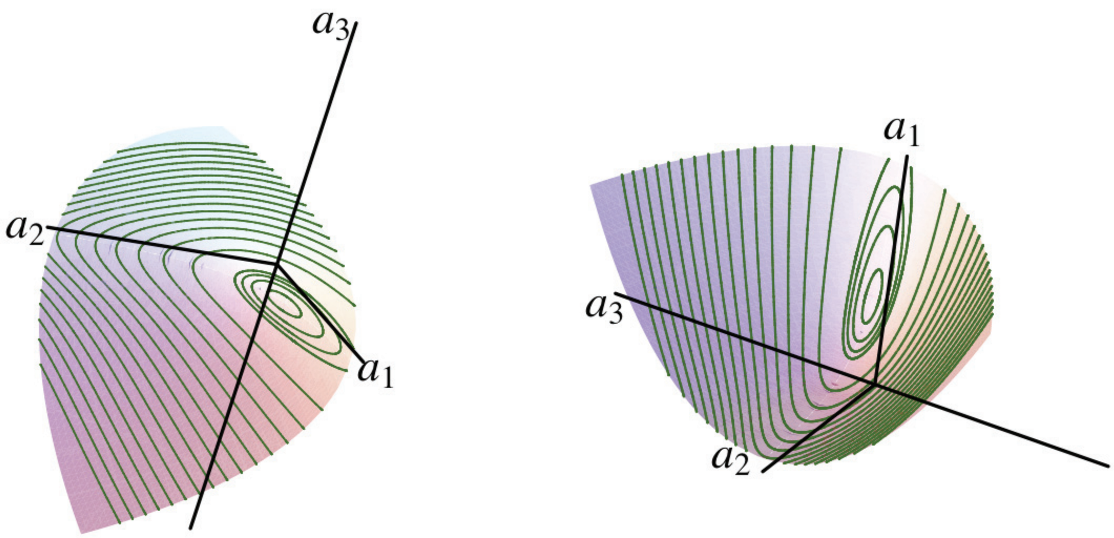

Figure 3. *Two views of the flow of the comet problem.*

The equations of motion are

$$\dot{a}_1 = 2a_3, \quad \dot{a}_2 = -2a_3(a_1)^{-3/2}, \quad \dot{a}_3 = -(a_1)^{-1/2} + a_2, \tag{3.7}$$

which can be obtained from $\dot{a}_i = \sum_j \{a_i, a_j\}\partial\bar{\mathcal{H}}/\partial a_j$ and the brackets in (2.4). Provided that $\gamma \neq 0$, these equations have a unique equilibrium point on $\mathcal{R}_\gamma$ at

$$a_1 = \gamma^4, \quad a_2 = \frac{1}{\gamma^2}, \quad a_3 = 0. \tag{3.8}$$

Note that $a_1 = r_1^2$ so $r_1 = \gamma^2$, and the equilibrium point is related to circular-like motions.

The procedures in subsections 2.2 and 2.3 of [65] show that this problem has two families of linearly stable (elliptic) periodic solutions, as we will detail below. We will also prove the existence of KAM 2-tori around the periodic solutions.

We start by introducing planar Delaunay coordinates [7], (ℓ, g, L, G), where ℓ represents the mean anomaly, g the argument of the pericenter, $L > 0$ the action related with the semimajor axis a by $L = \sqrt{a}$, and $G = x_1y_2 - x_2y_1$ the angular momentum (thus $G = -a_4 \equiv -\gamma$). We stress that $|G| \leq L$ and $|G| = L$ only for circular motions. The Hamiltonian (3.6) in terms of the Delaunay elements yields

$$\bar{\mathcal{H}} = -\frac{1}{2L^2}. \tag{3.9}$$

Now, we define Poincaré-like coordinates for one-degree-of-freedom systems as functions of Delaunay coordinates. In fact, as we want to analyze the dynamics in a neighborhood of the equilibrium point and this equilibrium is related with circular motions, we define

$$Q = \sqrt{2(L \pm \gamma)}\sin \ell, \quad P = \sqrt{2(L \pm \gamma)}\cos \ell, \tag{3.10}$$

where the sign "+" applies for $\gamma < 0$ (prograde motions), whereas the sign "−" is used when $\gamma > 0$ (retrograde motions).

The inverse of (3.10) is

$$L = \frac{1}{2}(Q^2 + P^2 \mp 2\gamma), \quad \ell = \pm\tan^{-1}\left(\frac{Q}{P}\right). \tag{3.11}$$

The sign of ℓ is taken to be positive or negative depending on the signs of Q and P. We remark that for the circular motions ℓ is not well defined, but then $Q = P = 0$. Indeed, the transformation (3.10) extends analytically to the origin of the QP-plane, provided that Q and P written in terms of ℓ and L and all the computations that we have to carry out satisfy the d'Alembert characteristic; see details in [32] and also a related example in [43]. As the d'Alembert characteristic is maintained, one can conclude that circular motions can be analyzed properly with these Poincaré-like coordinates and that all the expressions are valid in a neighborhood of the circular trajectories. It is also straightforward to prove that the Poisson bracket $\{Q, P\} = 1$, and thus they are symplectic variables.

Now, the Hamiltonian $\bar{\mathcal{H}}$ in terms of Q and P is

$$\bar{\mathcal{H}} = -\frac{2}{(P^2 + Q^2 \mp 2\gamma)^2}. \tag{3.12}$$

We take the convention that when the signs "±" or "∓" appear in a formula involving γ or G, the upper sign applies for $\gamma < 0$ $(G > 0)$, and the lower sign applies for $\gamma > 0$ $(G < 0)$.

The analysis of the stability of the equilibrium point (3.8) is translated to the study of the stability of the equilibrium $(0, 0)$ in the two sets of coordinates Q, P, but since

$$\bar{\mathcal{H}} = -\frac{2}{(P^2 + Q^2 \mp 2\gamma)^2} = -\frac{1}{2\gamma^2} \mp \frac{1}{2\gamma^3}(P^2 + Q^2) - \frac{3}{8\gamma^4}(P^2 + Q^2)^2 + O(6), \tag{3.13}$$

$\bar{\mathcal{H}}$ is a perturbation of the harmonic oscillator; therefore $(0,0)$ is parametrically stable. The parametric stability study of the equilibrium implies, following Theorem 2.2 and Corollaries 2.2 and 2.3 of [65] (see also the last paragraphs of section 1), that for Hamiltonian (3.5) there are two families of elliptic periodic motions, both nearly circular, one prograde and the other retrograde. The families of periodic motions are parameterized by γ. Since the eigenvalues of the quadratic part of $\bar{\mathcal{H}}$ are $\gamma^{-3}\imath$, $-\gamma^{-3}\imath$, the corresponding nontrivial characteristic multipliers of the periodic solutions are $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$ and $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$. Finally, after rescalings, the periods T of these periodic solutions are near $2\pi\varepsilon^{-1}$.

The existence of KAM 2-tori follows from the expression (3.13). Indeed, after introducing action-angle coordinates (I,φ) through $Q=\sqrt{2I}\sin\varphi$, $P=\sqrt{2I}\cos\varphi$, and taking into account that $\gamma=-G$, we can rewrite the Hamiltonian of the restricted problem in the comet case around the equilibrium point (3.8), arriving at

$$\mathcal{H}_\varepsilon = -G - \varepsilon^3\left(\frac{1}{2(I\pm G)^2}\right) + O(\varepsilon^5),$$

and the second derivative of the term factorized by ε^3 with respect to I yields

$$-\frac{3}{(I\pm G)^4},$$

which is a twist term that does not vanish. Therefore, by Theorem 2.5 of subsection 2.3 in [65] (see also the last paragraphs of section 1), there are invariant KAM tori of dimension two, and the periodic solutions of the previous paragraphs are orbitally stable. We remark that Theorem 2.5 of [65] applies for properly degenerate Hamiltonian systems such that the perturbation removes the degeneracy, a case studied by Arnol'd, Kozlov, and Neishtadt [4].

In particular, our procedure simplifies previous approaches dealing with the existence of a twist term that appeared in [37] for the restricted three-body problem. Besides, the treatment undertaken in the previous paragraphs is valid for the restricted N-body problem (see [46]) and generalizes the symmetric periodic solutions of large radii found by Moulton [50] in the planar restricted three-body problem. More specifically, we have proved the following result.

Theorem 3.1. *For $N\geq 3$, the Hamiltonian of the planar restricted N-body problem given by* (3.1) *with self-potential U given by* (3.4) *(or given by* (3.2) *if $N=3$, i.e., in the case of the planar circular restricted three-body problem) has two families of near-circular periodic solutions that are elliptic with characteristic multipliers 1, 1, $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$, and $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$. The radii and periods of these solutions are very large, $\|x\|\approx\varepsilon^{-2}G^2$ (or $\|x\|\approx\varepsilon^{-2}\gamma^2$) and $T(\varepsilon)\approx 2\pi\varepsilon^{-1}$. The families of periodic solutions are enclosed by KAM 2-tori; therefore these periodic solutions are also orbitally stable.*

This theorem can be applied to any (small) perturbation of the rotating planar Kepler problem since we have needed only to make use of the explicit expression of the terms factorized by ε^3 in (3.5), that is, the Kepler-like term. Thus, regardless of the higher-order terms, our deductions can be applied to other types of planar Hamiltonians that can be written in the form (3.5) where the terms of order ε^3 do not necessarily correspond to the restricted problem. Examples that can be cast in this form appear in many different contexts. We shall treat some of them in section 4 when dealing with some three-dimensional cases.

Recently Llibre and Stoica [41] considered planar comet- and Hill-type restricted N-body problems where the interaction potential between the infinitesimal particle and the primaries is taken to be a finite sum of terms of the form $r^{-\alpha_i}$ with $\alpha_i > 0\ \forall i$. In particular, they conclude that if the infinitesimal particle is far from the primaries, and the long range dominant term β/r^{α} of the potential is such that $\beta < 0$ and $\alpha \neq 2$, then there exist two one-parameter families of large nearly circular periodic solutions. These solutions are elliptic KAM stable for $0 < \alpha < 2$ and unstable for $\alpha > 2$. This result generalizes our Theorem 3.1, but our approach is more straightforward. In fact, the case we have considered corresponds to $\alpha = 1$ and $\beta = -1$, and then the families of periodic solutions and KAM tori and the stability are the same.

4. Three-degrees-of-freedom comet problems.

In this section we consider Hamiltonian systems that are three-dimensional perturbations of the term $\mathcal{G}$.

4.1. Reduced space.

We take $x \in \mathbb{R}^3$ representing the coordinates, and $y \in \mathbb{R}^3$ designating the associated momenta. We start by considering the invariant a_i's associated with a reduction process of a three-degrees-of-freedom Hamiltonian vector field. Apart from the "planar invariants" of section 2 we need to take into account possible invariants due to the introduction of x_3, y_3. Since both x_3 and y_3 are independent of $\mathcal{G}$, we can incorporate them into the list of invariants. We get

$$\begin{aligned} a_1 &= x_1^2 + x_2^2, & a_2 &= y_1^2 + y_2^2, \\ a_3 &= x_1 y_1 + x_2 y_2, & a_4 &= x_2 y_1 - x_1 y_2 = \mathcal{G}, \\ a_5 &= x_3, & a_6 &= y_3. \end{aligned} \tag{4.1}$$

We remark that for the spatial problems $-\mathcal{G}$ does not represent the magnitude of the angular momentum but its third component, that is, the scalar product of the angular momentum vector with the unit vector in the vertical direction x_3.

The constraints for the invariants are the same as for the planar case, i.e.,

$$a_3^2 + a_4^2 = a_1 a_2, \qquad a_1 \geq 0, \quad a_2 \geq 0. \tag{4.2}$$

Fixing the value of $a_4 = \gamma$, the reduced phase space is given by the identity and inequalities of (4.2); that is, after setting $\mathcal{G} = a_4 = \gamma$, one has

$$\mathcal{T}_\gamma = \{(a_1, a_2, a_3, a_5, a_6) : a_1 a_2 - a_3^2 = \gamma^2,\ a_1 \geq 0,\ a_2 \geq 0\}, \tag{4.3}$$

representing a four-dimensional symplectic manifold; see [11] and also [52]. As in the planar case, the manifold is noncompact and is regular provided that $\gamma \neq 0$. Hence, a reduction process that leads to this space lies in the context of singular reduction theory. It is also not hard to deduce that geometrically this manifold is the Cartesian product of $\mathcal{R}_\gamma$ and the $x_3 y_3$-plane; that is, $\mathcal{T}_\gamma = \mathcal{R}_\gamma \times \mathbb{R}^2$. In particular, after computing the gradient of the identity in (4.2), we deduce that when $\gamma = 0$ the gradient becomes zero if and only if $a_1 = a_2 = a_3 = 0$, whereas a_5 and a_6 can take any value. This means that the singular points on the manifold $\mathcal{T}_0$ are the points of the type $(0, 0, 0, a_5, a_6)$. Thus, the set of singular points is the two-dimensional set $\{(0, 0, 0, 0)\} \times \mathbb{R}^2$. The motions related to the points $\mathcal{T}_0$ are of polar type; i.e., their projections onto the $x_1x_2x_3$-space are perpendicular to the x_1x_2-plane. In particular, all rectilinear solutions correspond with points on the space $\mathcal{T}_0$, and the rectilinear solutions occurring in the x_3y_3-plane are in correspondence with points on $\{(0,0,0,0)\}\times\mathbb{R}^2$. The point $(0,0,0,0,0)$ corresponds to the origin of $\mathbb{R}^6$.

