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Periodic Solutions in Hamiltonian Systems, Averaging, and the Lunar Problem*

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Abstract. We investigate the existence, characteristic multipliers, and stability of periodic solutions to a Hamiltonian vector field which is a small perturbation of a vector field tangent to the fibers of a circle bundle. Our primary examples are the planar lunar and spatial lunar problems of celestial mechanics, i.e., the restricted three-body problem where the infinitesimal is close to one of the primaries. By averaging the perturbation over the fibers of the circle bundle one obtains a Hamiltonian system on the reduced (orbit) space of the circle bundle. Our goal in the first part of the paper is to state and prove results which have hypotheses on the reduced system and have conclusions about the full system. Starting with the classical work of Reeb, we give a summary of lemmas, corollaries, and theorems about the existence, characteristic multipliers, and stability of periodic solutions to Hamiltonian systems which are perturbations of circle bundle flows. By reformulating the classical results in modern language and giving alternative proofs in place of the original proofs, we are able to infer new consequences of these classical results. The second part of the paper is devoted to applications of the general results. We apply these general results to the planar and spatial lunar problem. After scaling, the lunar problem is a perturbation of the Kepler problem, which after regularization is a circle bundle flow. We find the classical near-circular periodic solutions and the near-rectilinear periodic solutions. Then we compute their approximate multipliers and show that there is a "twist." However, the twist is too degenerate to apply the classical KAM theorem on invariant tori. We also find symmetric periodic solutions which are continuations of elliptic solutions of the Kepler problem.

Key words. averaging, normalization, reduced space, N-body problem, periodic solutions, twist condition

AMS subject classifications. 34C20, 34C25, 37J40, 70F10, 70K65

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1. Introduction.

For us the lunar problem is the circular restricted three-body problem where the infinitesimal is close to one of the primaries. After scaling the restricted problem, the lunar problem is a perturbation of the Kepler problem, and Moser [39] has shown that the Kepler problem after regularization is a circle bundle flow. Thus, the lunar problem is a prototype for Hamiltonian systems that arise as perturbations of circle bundle flows.

By averaging the perturbation over the fibers of the circle bundle, Reeb [43] and Moser [39] obtained a Hamiltonian vector field on the base (or reduced) space; see also [32, 33]. They were able to give sufficient conditions for the existence of periodic solutions by looking at the system on the base alone.

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Since then a number of papers have appeared which analyze systems by looking at the reduced system only; see [7, 16, 21, 27, 28, 29, 41] and the many references therein. One starts with a small parameter which is a measure of the perturbation of an integrable system where all solutions are periodic. Then one normalizes (or averages) the perturbation term-by-term in the small parameter. After a finite number of terms have been normalized, the higher-order perturbations are truncated, giving an approximation of the full system. This approximation is well defined on the lower-dimensional reduced space. Being lower-dimensional, sometimes just two-dimensional, the system on the reduced space is easier to understand. But not all the features of the full system are accurately reflected by the reduced system; it typically does not display the breakdown of invariant tori, ergodic regions, solenoids, etc.

Our goal in the first part of the paper is to state, prove, and apply results which have hypotheses on the reduced system and have conclusions about the full system. Starting with the work of Reeb, we give in section 2 a summary of lemmas, corollaries, and theorems about the existence, characteristic multipliers, and parametric stability of periodic solutions for Hamiltonian systems which are perturbations of circle bundle flows. Some of the results are old, some are just extensions, and a few are new. Lemma 2.1 is the key to an original direct proof of Reeb's theorems using symplectic geometry arguments. Corollary 2.3 constitutes a new application of Krein–Gel'fand theory [47] about the stability of linear Hamiltonian vector fields with periodic coefficients. Theorem 2.5 connects, through Lemma 2.1, the theory of reduction in Hamiltonian systems with the existence of KAM tori in a simple way. Theorem 2.6, about the existence of symmetric periodic solutions, is also new.

The second part of the paper is devoted to applying these general results to the lunar problem. In section 3 we apply these general results to the planar lunar problem and in section 4 to the spatial lunar problem. In the planar problem we find Hill's classical near-circular periodic solutions, compute their approximate multipliers, and then show that there is a "twist" term. The twist is of too high an order in the perturbation parameter to apply the classical KAM theorem. We also find symmetric periodic solutions which are continuations of elliptic solutions of the Kepler problem.

For the spatial problem we again find the classical Hill periodic solutions, but also the near-rectilinear periodic solutions, and we compute their approximate multipliers. These solutions are shown to be parametrically stable and elliptic. Again we compute a twist term for all these periodic solutions. We pay particular attention to the near-rectilinear periodic solutions and show that they are not collision orbits.