The Poisson brackets of the invariants are the same as those given in (2.4), but incorporating a_5 and a_6. We get

$$\{a_i,a_5\}=\{a_i,a_6\}=0 \quad \text{for} \quad i=1,\dots,4, \quad \{a_5,a_6\}=1. \tag{4.4}$$

4.2. Spatial perturbations of $\mathcal{G}$ plus a Kepler term. We consider Hamiltonian systems of three degrees of freedom of the form

$$\mathcal{H}_\varepsilon=-(x_1y_2-x_2y_1)+\varepsilon^3\left(\frac{\|y\|^2}{2}-\frac{1}{\|x\|}\right)+O(\varepsilon^5). \tag{4.5}$$

That is, we study Hamiltonians that have as the dominant term the Coriolis force; its first perturbation the Kepler Hamiltonian and the rest of the terms appear as perturbations at higher order. We have placed the higher-order terms at $O(\varepsilon^5)$, but the power may change for other cases.

An example is the spatial circular restricted three-body problem in the comet case [47, 35], where the small parameter is introduced in exactly the same way as in the planar restricted three-body problem in section 3. Another example is the spatially restricted N-body problem with $N\geq 4$, which can be stated as its planar version of section 3. A particular case of the spatially restricted N-body problem considers that the $N-1$ primaries with equal mass m are in a central configuration at the vertices of an $(N-1)$-regular polygon; see [40]. Other related comet-like problems include the elliptic restricted three-body problem when the infinitesimal particle is very far from the two primaries [55], but in the present paper we do not deal with time-dependent Hamiltonians. We will say no more about these examples, but other examples will be discussed later in this section.

After dropping the terms of order $O(\varepsilon^5)$ and the constant $a_4=\gamma$ and multiplying by ε^{-3}, the Hamiltonian $\bar{\mathcal{H}}$ in terms of the invariants a_i is

$$\bar{\mathcal{H}}=\frac{1}{2}(a_2+a_6^2)-\frac{1}{\sqrt{a_1+a_5^2}}. \tag{4.6}$$

Now, making use of the Poisson brackets (2.4) and (4.4) yields the corresponding equations of motion,

$$\begin{gathered}\dot a_1=2a_3,\qquad \dot a_2=-\frac{2a_3}{(a_1+a_5^2)^{3/2}},\qquad \dot a_3=-\frac{a_1}{(a_1+a_5^2)^{3/2}}+a_2,\\ \dot a_5=a_6,\qquad \dot a_6=-\frac{a_5}{(a_1+a_5^2)^{3/2}},\end{gathered} \tag{4.7}$$

which have a unique relative equilibrium at

$$a_1=\gamma^4,\quad a_2=\frac{1}{\gamma^2},\quad a_3=0,\quad a_5=0,\quad a_6=0. \tag{4.8}$$

As in the planar case, we avoid $\gamma = 0$ because the infinitesimal particle should be far from the primaries. This equilibrium is related to circular coplanar motions (coplanar meaning that they lie in the equatorial plane defined by the motion of the primaries) as $a_5 = a_6 = 0$ and $r_1 \equiv r = \gamma^2$ is constant.

Polar-nodal coordinates [15, 16] are a set of symplectic variables $(r, \vartheta, \nu, R, \Theta, K)$, where r stands for the radial distance from the origin to the particle, ϑ represents the argument of latitude, ν accounts for the right ascension of the node, whereas R is the conjugate momentum of r. Additionally, $\Theta = \|\boldsymbol{\Theta}\|$ is the magnitude of the angular momentum vector, and K is the third component of $\boldsymbol{\Theta}$, so $0 \le |K| \le \Theta \le L$ and $K = -\gamma$. Thus, the spatial Delaunay coordinates are given by $(\ell, g, \nu, L, \Theta, K)$. We remark that K corresponds to the action G of the planar Delaunay elements.

We could introduce Poincaré-like coordinates in terms of the Delaunay elements, as in the previous section, but we prefer to work with Poincaré-like coordinates related to polar-nodal coordinates. The reason for our preference is that, in contrast to the planar situation, in order to obtain the twist condition for proving the existence of KAM 3-tori we need to normalize explicitly the terms of order $O(\varepsilon^5)$, and this will be executed more easily using polar-nodal coordinates, as we shall see in the examples. For the moment we drop the higher-order terms, although we will need them later on. As we deal with near-coplanar motions we define

$$\begin{aligned} Q_1 &= \sqrt{2(\Theta \pm \gamma)}\sin\vartheta, & Q_2 &= r, \\ P_1 &= \sqrt{2(\Theta \pm \gamma)}\cos\vartheta, & P_2 &= R, \end{aligned} \tag{4.9}$$

where the positive sign is used for prograde motions and the negative sign for retrograde motions. Note that for each sign (4.9) represents a set of symplectic coordinates.

The inverse of (4.9) is given by

$$\begin{aligned} \Theta &= \frac{1}{2}(Q_1^2 + P_1^2 \mp 2\gamma), & r &= Q_2, \\ \vartheta &= \pm\tan^{-1}\left(\frac{Q_1}{P_1}\right), & R &= P_2, \end{aligned} \tag{4.10}$$

where the sign of ϑ must be taken positive or negative depending on the signs of Q_1 and P_1.

For equatorial motions ($|\gamma| = \Theta$) the angle ϑ is undefined, but then $Q_1 = P_1 = 0$. As long as all the expressions we are handling as well as the transformation (4.9) exhibit the d'Alembert characteristic for Q_1 and P_1, the transformation (4.9) is extended analytically to the subset $Q_1 = P_1 = 0$. This was stated by Henrard [32] (see also [43]), and it is the case here. Hence (4.9) together with (4.10) are well suited for analyzing the relative equilibria related with equatorial periodic solutions. Another way of proving that the extension of (4.9) incorporating $Q_1 = P_1 = 0$ is analytic is by expressing the coordinates having singularities as explicit analytic functions of analytic first integrals. This approach was taken in [1], where a set of Poincaré-like coordinates were proved to be analytically extended to their singular points.

The relationship between the invariants a_i and the local coordinates of $\mathcal{T}_\gamma$ valid for motions of equatorial type is given through

$$
\begin{aligned}
Q_1 &= \frac{2^{1/2} a_5 \sqrt{a_2 a_5^2 - 2a_3 a_5 a_6 + a_1 a_6^2 + \gamma^2}\sqrt{\sqrt{a_2 a_5^2 - 2a_3 a_5 a_6 + a_1 a_6^2 + \gamma^2} \pm \gamma}}{\sqrt{a_1 + a_5^2}\sqrt{a_2 a_5^2 - 2a_3 a_5 a_6 + a_1 a_6^2}},\\
Q_2 &= \sqrt{a_1 + a_5^2},\\
P_1 &= \frac{2^{1/2}(a_1 a_6 - a_3 a_5)}{\sqrt{a_1 + a_5^2}\sqrt{\sqrt{a_2 a_5^2 - 2a_3 a_5 a_6 + a_1 a_6^2 + \gamma^2} \mp \gamma}},\\
P_2 &= \frac{a_3 + a_5 a_6}{\sqrt{a_1 + a_5^2}}.
\end{aligned} \tag{4.11}
$$

As in the planar case, we adopt the convention that when the signs "±" or "∓" appear in an expression involving γ or K, the upper sign applies for $\gamma < 0$ $(K > 0)$ and the lower sign applies for $\gamma > 0$ $(K < 0)$.

Using (4.10), the Hamiltonian (4.6) in terms of the polar-nodal coordinates is given by

$$\bar{\mathcal{H}} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r},$$

while in the variables Q_i, P_i the Hamiltonian $\bar{\mathcal{H}}$ reads as

$$\bar{\mathcal{H}} = \frac{P_2^2}{2} + \frac{(P_1^2 + Q_1^2 \mp 2\gamma)^2}{8Q_2^2} - \frac{1}{Q_2}. \tag{4.12}$$

Thus, we linearize $\bar{\mathcal{H}}$ around the equilibrium point (4.8), shifting the origin of the coordinates to the equilibrium point. This is achieved by the linear change

$$Q_1 = \varepsilon \bar{Q}_1 + Q_1^0, \quad Q_2 = \varepsilon \bar{Q}_2 + Q_2^0, \quad P_1 = \varepsilon \bar{P}_1 + P_1^0, \quad P_2 = \varepsilon \bar{P}_2 + P_2^0, \tag{4.13}$$

where Q_1^0, Q_2^0, P_1^0, and P_2^0 are the values of the Q_i's and P_i's at the equilibrium. The change is symplectic with multiplier ε^{-2}. Since the trajectory is of equatorial type, then $Q_1^0 = P_1^0 = 0$. It is easy to prove that $P_2^0 = 0$ and $Q_2^0 = K^2 = \gamma^2$ (as Q_2^0 and P_2^0 are respectively the values of r and R at the equilibrium). This implies that the equilibrium in the variables (4.9) is $(0, \gamma^2, 0, 0)$. After applying the linear change to $\bar{\mathcal{H}}$ and multiplying by ε^{-2}, we expand the result in powers of ε, taking only the terms independent of ε, which is equivalent to truncating $\mathcal{H}_\varepsilon$ at the power ε^3. Then we drop the constant terms, arriving at

$$\bar{\mathcal{H}}_2 = \frac{1}{2\gamma^6}\Big(\mp\gamma^3(\bar{P}_1^2 + \bar{Q}_1^2) + \gamma^6 \bar{P}_2^2 + \bar{Q}_2^2\Big), \tag{4.14}$$

which represents two quadratic Hamiltonian systems. We see that the convention for the sign as in the previous section implies that $\bar{\mathcal{H}}_2$ is positive definite. The eigenvalues are $\gamma^{-3}\imath, \gamma^{-3}\imath, -\gamma^{-3}\imath, -\gamma^{-3}\imath$ which, together with the feature that the eigenvectors form a basis of $\mathbb{R}^4$, reflect the fact that equations (4.14) are in semisimple $1:1$ resonance.

This is enough to conclude that the equilibrium point (4.8) is linearly and parametrically stable. Then, Theorem 2.2 and Corollaries 2.2 and 2.3 of [65] establish that the Hamiltonian (4.5) has two families of elliptic periodic motions, both nearly circular and coplanar, one prograde and the other retrograde. The families of periodic motions are parameterized by γ. The periods T of these periodic solutions are near $2\pi\varepsilon^{-1}$.

However, we cannot guarantee the existence of KAM tori around the equilibrium point as we did in the planar case. The reason is that, considering only terms up to the power three in ε, the associated twist term is still too degenerate, and we would need that the quadratic terms of (4.14) leave the $1:1$ resonance. In other words, we need to take into account the higher-order terms in order to get the appropriate twist condition. This means that we need to explicitly compute the terms factorized by ε^5 (or by ε^7) to conclude the existence of KAM tori. This will be done for the comet case of the restricted three-body problem and for two more cases in the forthcoming subsections.

At this point we have proved the following result.

Theorem 4.1. *The spatial Hamiltonian system defined by the Hamiltonian* (4.5) *(in particular the spatially restricted N-body problem and the spatial circular restricted three-body problem) has two families of near-circular near-coplanar elliptic periodic solutions. Their characteristic multipliers are* 1, 1, $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$, $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$, $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$, *and* $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^5)$. *The radii and periods of these solutions are very large,* $\|x\|\approx\varepsilon^{-2}K^2$ *(or* $\|x\|\approx\varepsilon^{-2}\gamma^2$*) and* $T(\varepsilon)\approx 2\pi\varepsilon^{-1}$.

In fact these are the same periodic solutions that we found in the planar problem in section 3. What is new is the characteristic multipliers and the parametric stability. Theorem 4.1 shows that one can take any perturbation of the rotating spatial Kepler problem and establish the existence and linear stability of the periodic solutions. So, regardless of the higher-order terms, the theory can be applied to other types of spatial Hamiltonians that can be cast in the form (4.5), where the terms of order ε^5 do not necessarily correspond to the restricted problem. To our knowledge this result is new.

If the problem has an additional discrete symmetry as in the restricted three-body problem, then the methods of [35, 40] can be used to establish the existence of inclined periodic solutions. See, for example, the arguments of subsection 2.4 in [65].

4.3. KAM 3-tori in the comet spatially restricted three-body problem.

In the spatially restricted three-body problem, the higher-order terms of (4.5) start at ε^7. Thence, we begin by computing the normalized Hamiltonian, that is, the average of (4.5) with respect to ν up to terms factorized by ε^7. By means of an averaging procedure based on Lie transformations [14], the resulting Hamiltonian is

$$\mathcal{H}_\varepsilon=-K+\varepsilon^3\left(\frac{1}{2}\Big(R^2+\frac{\Theta}{r^2}\Big)-\frac{1}{r}\right)-\varepsilon^7\frac{(1-\mu)\mu}{8\Theta^2r^3}\Big(\Theta^2-3K^2-3(\Theta^2-K^2)\cos(2\vartheta)\Big)+O(\varepsilon^{11}). \tag{4.15}$$

We remark that, although in the unnormalized Hamiltonian there are nonnull terms factorized by ε^9, after normalization over ν, the terms that follow those factorized by ε^7 appear at the eleventh power of ε.

Now we apply the change of coordinates (4.9) to the Hamiltonian $\mathcal{H}_\varepsilon$, dropping its first term and the remainder $O(\varepsilon^{11})$ (but we will incorporate them later), dividing by ε^3, and replacing K by $-\gamma$; we end up with the Hamiltonian

$$\begin{aligned}\bar{\mathcal{H}}_\varepsilon = &\frac{P_2^2}{2} + \frac{(P_1^2+Q_1^2 \mp 2\gamma)^2}{8Q_2^2} - \frac{1}{Q_2}\\ &- \varepsilon^4(1-\mu)\mu\frac{P_1^4 - P_1^2Q_1^2 - 2Q_1^4 \mp 4\gamma(P_1^2 - 2Q_1^2 \mp \gamma)}{4Q_2^3(P_1^2+Q_1^2 \mp 2\gamma)^2}.\end{aligned} \tag{4.16}$$

We apply the linear change (4.13) to the Hamiltonian $\bar{\mathcal{H}}_\varepsilon$. This time, since we are taking into account the terms of order seven in ε, in the Q_i and P_i coordinates the equilibrium point is no longer $(0,\gamma^2,0,0)$. Indeed, it is given by

$$(Q_1^0,\ Q_2^0,\ P_1^0,\ P_2^0) = \left(0,\ \frac{1}{2}\left(\gamma^2+\sqrt{\gamma^4-3\varepsilon^4\mu(1-\mu)}\right),\ 0,\ 0\right).$$

The above means that when incorporating the terms of order ε^4 in (4.16), the estimate of the radii of the near-circular near-coplanar solutions provided by Theorem 4.1 is slightly improved: the solutions remain circular and coplanar with radii $(\gamma^2+\sqrt{\gamma^4-3\varepsilon^4\mu(1-\mu)})/2$.

As before, we shift the origin to the equilibrium and scale variables with the multiplier ε^{-2}. The resulting Hamiltonian is expanded in powers of ε. After dropping constant terms, we get

$$\bar{\mathcal{H}}_\varepsilon = \sum_{j=2}^{9} \varepsilon^{j-2}\bar{\mathcal{H}}_j + O(\varepsilon^8), \tag{4.17}$$

where each $\bar{\mathcal{H}}_j$ is a homogeneous polynomial in $\bar{P}_i$'s and $\bar{Q}_i$'s of degree j. In particular,

$$\bar{\mathcal{H}}_2 = \mp\frac{2\gamma^4+3\varepsilon^4\mu(1-\mu)}{4\gamma^7}\bar{P}_1^2 \mp \frac{\gamma^4+3\varepsilon^4\mu(1-\mu)}{2\gamma^7}\bar{Q}_1^2 + \frac{1}{2}\bar{P}_2^2 + \frac{2\gamma^4+3\varepsilon^4\mu(1-\mu)}{4\gamma^{10}}\bar{Q}_2^2, \tag{4.18}$$

and we do not write down explicitly the higher-order terms that depend on γ. Additionally, $\bar{\mathcal{H}}_j$, $j=3,4,5$, include terms factorized by ε^4. We remark that if we drop in $\bar{\mathcal{H}}_2$ the terms factorized by ε^4, the resulting Hamiltonian is the same as that of (4.14); thus the inclusion of the terms $O(\varepsilon^7)$ in (4.15) (or equivalently, the inclusion of terms $O(\varepsilon^4)$ in (4.16)) makes sure that $\bar{\mathcal{H}}_2$ is no longer in $1:1$ resonance.

The next step is the passage to action-angle coordinates and the removal of the angles. This can be achieved by introducing the symplectic transformation

$$\begin{aligned}\bar{Q}_1 &= 2^{1/4}\sqrt{\frac{\sqrt{2\gamma^4+3\varepsilon^4(1-\mu)\mu}}{\sqrt{\gamma^4+3\varepsilon^4(1-\mu)\mu}}\,I_1}\sin\varphi_1,\\ \bar{Q}_2 &= 2^{3/4}(\mp\gamma)^{5/2}\sqrt{\frac{I_2}{\sqrt{2\gamma^4+3\varepsilon^4(1-\mu)\mu}}}\sin\varphi_2,\\ \bar{P}_1 &= 2^{3/4}\sqrt{\frac{\sqrt{\gamma^4+3\varepsilon^4(1-\mu)\mu}}{\sqrt{2\gamma^4+3\varepsilon^4(1-\mu)\mu}}\,I_1}\cos\varphi_1,\\ \bar{P}_2 &= \frac{2^{1/4}}{(\mp\gamma)^{5/2}}\sqrt{\sqrt{2\gamma^4+3\varepsilon^4(1-\mu)\mu}\,I_2}\cos\varphi_2,\end{aligned} \tag{4.19}$$

which converts $\bar{\mathcal{H}}_2$ into

$$\bar{\mathcal{H}}_2 = \mp\frac{4\gamma^4 + 9\varepsilon^4(1-\mu)\mu}{4\gamma^7} I_1 \mp \frac{4\gamma^4 + 3\varepsilon^4(1-\mu)\mu}{4\gamma^7} I_2.$$

The Hamiltonians $\bar{\mathcal{H}}_i$ with $i > 2$ are finite Fourier series in φ_1 and φ_2 whose coefficients are polynomials in $\sqrt{I_1}$ and $\sqrt{I_2}$.

We need to construct a normal form with the aim of eliminating the angles φ_1 and φ_2. This is performed through Lie transformations [14]. It takes seven steps in the Lie transformation to include the terms factorized by ε^7 in (4.17). In all the intermediate steps one needs to expand all the resulting expressions up to powers ε^7, then drop higher-order terms. We do not print the explicit expressions of the generating functions as they are very large finite Fourier series in φ_1 and φ_2, but the transformed Hamiltonian yields

$$\begin{aligned}\bar{\mathcal{H}}_\varepsilon = &\mp\frac{1}{\gamma^3}(I_1+I_2) - \frac{3\varepsilon^2}{2\gamma^4}(I_1+I_2)^2 \mp \frac{\varepsilon^4}{4\gamma^7}\Big(3(1-\mu)\mu(3I_1+I_2) + 8\gamma^2(I_1+I_2)^3\Big)\\ &- \frac{\varepsilon^6}{8\gamma^8}\Big(3(1-\mu)\mu(29I_1^2 + 20I_1I_2 + 4I_2^2) + 20\gamma^2(I_1+I_2)^4\Big) + O(\varepsilon^8).\end{aligned} \tag{4.20}$$

The normal form computed above does not substantially modify the higher-order terms in the sense that terms of order $O(\varepsilon^8)$ get converted into terms of the same order. Thus, after rescaling the actions by means of $I_i \to \varepsilon^{-2} I_i$ for $i = 1, 2$, multiplying $\bar{\mathcal{H}}_\varepsilon$ by ε^3, and dividing it by the multiplier ε^{-2}, incorporating the unperturbed part of the initial Hamiltonian and grouping terms factorizing by powers of ε, we arrive at

$$\begin{aligned}\mathcal{H}_\varepsilon = &-K - \frac{\varepsilon^3}{2K^6}(I_1+I_2)\Big(5(I_1+I_2)^3 \mp 4K(I_1+I_2)^2 + 3K^2(I_1+I_2) \mp 2K^3\Big)\\ &- \frac{3\varepsilon^7(1-\mu)\mu}{8K^8}\Big(29I_1^2 + 20I_1I_2 + 4I_2^2 \mp 2K(3I_1+I_2)\Big) + O(\varepsilon^{11}).\end{aligned} \tag{4.21}$$

The dependence of $\mathcal{H}_\varepsilon$ on the angles ν, φ_1, and φ_2 occurs for the first time at terms of the order $O(\varepsilon^{11})$. This Hamiltonian is valid in the neighborhood of the relative equilibrium (4.8).

The estimates of the solutions' periods and the characteristic multipliers of Theorem 4.1 can be improved if one uses the Hamiltonian (4.21), including the terms of order ε^7, but we do not compute them explicitly.

Another look at (4.21) shows that if the terms $O(\varepsilon^7)$ are dropped from the Hamiltonian, we end up with an analysis that would only involve terms of up to power three in ε, manifesting the degeneracy in the actions. That is to say, the corresponding determinant of the Hessian with respect to I_1 and I_2 is zero, giving the same conclusion as stated in the previous subsection. So, the terms factorized by ε^7 have to be computed explicitly in order to check the nondegeneracy condition needed to establish the existence of KAM tori.

However, due to the fact that the first two perturbations of $-K$ appear scaled at order three and seven (multiple scales) and the third order is degenerate, when applying the iso-energetic KAM theorem, the twist term is too high and one cannot conclude the existence of the KAM tori straightforwardly. The situation is similar to the Lunar case of the restricted three-body problem [62, 65]. For these problems, Han, Li, and Yi [29] have elaborated a theorem that works in the case of Hamiltonian systems with high-order proper degeneracy [29]. At this point we should also mention Féjoz's papers [25, 26] because they contain KAM theorems that look for many of the degenerate problems investigated here. However, we have preferred to use the main theorem of [29] as it can be applied directly to the cases we handle in this paper. We make use of it in order to prove the occurrence of the invariant 3-tori.

We state the result of Han, Li, and Yi in order to make it clear that our system satisfies the required hypotheses.

We start with a Hamiltonian system of the form

$$\mathcal{H}(I,\varphi,\varepsilon) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \cdots + \varepsilon^{m_a} h_a(I^{n_a}) + \varepsilon^{m_a+1} p(I,\varphi,\varepsilon), \tag{4.22}$$

where $(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables with the standard symplectic structure $dI \wedge d\varphi$, and $\varepsilon > 0$ is a sufficiently small parameter. The Hamiltonian $\mathcal{H}$ is real analytic, and the parameters a, m, n_i $(i = 0, 1, \ldots, a)$ and m_j $(j = 1, 2, \ldots, a)$ are positive integers satisfying $n_0 \leq n_1 \leq \cdots \leq n_a = n$, $m_1 \leq m_2 \leq \cdots \leq m_a = m$, $I^{n_i} = (I_1, \ldots, I_{n_i})$, for $i = 1, 2, \ldots, a$, and p depends on ε smoothly.

The Hamiltonian $\mathcal{H}(I,\varphi,\varepsilon)$ is considered in a bounded closed region $Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$. For each ε the integrable part of $\mathcal{H}$,

$$X_\varepsilon(I) = h_0(I^{n_0}) + \varepsilon^{m_1} h_1(I^{n_1}) + \cdots + \varepsilon^{m_a} h_a(I^{n_a}),$$

admits a family of invariant n-tori $T^\varepsilon_\zeta = \{\zeta\} \times \mathbb{T}^n$, with linear flows $\{x_0 + \omega^\varepsilon(\zeta)t\}$, where, for each $\zeta \in Z$, $\omega^\varepsilon(\zeta) = \nabla X_\varepsilon(\zeta)$ is the frequency vector of the n-torus T^ε_ζ and ∇ is the gradient operator. When $\omega^\varepsilon(\zeta)$ is nonresonant, the n-torus T^ε_ζ becomes quasi-periodic with slow and fast frequencies of different scales. We refer to the integrable part X_ε and its associated tori $\{T^\varepsilon_\zeta\}$ as the intermediate Hamiltonian and intermediate tori, respectively.

Let $\bar{I}^{n_i} = (I^{n_{i-1}+1}, \ldots, I^{n_i})$, $i = 0, 1, \ldots, a$ (where $n_{-1} = 0$, hence $\bar{I}^{n_0} = I^{n_0}$), and define

$$\Omega = \Big(\nabla_{\bar{I}^{n_0}} h_0(I^{n_0}), \ldots, \nabla_{\bar{I}^{n_a}} h_{n_a}(I^{n_a})\Big)$$

such that, for each $i = 0, 1, \ldots, a$, $\nabla_{\bar{I}^{n_i}}$ denotes the gradient with respect to $\bar{I}^{n_i}$.

We assume the following high-order degeneracy-removing condition of Bruno–Rüssmann type (so named by Han, Li, and Yi, giving credit to Bruno [8] and Rüssmann [59, 60], who provided weak conditions on the frequencies guaranteeing the persistence of the invariant tori, although V. I. Arnold seems to have been the first who pointed out the importance of this type of nondegeneracy condition), the so-called condition (A): there is a positive integer s such that

$$\text{Rank}\Big\{\partial_I^\alpha \Omega(I) \;:\; 0 \leq |\alpha| \leq s\Big\} = n \qquad \forall\, I \in Z.$$

For the usual case of a nearly integrable Hamiltonian system of the type

$$\mathcal{H}(I,\varphi,\varepsilon) = X(I) + \varepsilon p(I,\varphi,\varepsilon), \quad (I,\varphi) \in Z \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n, \tag{4.23}$$

condition (A) given above generalizes the classical Kolmogorov nondegenerate condition that $\partial\omega(I)$ be nonsingular over Z, where $\omega(I) = \nabla X(I)$; Bruno's nondegenerate condition that $\text{Rank}\,\{\omega(I), \partial\omega(I)\} = n \;\forall\, I \in Z$; and the weakest nondegenerate condition guaranteeing such persistence, provided by Rüssmann, that $\omega(Z)$ should not lie in any $(n-1)$-dimensional subspace. The Rüssmann condition is equivalent to condition (A) for systems like (4.23); see [29] for details. However, Bruno or Rüssmann conditions do not apply to Hamiltonian (4.22), as it is too degenerate.

The following theorem gives the right setting in which one can ensure the persistence of KAM tori for Hamiltonians like (4.22).

Theorem 4.2 (Han, Li, and Yi [29]). *Assume the condition* (A), *and let* δ *with* $0 < \delta < 1/5$ *be given. Then there exists an* $\varepsilon_0 > 0$ *and a family of Cantor sets* $Z_\varepsilon \subset Z$, $0 < \varepsilon < \varepsilon_0$, *with* $|Z \setminus Z_\varepsilon| = O(\varepsilon^{\delta/s})$, *such that each* $\zeta \in Z_\varepsilon$ *corresponds to a real analytic, invariant, quasi-periodic* n*-torus* $\bar{T}^\varepsilon_\zeta$ *of the Hamiltonian* (4.22), *which is slightly deformed from the intermediate* n*-torus* T^ε_ζ. *Moreover, the family* $\{\bar{T}^\varepsilon_\zeta : \zeta \in Z_\varepsilon,\ 0 < \varepsilon < \varepsilon_0\}$ *varies Whitney smoothly.*

We have been assured by the authors of [29] that the above theorem can be applied to Hamiltonian systems with finite smoothness using standard arguments of KAM theory; thus we can use it for the examples of this subsection.

At this point we can apply the above result to our Hamiltonian (4.21). We have the following numbers: $n_0 = 1$, $n_1 = n_2 = 3$, $a = 2$, $m_1 = 3$, $m_2 = m = 7$. Moreover, $h_0 = -K$, h_1 is composed of the terms of $\mathcal{H}_\varepsilon$ factorized by ε^3, whereas the terms of h_2 are given by the terms of $\mathcal{H}_\varepsilon$ factorized by ε^7. Then,

$$\Omega \equiv (\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4,\ \Omega_5) = \left(\frac{\partial h_0}{\partial K},\ \frac{\partial h_1}{\partial I_1},\ \frac{\partial h_1}{\partial I_2},\ \frac{\partial h_2}{\partial I_1},\ \frac{\partial h_2}{\partial I_2}\right).$$

Now, we form the matrix

$$\begin{bmatrix} \Omega_1 & \dfrac{\partial \Omega_1}{\partial K} & \dfrac{\partial \Omega_1}{\partial K} \\ \Omega_2 & \dfrac{\partial \Omega_2}{\partial I_1} & \dfrac{\partial \Omega_2}{\partial I_2} \\ \Omega_3 & \dfrac{\partial \Omega_3}{\partial I_1} & \dfrac{\partial \Omega_3}{\partial I_2} \\ \Omega_4 & \dfrac{\partial \Omega_4}{\partial I_1} & \dfrac{\partial \Omega_4}{\partial I_2} \\ \Omega_5 & \dfrac{\partial \Omega_5}{\partial I_1} & \dfrac{\partial \Omega_5}{\partial I_2} \end{bmatrix}.$$

After replacing the concrete values of the Hamiltonians, h_i, and the partial derivatives in the above matrix, using (4.21), we deduce that its rank is three. This readily implies that there are KAM 3-tori related with the equilibrium point (4.8).

Setting $b = \sum_{i=1}^{a} m_i(n_i - n_{i-1})$, we obtain $b = 6$. Further, we have got $s = 1$, and thus $\varepsilon^{sb+\delta} = \varepsilon^{6+\delta} < \varepsilon^7$. According to Remark (2) of [29] (p. 1422), the excluded measure for the existence of quasi-periodic invariant tori is improved to an order of ε^b, that is, to ε^6.

Thus, we have proved the following result that, so far, was not known.

Theorem 4.3. *The Hamiltonian of the spatial restricted three-body problem has invariant KAM* 3*-tori surrounding the near-circular near-coplanar periodic solutions of very large radii encountered in subsection* 4.2. *If* ε *is the small parameter introduced in* (4.5) *to measure the length of the periodic solutions (i.e.,* $\|x\| \approx \varepsilon^{-2}K^2$ *or its improved value* $\|x\| \approx \varepsilon^{-2}(K^2 +$

$\sqrt{K^4 - 3\varepsilon^4\mu(1-\mu)})/2)$, *the excluding measure for the existence of invariant tori is of the order of* ε^6.

4.4. KAM 3-tori in other spatial problems. In this subsection we present two examples of Hamiltonian systems with three degrees of freedom that can be cast in the form (4.5), and therefore we can apply the same techniques we have introduced in this section to prove the existence of periodic solutions and invariant tori.

As mentioned in the introduction, Hamiltonians of type (4.5) appear in many situations. Apart from the spatially restricted three-body problem, one also has the attitude of a non-spherical spacecraft [22], the trapping mechanisms of hydrogen atoms in crossed or parallel electric and magnetic fields [18, 63], the artificial satellite with tesseral harmonics [53], the radiation pressure problem (or orbiting dust) [17], and the motion of a particle subject to the gravity potential of two orthogonal rotating straight lines [5]. The last two will be treated in this subsection.

4.4.1. Radiation pressure. The orbiting dust problem models the effect of radiation pressure on dust particles revolving around an idealized planet itself in a planar circular orbit around a star. The origin is set at the center of mass of the star-planet pair. The x_1x_2-plane is identified with the planet's orbital plane, and the mean motion of the planet around its star is normalized to one.

The Hamiltonian of the problem was derived in [17] (see also [19] for a different but related treatment). In a synodic frame that rotates with the dust particle around the planet, in such a way that the dust particle lies on the axis x_1, the Hamiltonian is a conservative system and is given by

$$\mathcal{H} = \frac{\|y\|^2}{2} - \frac{1}{\|x\|} - (x_1y_2 - x_2y_1) + kx_1, \tag{4.24}$$

where $k > 0$ is a parameter designating the strength of the radiation. The introduction of the Coriolis term is the price paid for making the system autonomous. An illustration of the problem is given in Figure 4.

The expression of the Hamiltonian is equivalent to the so-called microwave ionization problem [34, 66], in which a hydrogen atom is perturbed by a circularly polarized microwave field with the electron orbit lying in the plane of polarization. This system is described in a co-rotating frame by the autonomous Hamiltonian (4.24).

We assume that the particle is far away from the planet to see "comet"-type behavior. We also assume that the strength provoked by the radiation pressure is small, so that we define $k = \varepsilon^3 f$. Use the same scaling as in the spatially restricted three-body problem in the comet case, i.e., scale by $x \to \varepsilon^{-2}x$, $y \to \varepsilon y$ and multiply $\mathcal{H}$ by the multiplier ε. Rearranging terms in powers of ε, we get

$$\mathcal{H}_\varepsilon = -(x_1y_2 - x_2y_1) + \varepsilon^2 f x_1 + \varepsilon^3 \left(\frac{\|y\|^2}{2} - \frac{1}{\|x\|} \right). \tag{4.25}$$

Note that the resulting Hamiltonian corresponds to the class of systems that are perturbations of the Coriolis term. However, it is not in the same form as (4.5) since the perturbing term $\varepsilon^2 f x_1$ is in between.

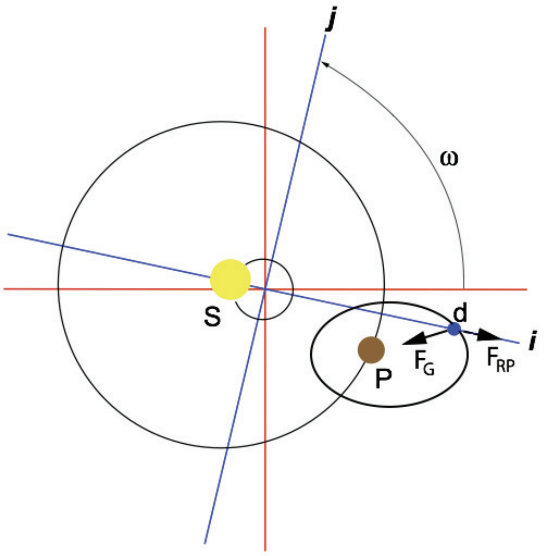

Figure 4. *Orbiting dust problem. The points S, P, and d stand respectively for the star, the planet, and the dust particle. The force F_{RP} refers to the radiation pressure of the star acting on the particle, whereas F_G refers to the gravitational force of the planet on the particle. The angular velocity ω is taken as equal to one. While the motions of the star and the planet occur in the same plane, d is allowed to move in the three-dimensional space around P in a "perturbed" ellipse.*

We normalize $\mathcal{H}_\varepsilon$ with respect to ν, arriving at order ε^{16} through a Lie transformation. The averaged Hamiltonian reads as

$$\begin{aligned}\mathcal{H}_\varepsilon = &-K + \varepsilon^3\left(\frac{1}{2}\Big(R^2 + \frac{\Theta}{r^2}\Big) - \frac{1}{r}\right) + \frac{\varepsilon^7 f^2}{2}\\ &+ \varepsilon^{13}\frac{f^2}{8\Theta^2 r^3}\Big(\Theta^2 - 3K^2 - 3(\Theta^2 - K^2)\cos(2\vartheta)\Big) + O(\varepsilon^{17}).\end{aligned} \tag{4.26}$$

A glimpse at $\mathcal{H}_\varepsilon$ reveals now that the averaged Hamiltonian is of the required format, the same as in the comet case of the spatially restricted three-body problem, since the term of order ε^7 can be dropped. Thus, after removing the constants and dropping higher-order terms, the Hamiltonian vector field corresponding to (4.26) written in the invariants a_i has a unique relative equilibrium in $\mathcal{T}_\gamma$ related to circular equatorial motions. As a consequence, the theory of [65] applies, and Theorem 4.1 also holds, leading to the existence of two families of elliptic near-circular near-equatorial periodic solutions with very large radii and very large periods.

The next goal is the analysis of the invariant KAM tori surrounding the periodic solutions. We can apply mutatis mutandis the same steps as in subsection 4.3, although with some slight variations. We do not need to give the full details. For instance, the linear transformation (4.13) is replaced by

$$Q_1 = \varepsilon^2\bar{Q}_1 + Q_1^0, \quad Q_2 = \varepsilon^2\bar{Q}_2 + Q_2^0, \quad P_1 = \varepsilon^2\bar{P}_1 + P_1^0, \quad P_2 = \varepsilon^2\bar{P}_2 + P_2^0, \tag{4.27}$$

where

$$(Q_1^0,\ Q_2^0,\ P_1^0,\ P_2^0) = \left(0,\ \frac{1}{2}(\gamma^2 + \sqrt{\gamma^4 - 3\varepsilon^{10} f^2}),\ 0,\ 0\right).$$

First of all, the Hamiltonian (4.26) is written in terms of Q_i's and P_i's. Then, using (4.27), the Hamiltonian is expressed as a function of $\bar{Q}_i$'s and $\bar{P}_i$'s. Next, the corresponding Hamiltonian is expanded in powers of ε up to the power sixteen, and the Hamiltonian in the coordinates $\bar{Q}_i$ and $\bar{P}_i$ is formed by the sum of homogeneous polynomials up to degree nine in $\bar{Q}_i$'s and $\bar{P}_i$'s. Then we use action-angle coordinates in order to prepare the Hamiltonian with the aim of eliminating the angles φ_1 and φ_2 through a Lie transformation, in a fashion similar to that given in subsection 4.3. The Lie transformation is carried out to order eight; that is, eight steps are performed. This time in the intermediate process all the expansions are truncated at order ε^{17}.

After rearranging the final Hamiltonian where we have rescaled conveniently, we get

$$\begin{aligned}\mathcal{H}_\varepsilon = &-K - \frac{\varepsilon^3}{2K^6}(I_1 + I_2)\Big(5(I_1 + I_2)^3 \mp 4K(I_1 + I_2)^2 + 3K^2(I_1 + I_2) \mp 2K^3\Big) \\ &- \frac{3\varepsilon^{13} f^2}{8K^8}\Big(29I_1^2 + 20I_1I_2 + 4I_2^2 \mp 2K(3I_1 + I_2)\Big) + O(\varepsilon^{17}).\end{aligned} \tag{4.28}$$

The Hamiltonian (4.28) is the same as (4.21) except for a few constants and also for the fact that the third term of $\mathcal{H}_\varepsilon$ is of order thirteen in ε instead of order seven.

The application of Han, Li, and Yi's theorem is almost the same as that of subsection 4.3 but with $m_2 = m = 13$. However, $b = 6$ and $s = 1$; thus the estimate of the excluding measure for the existence of quasi-periodic invariant tori is the same as in the comet case.

We notice that, due to the expression of P_1^0, the radii of the periodic solutions are slightly modified when one includes the terms factorized by ε^{13}. The periods and characteristic multipliers can also be improved, but we do not write down the explicit formulae for them. We have obtained the following result.

Theorem 4.4. *The Hamiltonian of the radiation pressure* (4.24) *has two families of near-circular near-equatorial periodic solutions that are elliptic with characteristic multipliers* 1, 1, $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^{13})$, $1+\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^{13})$, $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^{13})$, *and* $1-\varepsilon^3\gamma^{-3}T\imath+O(\varepsilon^{13})$, *where the periods are* $T(\varepsilon) \approx 2\pi\varepsilon^{-1}$. *If* $\eta = (\gamma^2 + \sqrt{\gamma^4 - 3\varepsilon^{10} f^2})/2$, *the radii of these solutions are very large,* $\|x\| \approx \varepsilon^{-2}\eta$.

Moreover, there are invariant KAM 3*-tori surrounding the near-circular near-equatorial periodic solutions of very large radii, and if* ε *is the small parameter introduced in* (4.5) *to measure the length of the periodic solutions, the excluding measure for the existence of the invariant tori is of the order of* ε^6.

4.4.2. A rotating double material segment. In order to approximate the gravity field of irregular nonspherical celestial bodies, Bartczak and Breiter [5] introduced a model that consists of two perpendicular segments that can have different lengths and masses. It extends the case of the potential produced by a finite straight line [57, 58]. By irregular bodies we mean cigar-shaped or ellipsoid-resembling objects.

We consider two segments of lengths l_1 and l_2 and masses m_1 and m_2. The segment of length l_1 is oriented along the axis x_1, while the one of length l_2 is oriented along the axis x_2.

We assume that the double segment rotates uniformly with constant angular velocity $\omega = 1$ about an axis perpendicular to the x_1x_2-plane, that is, about the (fixed) axis x_3. This is a natural assumption since the celestial body mimicked by the perpendicular straight segments is supposed to rotate. We fix the center of mass O (the intersection of the two segments) at the origin of the frame. Then $Ox_1x_2x_3$ is a synodic frame, and the x_1x_2-plane is fixed in it. A sketch of the system is given in Figure 5.

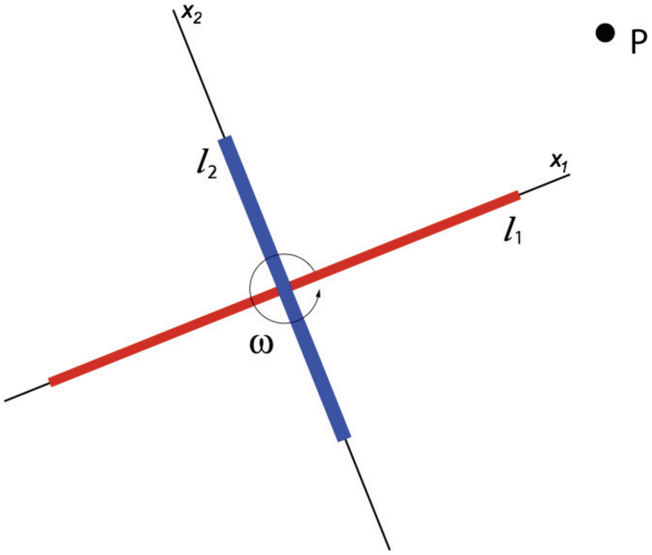

Figure 5. *Double material segment in rotation. We set the angular velocity ω to 1. It is supposed that the two segments remain in the x_1x_2-plane, whereas P moves in three-dimensional space.*

In this frame, the potential of the double segment is given by

$$V = -\frac{\mu_1}{2l_1}\log\left(\frac{s_1+2l_1}{s_1-2l_1}\right) - \frac{\mu_2}{2l_2}\log\left(\frac{s_2+2l_2}{s_2-2l_2}\right), \tag{4.29}$$

where

$$s_1 = \sqrt{\|x\|^2 + 2l_1x_1 + l_1^2} + \sqrt{\|x\|^2 - 2l_1x_1 + l_1^2},$$

$$s_2 = \sqrt{\|x\|^2 + 2l_2x_2 + l_2^2} + \sqrt{\|x\|^2 - 2l_2x_2 + l_2^2},$$

and $\mu_1 = \Gamma^2 m_1$, $\mu_2 = \Gamma^2 m_2$ with Γ the universal gravitational constant that can be normalized to one. Moreover, $m = m_1 + m_2$ is supposed to be the total mass of the celestial body approximated by the double segment, and we can rescale it to be 1. Finally, the center of mass is taken as the intersection of the two straight lines.

The interest of this problem lies in the mission analysis of spacecraft to very irregular comets and asteroids and the question of how to efficiently approximate their potentials. After defining appropriate momenta y associated with x, the Hamiltonian of a massless particle (typically the spacecraft) attracted by the double segment is

$$\mathcal{H} = \frac{\|y\|^2}{2} - (x_1y_2 - x_2y_1) + V, \tag{4.30}$$

with V given by (4.29).

If we assume that the massless particle is very far away from the segment, we use the same scale as used in the comet case of the restricted three-body problem; i.e., we scale by $x \to \varepsilon^{-2}x$, $y \to \varepsilon y$ and multiply $\mathcal{H}$ by the multiplier ε. Expanding $\mathcal{H}$ in powers of ε, we arrive at a Hamiltonian that reproduces the same pattern as (4.5). Hence, a first consequence of the dynamics of Hamiltonian (4.30) is that there is a unique relative equilibrium of the averaged Hamiltonian in the reduced space $\mathcal{T}_\gamma$ related to circular equatorial trajectories. Specifically, we can apply the theory of [65] and Theorem 4.1 holds, leading to the existence of two families of elliptic near-circular near-coplanar periodic solutions with very large radii and periods.

Next we try to obtain KAM 3-tori around the periodic solutions by applying the same steps as in subsection 4.3.

The average of $\mathcal{H}$ with respect to the argument of the node is given by

(4.31)
$$\mathcal{H}_\varepsilon = -K + \varepsilon^3 \left(\frac{1}{2}\Big(R^2 + \frac{\Theta}{r^2}\Big) - \frac{1}{r} \right) + \varepsilon^7 \frac{k}{24\Theta^2 r^3}\Big(\Theta^2 - 3K^2 - 3(\Theta^2 - K^2)\cos(2\vartheta)\Big) + O(\varepsilon^{11}),$$

with $k = l_1^2 m_1 + l_2^2(1-m_1)$. This is very similar to the averaged Hamiltonian of the spatially restricted three-body problem in the comet case and to that of the radiation pressure.

The computations that follow are very similar to those of subsection 4.3. Indeed, we need to push the calculations to the same power in ε and the same degree in the actions.

When taking into account the terms of order ε^7, the expression of the equilibrium point in the local coordinates Q_i and P_i is

$$(Q_1^0,\ Q_2^0,\ P_1^0,\ P_2^0) = \left(0,\ \frac{1}{2}(\gamma^2 + \sqrt{\gamma^4 - \varepsilon^4 k}),\ 0,\ 0\right).$$

Therefore, the radii of the periodic solutions are modified after incorporating the terms of order ε^7. The periods and characteristic multipliers of the periodic solutions are also modified, although we do not write down the new values.

The resulting Hamiltonian (compare with (4.21) and (4.28)) is

(4.32)
$$\begin{aligned}\mathcal{H}_\varepsilon = -K &- \frac{\varepsilon^3}{2K^6}(I_1+I_2)\Big(5(I_1+I_2)^3 \mp 4K(I_1+I_2)^2 + 3K^2(I_1+I_2) \mp 2K^3\Big)\\ &- \frac{\varepsilon^7 k}{8K^8}\Big(29I_1^2 + 20I_1I_2 + 4I_2^2 \mp 2K(3I_1+I_2)\Big) + O(\varepsilon^{11}).\end{aligned}$$

We apply again Han, Li, and Yi's theorem, as for the comet case, and the radiation pressure with the same values of b and s. Thus, we conclude the following.

Theorem 4.5. *The Hamiltonian of the double material segment given by* (4.30) *has two families of near-circular near-equatorial periodic solutions that are elliptic with characteristic multipliers* $1,\ 1,\ 1+\varepsilon^3\gamma^{-3}T\imath + O(\varepsilon^7),\ 1+\varepsilon^3\gamma^{-3}T\imath + O(\varepsilon^7),\ 1-\varepsilon^3\gamma^{-3}T\imath + O(\varepsilon^7)$, *and* $1 - \varepsilon^3\gamma^{-3}T\imath + O(\varepsilon^7)$, *where the periods are* $T(\varepsilon) \approx 2\pi\varepsilon^{-1}$. *If* $\eta = (\gamma^2 + \sqrt{\gamma^4 - \varepsilon^4 k})/2$, *the radii of the periodic solutions are very large,* $\|x\| \approx \varepsilon^{-2}\eta$.

Moreover, there are invariant KAM 3*-tori surrounding the near-circular near-equatorial periodic solutions of very large radii, and if* ε *is the small parameter introduced in* (4.5) *to measure the length of the periodic solutions, the excluding measure for the existence of the invariant tori is of the order of* ε^6.

5. Two Hamiltonians invariant under the symmetry induced by $\mathcal{G}$.

5.1. The isochrone model. The isochrone model was introduced to the astronomical field by Hénon in 1959 [31] as a model for globular clusters with spherical symmetry. Since then it has been used by many authors in astronomy. The isochrone was used in [21] to model the galaxy and extract conclusions concerning its formation process. They chose the isochrone to represent the motion of stars for the case where the galaxy is in dynamical equilibrium. Then, they studied the dynamics in a contracting galaxy, taking into account what properties of a stellar orbit are preserved throughout time. From these studies and the data available at the moment they concluded that the galaxy was formed by a very rapid collapse. For a few other works where the isochrone has been used in galactic dynamics, the reader is referred to [23, 13, 64].

The Hamiltonian of the problem is given by

$$\mathcal{H} = \frac{1}{2}\|y\|^2 + V, \tag{5.1}$$

where V is the potential

$$V = -\frac{1}{b + \sqrt{\|x\|^2 + b^2}} \tag{5.2}$$

and $x, y \in \mathbb{R}^3$ are conjugate coordinates. The parameter $b > 0$ is a scaling of the radius $\|x\|$ that gives the extent of the region where the potential resembles that of a homogeneous body. In the case where $b = 0$ the Kepler potential is recovered. The isochrone model has a spherical potential, and spherical potentials are very useful when modeling the motions of stars belonging to globular clusters, which are nearly spherical. If we fix a value for the energy $\mathcal{H} = h$, the radial period is given by

$$T_r = \frac{2\pi}{(-2h)^{3/2}}.$$

The isochrone is a minimally superintegrable model [6, 64]. This means that as the system is defined in three-dimensional space (six-dimensional taking into account positions and momenta), it has four functionally independent integrals of motion, which are the energy and the three components of the angular momentum.

In all spherical potentials a typical orbit is a planar rosette that does not close; i.e., it is a quasi-periodic orbit. The motion is confined to a ring determined by the pericenter and apocenter distances. For the isochrone model, the commensurability relation between the radial ω_r and the angular ω_ϕ frequencies is given by

$$\frac{\omega_\phi}{\omega_r} = \frac{1}{2}\left(1 + \frac{\Theta}{\sqrt{\Theta^2 + 4b}}\right),$$

where Θ denotes the modulus of the angular momentum vector. This relation is a rational number only for some particular values of Θ and b.

Our aim in this paper is to obtain periodic solutions for any value of $\Theta \neq 0$ and b, using the techniques developed in this paper. We treat the Hamiltonian as a three-degrees-of-freedom system, as the isochrone is usually taken as the unperturbed part of a Hamiltonian whose perturbation terms are of three degrees of freedom [13, 64].

We start by pointing out that $\mathcal{H}$ is invariant with respect to the symmetry induced by $\mathcal{G}$. In fact, $\mathcal{H}$ is invariant under the group $SO(3)$ of rotations, which implies that the angular momentum vector is kept fixed by the flow, but we are interested in the axial symmetry. This allows us to rewrite $\mathcal{H}$ in terms of the invariants a_i without resorting to averaging theory. The reduced Hamiltonian in the space $\mathcal{T}_\gamma$ reads

$$\mathcal{H} = \frac{1}{2}(a_2 + a_6^2) - \frac{1}{b + \sqrt{a_1 + a_5^2 + b^2}}. \tag{5.3}$$

The equations of motion related to this Hamiltonian have two equilibria. One is the point with coordinates $a_i = 0\ \forall\, i \in \{1, \dots, 6\}$, which is an equilibrium provided that $\gamma = 0$, and the other one is given by

$$\begin{aligned} a_1 &= \frac{\gamma}{2}\left(\gamma(\gamma^2+4b) \mp (\gamma^2+2b)\sqrt{\gamma^2+4b}\right), \\ a_2 &= \frac{2\gamma}{\gamma(\gamma^2+4b) \mp (\gamma^2+2b)\sqrt{\gamma^2+4b}}, \\ a_3 &= 0, \quad a_5 = 0, \quad a_6 = 0, \end{aligned} \tag{5.4}$$

where the upper sign is taken if $\gamma > 0$ and the lower when $\gamma < 0$. The point $a_i = 0\ \forall\, i$ corresponds with the origin of $\mathbb{R}^6$ and hence does not lead to any periodic solution, whereas the other point gets reduced to the origin of $\mathbb{R}^6$ when $\gamma = 0$. Thus, from now on we continue our analysis with $\gamma \neq 0$.

We apply Theorem 2.2 and Corollaries 2.2 and 2.3 of [65] (see also the last paragraphs of section 1) and conclude that the equilibrium (5.4) is related to a family of periodic solutions of the Hamiltonian (5.1) that depends on the parameter $\gamma \neq 0$. The periodic solutions are either prograde if $\gamma < 0$ or retrograde if $\gamma > 0$. We stress that, as we have not performed any perturbative analysis so far, the solutions are indeed truly circular and equatorial with their radii given by $\sqrt{a_1}$.

Now we study the stability of the periodic solutions through the stability analysis of the equilibrium point (5.4) in $\mathcal{T}_\gamma$. Since the periodic solutions are circular and equatorial, we begin by making use of the coordinates (4.9). The Hamiltonian $\mathcal{H}$ reads as

$$\mathcal{H} = \frac{P_2^2}{2} + \frac{(Q_1^2 + P_1^2 \mp 2\gamma)^2}{8Q_2^2} - \frac{1}{b + \sqrt{Q_2^2 + b^2}}, \tag{5.5}$$

and the equilibrium point (5.4) is

$$(Q_1^0,\ Q_2^0,\ P_1^0,\ P_2^0) = \left(0,\ \sqrt{\frac{\gamma}{2}\left(\gamma(\gamma^2+4b) \mp (\gamma^2+2b)\sqrt{\gamma^2+4b}\right)},\ 0,\ 0\right).$$

Next we shift the origin of the coordinates to the equilibrium, introducing a small parameter ε through

$$Q_1 = \varepsilon\bar{Q}_1 + Q_1^0, \quad Q_2 = \varepsilon\bar{Q}_2 + Q_2^0, \quad P_1 = \varepsilon\bar{P}_1 + P_1^0, \quad P_2 = \varepsilon\bar{P}_2 + P_2^0. \tag{5.6}$$

Introducing the change (5.6) into the Hamiltonian (5.5), expanding the result in powers of ε including terms factorized by ε^2, and dropping constant terms, we arrive at a Hamiltonian whose quadratic terms are

$$\mathcal{H}_2 = \pm\frac{2}{B(B\mp\gamma)^2}(\bar{P}_1^2+\bar{Q}_1^2)+\frac{1}{2}\bar{P}_2^2+\frac{32}{(B\mp\gamma)^6}\bar{Q}_2^2, \tag{5.7}$$

where we have used $B=\sqrt{\gamma^2+4b}$. Note that $B>|\gamma|$ as $b>0$.

The characteristic exponents of the linear Hamiltonian vector field related to (5.7) are

$$\lambda_{1,3}=\pm\frac{4\imath}{B(B\mp\gamma)^2},\quad \lambda_{2,4}=\pm\frac{8\imath}{(B\mp\gamma)^3}, \tag{5.8}$$

which implies that the point (5.4) is linearly stable in $\mathcal{T}_\gamma$. Moreover, since the $1:-1$ resonance cannot occur as it would imply that $B=|\gamma|$, we conclude that the equilibrium is parametrically stable. The linear stability analysis implies that the periodic solutions are elliptic. Indeed, we do not need to have parametric stability, as the reduction we have performed was exact; that is, it does not involve any truncation process. Furthermore, for the same reason we conclude that the nontrivial characteristic multipliers of the periodic solutions are known exactly, and their values are e^{λ_1}, e^{λ_2}, $e^{-\lambda_1}$, and $e^{-\lambda_2}$.

Finally, by inspection of the coefficients of (5.7) it is easy to see that $\mathcal{H}_2$ is positive definite when $\gamma<0$, whereas it is indefinite for $\gamma>0$. Therefore, applying the Dirichlet criterion [49] for negative values of γ, the equilibrium point is nonlinearly stable for the prograde periodic solutions. This implies that the family of periodic solutions with γ negative is also nonlinearly stable.

The periodic solutions have period $T_\phi=2\pi/\omega_\phi$. Taking that into account for the periodic solution $\Theta=|\gamma|$, and fixing a value of $\mathcal{H}$ in (5.1), say $h<0$, we get

$$T_\phi=\frac{\sqrt{2}\pi B}{(-h)^{3/2}(B\mp\gamma)}. \tag{5.9}$$

Finally we prove the existence of invariant 2-tori surrounding the periodic solutions. In general, an integrable Hamiltonian of n degrees of freedom has $2n-d$ independent integrals of motion, and the system is superintegrable if $n>d$. Thus, the phase space of an integrable Hamiltonian is fibered by invariant d-tori [51, 24, 30]. Moreover, the local representative $\mathcal{H}$ of the Hamiltonian in any local system of (generalized) action-angle coordinates depends on the d actions alone. For the isochrone problem we get $n=3$ and $d=2$, and explicit expressions of the action-angle coordinates have been found in [27] (see also [6]) after solving the corresponding Hamiltonian–Jacobi equation. In particular, a set of actions is defined through

$$I_1=\frac{1}{\sqrt{-2h}}+\frac{1}{2}\left(\Theta-\sqrt{\Theta^2+4b}\right),\quad I_2=\Theta,\quad I_3=K, \tag{5.10}$$

that, together with their conjugate angles that we do not write down, are derived with detail in [27]. In particular, when $b\to 0$ these action-angle coordinates reduce to the spatial Delaunay elements.

Then, the isochrone Hamiltonian (5.1) can be written as

$$\mathcal{H}=-\frac{2}{(2I_1-I_2+\sqrt{I_2^2+4b})^2}. \tag{5.11}$$

Now, in order to prove the existence of 2-tori, it is enough to check the usual nondegeneracy condition on Hamiltonian (5.11). Thus, since the determinant

$$\det \begin{bmatrix} \dfrac{\partial^2 \mathcal{H}}{\partial I_1^2} & \dfrac{\partial^2 \mathcal{H}}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 \mathcal{H}}{\partial I_2 \partial I_1} & \dfrac{\partial^2 \mathcal{H}}{\partial I_2} \end{bmatrix} = -\frac{768b}{(I_2^2+4b)^{3/2}(2I_1 - I_2 + \sqrt{I_2^2+4b})^7} \tag{5.12}$$

is different from zero, it is enough to ensure that there are 2-tori surrounding the periodic orbits. The frequencies of these tori are $\omega_i = \partial\mathcal{H}/\partial I_i$ for $i = 1, 2$.

Thus, we end up with the following result.

Theorem 5.1. *The Hamiltonian of the isochrone model given by* (5.1) *has two families of circular equatorial (prograde and retrograde) periodic solutions that are elliptic with characteristic multipliers* 1, 1, e^{λ_1}, e^{λ_2}, $e^{-\lambda_1}$, *and* $e^{-\lambda_2}$, *where the* λ_i*'s are defined in* (5.8). *The periodic solutions are also nonlinearly stable for* $\gamma < 0$. *The periods of the solutions are the values* T_ϕ *given in* (5.9), *and their radii are* $((\gamma^2(\gamma^2+4b) \pm \gamma(\gamma^2+2b)\sqrt{\gamma^2+4b})/2)^{1/2}$. *Moreover, there are invariant* 2*-tori surrounding the circular equatorial periodic solutions.*

We could explore the existence of periodic solutions and invariant tori by reducing out the $SO(3)$ symmetry instead of the axial symmetry. However, by doing so we end up with a Hamiltonian system of two degrees of freedom that has no isolated equilibria, a fact which prevents us from applying the techniques we have used above to get the periodic solutions together with their stability character. Alternatively, we could apply two reductions (related to the axial and spherical symmetries) to Hamiltonian (5.1) in order to obtain a system of one degree of freedom and then study it. Thus, we would get the same periodic solutions and 2-tori.

5.2. The spring pendulum. The spring pendulum, also called swing spring or elastic pendulum, is a mechanical system that exemplifies the motion of a point particle attached to a spring under a constant vertical gravitation field. The spring has one end fixed, a mass attached at the other end, and a constant vertical gravitation field acting upon it. The swing spring owes its name to the fact that, for appropriate initial conditions, the mass can either swing like a pendulum or bounce up and down like a spring. However, if in linear approximation near the equilibrium, the frequencies of the swinging and springing motion are in resonance, and the two types of motions are intricately intertwined. The long history of the model is well described in [42]; see also [33].

The system is represented through the Hamilton function

$$\mathcal{H} = \frac{1}{2}\|y\|^2 + x_3 + \frac{\kappa^2}{2}\left(1 - \frac{1}{\kappa^2} - \|x\|\right)^2. \tag{5.13}$$

The parameter κ is related to the equilibrium and unstretched lengths of the spring, respectively l and l_0, by $\kappa = l/(l - l_0)$. Thus, $\kappa > 1$ since the frequency of the spring oscillation is bigger than the frequency of the small amplitude pendulum oscillations; that is, $l \geq l_0$. Dullin, Giacobbe, and Cushman [20] performed the analysis of monodromy for the spring pendulum when $\kappa = 2$. Gutiérrez-Romero, Palacián, and Yanguas [28] showed the occurrence of Hamiltonian Hopf bifurcations when $\kappa > 1$ and $\kappa \in \mathbb{R} \setminus \mathbb{Q}$ using generalized normal forms. Furthermore, the latter also showed that the dynamical system of the swing spring has monodromy.

The Hamiltonian (5.13) is invariant under rotation abut the axis x_3; therefore $\mathcal{H}$ defines, in fact, a two-degrees-of-freedom system. Here our task is to discuss the existence of periodic solutions and KAM tori for any value of $\kappa > 1$ using reduction and KAM theories.

The reduced Hamiltonian in the invariants a_i and the reduced phase space $\mathcal{T}_\gamma$ reads

$$\mathcal{H} = \frac{1}{2}(a_2 + a_6^2) + a_5 + \frac{\kappa^2}{2}\left(1 - \frac{1}{\kappa^2} - \sqrt{a_1 + a_5^2}\right)^2. \tag{5.14}$$

The associated vector field is

$$\begin{gathered} \dot{a}_1 = 2a_3, \quad \dot{a}_2 = -\frac{2(1-\kappa^2(1-\sqrt{a_1+a_5^2}))a_3}{\sqrt{a_1+a_5^2}}, \quad \dot{a}_3 = a_2 - \frac{(1-\kappa^2(1-\sqrt{a_1+a_5^2}))a_1}{\sqrt{a_1+a_5^2}}, \\ \dot{a}_5 = a_6, \qquad \dot{a}_6 = -1 - \frac{(1-\kappa^2(1-\sqrt{a_1+a_5^2}))a_5}{\sqrt{a_1+a_5^2}}. \end{gathered} \tag{5.15}$$

The possible critical points of (5.15) have to satisfy the conditions $a_3 = a_6 = 0$; additionally, $a_5 \neq 0$, as then $\dot{a}_6 = -1$. Thus, there cannot be equatorial motions. A closer look at (5.15) yields the points (i) $a_1 = a_2 = a_3 = a_6 = 0$, $a_5 = -1$, which is valid if and only if $\gamma = 0$, and (ii) $a_1 = a_2 = a_3 = a_6 = 0$, $a_5 = 1 - 2/\kappa^2$, valid if and only if $\gamma = 0$ and $\kappa > \sqrt{2}$. Both are isolated equilibria. For both equilibria the modulus of the angular momentum, which in terms of the invariants a_i is given through

$$\Theta = \sqrt{a_1 a_6^2 + a_2 a_5^2 + a_4^2 - 2a_3 a_5 a_6},$$

is zero. Thus these points are in the singular part of the reduced space, i.e., are points on $\mathcal{T}_0$, so they are reconstructed to families of equilibrium points of the system defined by (5.13). In particular, (i) corresponds to the point $(x, y) = (0, 0, -1, 0, 0, 0)$, whereas (ii) corresponds to the family of points $(x, y) = (0, 0, 1 - 2/\kappa^2, 0, 0, 0)$ parameterized by κ. We also check that there are no other types of motions related to $\gamma = 0$. We recall that the point (i) is precisely the one used to study the characteristic feature of this system's dynamics, namely, the stepwise precession of its azimuthal angle [42, 33, 20, 28].

The search of other possible relative equilibria for $\gamma = 0$ does not give more points than those given above, so from now on we concentrate on the existence of possible relative equilibria when $|\gamma| > 0$ and $\kappa > 1$. We start by writing down a_1 and a_2 in terms of a_5, using that the a_i's are critical points of (5.15) and satisfy (4.2):

$$\begin{aligned} a_1 &= -\frac{\kappa^2 a_5^2 (a_5 + 1)(\kappa^2(a_5 - 1) + 2)}{(\kappa^2 a_5 + 1)^2}, \\ a_2 &= \frac{\kappa^2 |a_5| (a_5 + 1)(\kappa^2(a_5 - 1) + 2)(|\kappa^2 a_5 + 1| - \kappa^2 |a_5|)}{(\kappa^2 a_5 + 1)^2}. \end{aligned} \tag{5.16}$$

Imposing that $a_1 > 0$ ($a_1 = 0$ would lead to the points (i) and (ii)) and $a_2 \geq 0$, it is easy to conclude that $-1 < a_5 < \min\{0, 1 - 2/\kappa^2\}$.

At this point the computations become very cumbersome using the invariants, so we resort to local coordinates. We distinguish between near-equatorial and nonequatorial types of motions. For the former we express the Hamiltonian $\mathcal{H}$ in the coordinates (4.9), finding no solutions, which is in agreement with the fact that $a_5 < 0$. For the latter we use the set of (analytic) polar-nodal coordinates that act as rectangular coordinates, from which we will define action-angle coordinates

$$Q_1 = r, \qquad Q_2 = \vartheta, \qquad P_1 = R, \qquad P_2 = \Theta. \tag{5.17}$$

The Hamiltonian (5.13) in the coordinates Q_i, P_i is

$$\mathcal{H} = \frac{1}{2}\left(P_1^2 + \frac{P_2^2}{Q_1^2}\right) + Q_1\sqrt{1-\frac{\gamma^2}{P_2^2}}\sin Q_2 + \frac{\kappa^2}{2}\left(1-\frac{1}{\kappa^2}-Q_1\right)^2, \tag{5.18}$$

and its related vector field yields

$$\begin{aligned}
\dot{Q}_1 &= P_1,\\
\dot{Q}_2 &= \frac{P_2}{Q_1^2} + \frac{\gamma^2 Q_1}{P_2^2\sqrt{P_2^2-\gamma^2}}\sin Q_2,\\
\dot{P}_1 &= \kappa^2\left(1-\frac{1}{\kappa^2}-Q_1\right) + \frac{P_2^2}{Q_1^3} - \sqrt{1-\frac{\gamma^2}{P_2^2}}\sin Q_2,\\
\dot{P}_2 &= -\sqrt{1-\frac{\gamma^2}{P_2^2}}Q_1\cos Q_2.
\end{aligned} \tag{5.19}$$

Taking into account that P_2 cannot reach the value $\mp\gamma$ as $0 < |\gamma| < \Theta$ and that Q_1 is strictly positive, an inspection on (5.19) reveals that the equilibrium points satisfy $Q_2^0 = -\pi/2$, $P_1^0 = 0$, whereas P_2^0 is a root of the equation

$$1 - \frac{P_2^0}{\sqrt{(P_2^0)^2-\gamma^2}} - \kappa^2\left(1 - \frac{P_2^0((P_2^0)^2-\gamma^2)^{1/6}}{(\mp\gamma)^{2/3}}\right) = 0, \tag{5.20}$$

while Q_1^0 has to be taken as

$$Q_1^0 = \frac{P_2^0((P_2^0)^2-\gamma^2)^{1/6}}{(\mp\gamma)^{2/3}}. \tag{5.21}$$

In the above formulae the positive sign is valid for $\gamma < 0$ (prograde solutions), and the negative sign applies when $\gamma > 0$ (retrograde solutions). Our aim is to prove that (5.20) has a unique root P_2^0 when $P_2^0 > |\gamma| > 0$ and $\kappa > 1$; thus we will show that there is a unique relative equilibrium of the equations of motion (5.19).

Instead of seeking a value P_2^0 we write κ as a function of it. The concrete value is

$$\kappa = (\mp\gamma)^{1/3}\sqrt{\frac{P_2^0(\sqrt{(P_2^0)^2-\gamma^2}-P_2^0)+\gamma^2}{((P_2^0)^2-\gamma^2)(P_2^0((P_2^0)^2-\gamma^2)^{1/6}-(\mp\gamma)^{2/3})}}. \tag{5.22}$$

Now, from (5.22), the condition $\kappa > 1$ is equivalent to saying that

$$P_2^0 < \sqrt{\gamma^2 \mp \gamma}. \tag{5.23}$$

On the other hand, the denominator of (5.22) does not vanish if and only if

$$\mp\gamma \neq \frac{(P_2^0)^2}{\sqrt{2}}\sqrt{\sqrt{(P_2^0)^4+4}-(P_2^0)^2}. \tag{5.24}$$

Thus, both conditions (5.23) and (5.24) are satisfied, provided that

$$0 < \frac{1}{2}\left(\sqrt{4(P_2^0)^2+1}-1\right) < \mp\gamma < \frac{(P_2^0)^2}{\sqrt{2}}\sqrt{\sqrt{(P_2^0)^4+4}-(P_2^0)^2} < P_2^0. \tag{5.25}$$

The value of κ in terms of γ and P_2^0 is valid if and only if (5.25) holds; in other words, as $\kappa > 1$ is a fixed value, there is a unique P_2^0 satisfying the constraints (5.22) and (5.25). Hence, we conclude that apart from the equilibria (i) and (ii), there is a relative equilibrium point that is obtained as a root of the equations of motion (5.15) with $a_3 = a_6 = 0$, while a_1, a_2, and a_5 can be approximated numerically for concrete values of γ and κ combining (5.20), (5.21), and (5.22) with (5.15). This relative equilibrium does not bifurcate for any specific combination of κ and γ.

Further, the equilibrium point is isolated, depends on $\kappa > 1$ and $\gamma \neq 0$, and, by Theorem 2.2 and Corollaries 2.2 and 2.3 of [65] (see also the last paragraphs of section 1), leads to two families of periodic solutions, one of prograde type if $\gamma < 0$ and another of retrograde type when $\gamma > 0$. These periodic solutions are circular, their projections in the coordinate space are parallel to the x_1x_2-plane and are located below this plane, and they are usually called halo orbits. The distances between the planes where the orbits live and the equatorial plane is given by the value that a_5 takes, which is obtained using (5.16), (5.21), and (5.22), yielding

$$a_5 = -\left(\pm\gamma \mp \frac{(P_2^0)^2}{\gamma}\right)^{2/3}; \tag{5.26}$$

this value is limited to the interval $(-1, \min\{0, 1-2/\kappa^2\})$. The corresponding values of a_1 and a_2 in terms of γ and P_2^0 are

$$a_1 = (\mp\gamma)^{2/3}((P_2^0)^2-\gamma^2)^{1/3}, \qquad a_2 = \frac{(\mp\gamma)^{4/3}}{((P_2^0)^2-\gamma^2)^{1/3}}. \tag{5.27}$$

More specifically, when $\mp\gamma$ tends to the right-hand side of (5.24), the distances of the periodic solutions from the x_1x_2-plane vary in $(\max\{0, 2/\kappa^2-1\}, 1)$, while their radii (given by $\sqrt{a_1}$) are cumbersome expressions in terms of κ and γ. Additionally, the periodic solutions end up in the equilibrium point (i) as long as $\gamma \to 0$. On the other hand, when $\mp\gamma$ approaches $(\sqrt{4(P_2^0)^2+1}-1)/2$, the distances of the periodic solutions from the x_1x_2-plane tend to 1 and their radii tend to $\sqrt{\mp\gamma}$. We conclude that for each value of P_2^0 obtained from (5.20) there are two families of periodic solutions (prograde and retrograde) parameterized by γ and restricted to the constraints (5.25). The period of the solutions is 2π.

We need to manipulate the Hamiltonian (5.18) in order to analyze the stability of the periodic solutions. The origin of the coordinates is shifted to the equilibrium, introducing at the same time a small parameter ε through

$$Q_1 = \varepsilon\bar{Q}_1 + Q_1^0, \quad Q_2 = \varepsilon\bar{Q}_2 + Q_2^0, \quad P_1 = \varepsilon\bar{P}_1 + P_1^0, \quad P_2 = \varepsilon\bar{P}_2 + P_2^0, \tag{5.28}$$

where the values of Q_1^0, Q_2^0, P_1^0, and P_2^0 are those of the equilibrium point.

Next $\mathcal{H}$ is expanded in powers of ε including terms factorized by ε^2, the constant terms are dropped from the Hamiltonian, and we arrive at a Hamiltonian whose quadratic terms are

$$
\begin{aligned}
\mathcal{H}_2 &= \frac{1}{2}\bar{P}_1^2 \\
&+ \frac{(\mp\gamma)^{2/3}(3(\mp\gamma)^{8/3} - P_2^0((P_2^0)^2-\gamma^2)^{1/6}((P_2^0)^2+3\gamma^2) + (P_2^0)^2((P_2^0)^2-\gamma^2)^{2/3})}{2(P_2^0)^2((P_2^0)^2-\gamma^2)^{2/3}((\mp\gamma)^{2/3} - P_2^0((P_2^0)^2-\gamma^2)^{1/6})}\bar{Q}_1^2 \\
&+ \frac{(\mp\gamma)^{4/3}(4(P_2^0)^2-3\gamma^2)}{2(P_2^0)^2((P_2^0)^2-\gamma^2)^{4/3}}\bar{P}_2^2 + \frac{((P_2^0)^2-\gamma^2)^{2/3}}{2(\mp\gamma)^{2/3}}\bar{Q}_2^2 - \frac{3\gamma^2}{(P_2^0)^2\sqrt{(P_2^0)^2-\gamma^2}}\bar{Q}_1\bar{P}_2,
\end{aligned} \tag{5.29}
$$

where κ has be expressed in terms of P_2^0 and γ by means of (5.22).

The eigenvalues related to $\mathcal{H}_2$ are

$$
\lambda_{1,3} = \pm\sqrt{\frac{A+\sqrt{B}}{C}}, \qquad \lambda_{2,4} = \pm\sqrt{\frac{A-\sqrt{B}}{C}}, \tag{5.30}
$$

with

$$
\begin{aligned}
A &= (\mp\gamma)^{2/3}D^{7/3}(P_2^0)^4(P_2^0+D)\big(4(\mp\gamma)^{2/3} - D^{1/3}(5P_2^0 - D)\big), \\
B &= (\mp\gamma)D^{14/3}(P_2^0)^6(P_2^0+D)^2\Big(12(\mp\gamma)^{5/3}D^2 + 12(\mp\gamma)^{1/3}D^{11/3}P_2^0 + (\mp\gamma)^{1/3}D^{8/3}(P_2^0)^2 \\
&\qquad + 4(\mp\gamma)^{5/3}(P_2^0)^2 - 6(\mp\gamma)^{1/3}D^{5/3}(P_2^0)^3 \\
&\qquad + 9(\mp\gamma)^{1/3}D^{2/3}(P_2^0)^4 \\
&\qquad - 4(\mp\gamma)D^{1/3}\big(3D^3 + 3D^2P_2^0 - D(P_2^0)^2 + 3(P_2^0)^3\big)\Big), \\
C &= 2D^{11/3}(P_2^0)^4(P_2^0+D)(D^{1/3}P_2^0 - (\mp\gamma)^{2/3}),
\end{aligned} \tag{5.31}
$$

and $D = \sqrt{(P_2^0)^2 - \gamma^2} > 0$. An important feature is that for all the admissible values of γ and P_2^0 one has that $A < 0$, while B, $C > 0$. Moreover, $A^2 > B$; thence the characteristic exponents $\lambda_{1,3}$ and $\lambda_{2,4}$ are pure imaginary numbers. Indeed, it is not easy to prove this assertion, but as we know that the relative equilibrium does not bifurcate, the characteristic exponents remain purely imaginary for all possible combinations, and this is enough to ensure that A, B, C, and D do not change sign for values of P_2^0 and γ where (5.25) holds. So, we give particular values of $\kappa > 1$ and $\gamma \neq 0$, compute P_2^0 from (5.20), and check that the values of A, B, and C are as expected and, more importantly, that the corresponding signs are not going to change for all the admissible values of the parameters.

The eigenvectors of the vector field related to $\mathcal{H}_2$ form a basis of $\mathbb{R}^4$, and we can build a linear change of coordinates to bring $\mathcal{H}_2$ to diagonal form. We do not detail the computations, as they are customary but long. The final form of the quadratic Hamiltonian is

$$
\mathcal{H}_2 = \frac{\omega_1}{2}(\tilde{P}_1^2 + \tilde{Q}_1^2) + \frac{\omega_2}{2}(\tilde{P}_2^2 + \tilde{Q}_2^2), \tag{5.32}
$$

where $\omega_1 \imath = \lambda_1$, $\omega_2 \imath = \lambda_2$, and $\omega_2 > \omega_1 > 0 \; \forall \; \kappa > 1$ and $\gamma \neq 0$. Higher-order terms given by $\mathcal{H}_3, \mathcal{H}_4, \ldots$ are transformed using the same linear change, and they are factorized respectively by $\varepsilon, \varepsilon^2, \ldots$.

As a consequence, the equilibrium point $(Q_1^0, -\pi/2, 0, P_2^0)$, with Q_1^0 and P_2^0 given in (5.20) and (5.21), is linearly and parametrically stable. The linear stability means that the families of periodic solutions are elliptic. Note that as in the isochrone model, we have not used averaging theory; thus we do not really need the parametric stability to conclude the linear stability of the periodic solutions. Hence, we can say that the exact nontrivial characteristic multipliers of the periodic solutions are e^{λ_1}, e^{λ_2}, $e^{-\lambda_1}$, and $e^{-\lambda_2}$.

The Hamiltonian $\mathcal{H}_2$ is positive definite; thus the equilibrium point is always nonlinearly stable. This means that the periodic solutions are nonlinearly stable for all possible values of $\kappa > 1$ and $\gamma \neq 0$.

Finally, we remark that $\mathcal{H}_2$ can be in resonance; i.e., there can be positive integers p and q such that $q\omega_2 = p\omega_1$. In particular, when the distance of a halo orbit from the equatorial plane approaches 1, then ω_2/ω_1 tends to 2, although this distance cannot occur. However, other resonances can also take place. For instance, looking for a possible value of γ such that the resonance 3 : 2 holds, we pick $b = 2$ and solve the nonlinear system given by the two equations $3\omega_1 = 2\omega_2$ and (5.20) for the unknowns γ and P_2^0, yielding $\gamma = -0.30947808\ldots$, $P_2^0 = 0.59681220\ldots$. The nonnull invariants of the corresponding periodic solution are $a_1 = 0.29217167\ldots$, $a_2 = 0.32780962\ldots$, $a_4 = \gamma$, and $a_5 = -0.89128462\ldots$.

In order to apply KAM theory we need to introduce action-angle coordinates as follows:

$$\begin{aligned} \tilde{Q}_1 &= \sqrt{2I_1}\sin\varphi_1, \qquad & \tilde{P}_1 &= \sqrt{2I_1}\cos\varphi_1, \\ \tilde{Q}_2 &= \sqrt{2I_2}\sin\varphi_2, \qquad & \tilde{P}_2 &= \sqrt{2I_2}\cos\varphi_2. \end{aligned} \tag{5.33}$$

These transform $\mathcal{H}$ into a new Hamiltonian such that its quadratic terms are given by

$$\mathcal{H}_2 = \omega_1 I_1 + \omega_2 I_2. \tag{5.34}$$

Now, KAM theory can be applied if we exclude the resonant cases. We set $\gamma = -K$ and assume that the transformed Hamiltonian expressed in terms of the actions I_1, I_2, and K is given by $\mathcal{H}_\varepsilon = \mathcal{H}_2 + O(\varepsilon^2)$. The Hamiltonian $\mathcal{H}_2$ is degenerate in the sense of Arnold, but we apply the iso-energetic version of the KAM theorem, determining

$$\det \begin{bmatrix} \dfrac{\partial^2 \mathcal{H}_2}{\partial I_1^2} & \dfrac{\partial^2 \mathcal{H}_2}{\partial I_1 \partial I_2} & \dfrac{\partial^2 \mathcal{H}_2}{\partial I_1 \partial K} & \dfrac{\partial \mathcal{H}_2}{\partial I_1} \\ \dfrac{\partial^2 \mathcal{H}_2}{\partial I_2 \partial I_1} & \dfrac{\partial^2 \mathcal{H}_2}{\partial I_2^2} & \dfrac{\partial^2 \mathcal{H}_2}{\partial I_2 \partial K} & \dfrac{\partial \mathcal{H}_2}{\partial I_2} \\ \dfrac{\partial^2 \mathcal{H}_2}{\partial K \partial I_1} & \dfrac{\partial^2 \mathcal{H}_2}{\partial K \partial I_2} & \dfrac{\partial^2 \mathcal{H}_2}{\partial K^2} & \dfrac{\partial \mathcal{H}_2}{\partial K} \\ \dfrac{\partial \mathcal{H}_2}{\partial I_1} & \dfrac{\partial \mathcal{H}_2}{\partial I_2} & \dfrac{\partial \mathcal{H}_2}{\partial K} & 0 \end{bmatrix} = \frac{X}{Y}, \tag{5.35}$$

and one needs to study whether X/Y can be zero or not. In particular the specific values of

X and Y are

$$
\begin{aligned}
X \;=\; (\pm K)^{4/3}(P_2^0 + D)^2 \Big(& -9D^6\left((P_2^0)^2 - 10\right) + 21(\pm K)^{2/3}D^{17/3}P_2^0 - 12(\pm K)^{4/3}D^{16/3} \\
& + 18D^5P_2^0\left(7(P_2^0)^2 + 19\right) - 3(\pm K)^{2/3}D^{14/3}\left(115(P_2^0)^2 + 24\right) \\
& + 309(\pm K)^{4/3}D^{13/3}P_2^0 + 2D^4(P_2^0)^2\left(69(P_2^0)^2 + 61\right) \\
& - 2(\pm K)^{2/3}D^{11/3}P_2^0\left(235(P_2^0)^2 + 36\right) + 602(\pm K)^{4/3}D^{10/3}(P_2^0)^2 \\
& - 6D^3(P_2^0)^3\left((P_2^0)^2 + 58\right) - 12(\pm K)^{2/3}D^{8/3}(P_2^0)^2\left(5(P_2^0)^2 - 6\right) \\
& + 206(\pm K)^{4/3}D^{7/3}(P_2^0)^3 - 3D^2(P_2^0)^4\left(3(P_2^0)^2 + 70\right) \\
& + (\pm K)^{2/3}D^{5/3}(P_2^0)^3\left(25(P_2^0)^2 + 72\right) - 22(\pm K)^{4/3}D^{4/3}(P_2^0)^4 \\
& + 6D(P_2^0)^5 - 3(\pm K)^{2/3}D^{2/3}(P_2^0)^6 + 5(\pm K)^{4/3}D^{1/3}(P_2^0)^5 \\
& - 2(P_2^0)^6\Big)^2
\end{aligned}
$$

and

$$
\begin{aligned}
Y \;=\; & 36D^6\left(D^{1/3}P_2^0 - (\pm K)^{2/3}\right)^4 \\
& \times \Big(-2304(\pm K)^{2/3}D^{10} + 36D^{28/3}\left((P_2^0)^2 + 2\right) - 2304(\pm K)^{2/3}D^9P_2^0 \\
& + 3072(\pm K)^{4/3}D^{26/3} + 3D^{25/3}P_2^0\left(49(P_2^0)^2 + 120\right) - 648(\pm K)^{8/3}D^8 \\
& + 3072(\pm K)^{4/3}D^{23/3}P_2^0 + 6D^{22/3}(P_2^0)^2\left(43(P_2^0)^2 + 116\right) - 1400(\pm K)^{8/3}D^7P_2 \\
& + 6176(\pm K)^{10/3}D^{20/3} + D^{19/3}(P_2^0)^3\left(265(P_2^0)^2 + 552\right) \\
& - (\pm K)^{8/3}D^6\left(4888(P_2)^2 - 2021K^2 - 384\right) + 4640(\pm K)^{10/3}D^{17/3}P_2^0 \\
& + 4D^{16/3}(P_2^0)^4\left(45(P_2^0)^2 - 14\right) - (\pm K)^{8/3}D^5P_2^0\left(2984(P_2^0)^2 - 943K^2 - 384\right) \\
& + 4238(\pm K)^{16/3}D^{14/3} + 3D^{13/3}(P_2^0)^5\left(39(P_2^0)^2 - 152\right) \\
& - (\pm K)^{14/3}D^4\left(2021(P_2^0)^2 - 365K^2 - 480\right) + 2302(\pm K)^{16/3}D^{11/3}P_2^0 \\
& + 2(P_2^0)^6D^{10/3}\left(65(P_2^0)^2 - 236\right) - (\pm K)^{14/3}D^3P_2^0\left(943(P_2^0)^2 - 111K^2 - 288\right) \\
& + 1225(\pm K)^{22/3}D^{8/3} + D^{7/3}(P_2^0)^7\left(111(P_2^0)^2 - 392\right) \\
& - (\pm K)^{20/3}D^2\left(365(P_2^0)^2 - 144\right) + 446(\pm K)^{22/3}D^{5/3}P_2^0 \\
& + 12D^{4/3}(P_2^0)^8\left(3(P_2^0)^2 - 20\right) - 3(\pm K)^{20/3}DP_2^0\left(37(P_2^0)^2 - 16\right) \\
& + 127(\pm K)^{28/3}D^{2/3} - 64D^{1/3}(P_2^0)^9 + 12(\pm K)^{26/3}\Big).
\end{aligned}
$$

At this point it is hard to check when X/Y is properly defined and when it can be zero, although generically it does not vanish. We stress that $D = \sqrt{(P_2^0)^2 - K^2}$ and P_2^0 is a root of (5.20); thus, given κ and K, the determinant takes a unique value X/Y. We have plotted in Figure 6 the quotient X/Y for $0 < |K| < P_2^0$. The curves marked by the red arrows show the

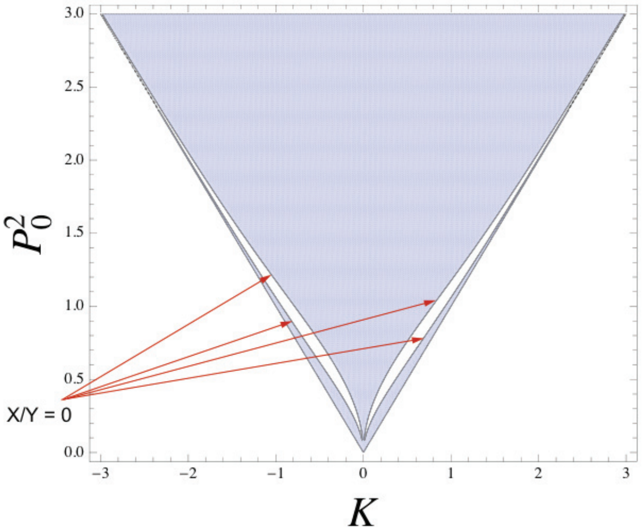

Figure 6. *The sign of the quotient X/Y in terms of K and P_2^0.*

values for which $X/Y = 0$. These curves separate the regions for which $X/Y > 0$ (in blue) and the two regions for which $X/Y < 0$ (the small white strips inside the curves $X/Y = 0$).

We conclude that, excepting the curve where $X/Y = 0$ and the combinations of κ and γ that lead to resonances, the Hamiltonian of the spring pendulum is iso-energetically nondegenerate, and there are families of KAM 3-tori surrounding the periodic solutions we have discussed above. Indeed, the invariant tori form a majority of each level of the energy. Thus, we end up with the following result, which to the best of our knowledge is new.

Theorem 5.2. *For $\kappa > 1$ and $\gamma \neq 0$, the Hamiltonian of the spring pendulum given in (5.13) has two families (prograde and retrograde) of circular periodic solutions in the phase space $T\mathbb{R}^3$ such that their projections in the coordinate space are in planes parallel to the x_1x_2-plane and below this plane. The distances of these planes to the x_1x_2-plane are given by (5.26), where a_5 is in the interval $(-1, \min\{0, 1-2\kappa^2\})$ and P_2^0 is a root of (5.20). The radii of these solutions tend to $\sqrt{\mp\gamma}$ when a_5 approaches -1 and tend to zero if a_5 tends to zero. The periods of the periodic solutions are 2π. These solutions are elliptic with characteristic multipliers 1, 1, e^{λ_1}, e^{λ_2}, $e^{-\lambda_1}$, and $e^{-\lambda_2}$, where the λ_i's are defined in (5.30). The periodic solutions remain also nonlinearly stable.*

Moreover, when the quotient X/Y of (5.35) does not vanish and the values of κ and γ (or K) do not lead to a resonant situation, there are invariant KAM 3-tori surrounding the circular halo periodic solutions, and these tori form a majority of each level of the energy. If ε is the small parameter introduced in (5.28), the excluding measure for the existence of the invariant tori is small when ε is small.

6. Concluding remarks. We study Hamiltonian systems of two and three degrees of freedom that are invariant with respect to rotation about the vertical axis. The systems can enjoy this symmetry originally, or averaging theory can be used so that after truncation of higher-order terms the resulting systems become axially symmetric.

We discuss the existence of relative equilibria and their stability on the reduced spaces which are unbounded symplectic manifolds with singular points whose dimension is either two or four. Once the flows on the reduced spaces are understood we reconstruct the dynamics of the unreduced systems, establishing the existence and stability character of the families of periodic solutions related to the relative equilibria and with the application of some KAM theorems to conclude the persistence of families of KAM 2-tori or 3-tori surrounding the families of periodic solutions.

The main features of our paper are the following:

- Reduction theory has been used to analyze the dynamics of different problems on the reduced spaces $\mathcal{R}_\gamma$ and $\mathcal{T}_\gamma$. This has been possible because the Hamiltonian functions have been written in terms of the invariants related to the axial symmetry, i.e., the polynomials a_i, which globally parameterize the reduced spaces.
- We have dealt with some problems of different types. First we treated the comet cases of the circular restricted three-body and the restricted N-body problems when the infinitesimal particle is supposed to move near infinity. We have also dealt with the radiation pressure problem and the double material segment, which can be modeled similarly to the spatial comet case of the circular restricted three-body problem. The theory has also been applied to two other problems that enjoy axial symmetry and do not need any normalization procedure, the so-called Hénon's isochrone and the spring pendulum. We have obtained new periodic solutions for all the problems considered, and KAM tori in all the cases except the isochrone problem.
- Local canonical coordinates have been used in combination with global invariants in order to deal with the stability of the relative equilibria and as a first step towards the construction of action-angle coordinates needed to apply the KAM techniques. In particular we have used Delaunay-like planar and polar coordinates for the problems of two degrees of freedom, and polar-nodal coordinates for three-degrees-of-freedom systems. These variables have been very useful as they usually lead to short expressions for the expansions around the relative equilibria.
- Three versions of KAM theorems have been applied in order to deal with the different degeneracies of the problems. In particular, we have used an iso-energetic version of the KAM theorem, another version to deal with properly degenerate Hamiltonian systems, and a recent result of Han, Li, and Yi [29] that works well in the case of Hamiltonian systems with high-order proper degeneracy. In particular for systems with three or more degrees of freedom, Han, Li, and Yi's theorem is crucial in order to achieve the existence of KAM tori when the perturbation appears at different scales.

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