2. Averaging theorems.

Here we summarize some general results from the classic paper by Reeb [43] on averaging Hamiltonian systems on manifolds, along with some obvious corollaries. We also extend these results to systems with discrete symmetries.

Let (M, Ω) be a symplectic manifold of dimension $2n$, $\mathcal{H}_0 : M \to \mathbb{R}$ a smooth Hamiltonian which defines a Hamiltonian vector field $Y_0 = (d\mathcal{H}_0)^\#$ with symplectic flow ϕ_0^t (see [1]). Let $\mathbb{I} \subset \mathbb{R}$ be an interval such that each $h \in \mathbb{I}$ is a regular value of $\mathcal{H}_0$ and $\mathcal{N}_0(h) = \mathcal{H}_0^{-1}(h)$ is a compact connected circle bundle over a base space $B(h)$ with projection $\pi : \mathcal{N}_0(h) \to B(h)$. Assume the vector field Y_0 is everywhere tangent to the fibers of $\mathcal{N}_0(h)$; i.e., assume that all the solutions of Y_0 in $\mathcal{N}_0(h)$ are periodic. There is no loss of generality [22] in assuming that all these periodic solutions have periods smoothly depending only on the value of the

Hamiltonian; i.e., the period is a smooth function $T = T(h)$ (sometimes the dependence on h will be omitted in the notation).

For example, consider a pair of harmonic oscillators

$$\ddot{x} + x = 0, \qquad \ddot{y} + y = 0,$$

which may be written as the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial u} = u, \qquad \dot{u} = -\frac{\partial H}{\partial x} = -x, \qquad \dot{y} = \frac{\partial H}{\partial v} = v, \qquad \dot{v} = -\frac{\partial H}{\partial y} = -y,$$

with Hamiltonian

$$H = \frac{1}{2}(x^2 + u^2) + \frac{1}{2}(y^2 + v^2).$$

In polar coordinates

$$r^2 = x^2 + u^2, \qquad \theta = \tan^{-1} u/x, \qquad \rho^2 = y^2 + v^2, \qquad \phi = \tan^{-1} v/y,$$

the equations become

$$\dot{r} = 0, \qquad \dot{\theta} = -1, \qquad \dot{\rho} = 0, \qquad \dot{\phi} = -1,$$

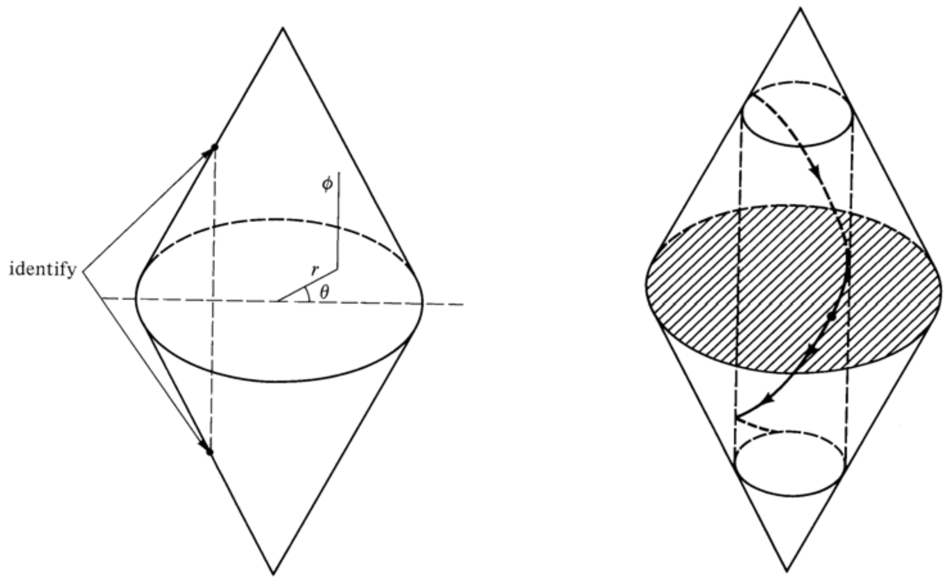
and they admit the two integrals r and ρ.

The energy level $E = H^{-1}(\frac{1}{2})$ is a 3-sphere and is invariant under the flow. All the solutions are 2π-periodic, and so the orbits are circles. Thus the 3-sphere is a union of circles. We can use polar coordinates to coordinatize the sphere provided we are careful to observe the proper conventions.

Starting with the polar coordinates r, θ, ρ, ϕ for $\mathbb{R}^4$, we note that on the 3-sphere, $E = r^2 + \rho^2 = 1$; so we may discard ρ and take $0 \le r \le 1$. We will use r, θ, ϕ as coordinates on S^3. Now r, θ with $0 \le r \le 1$ are just polar coordinates for the closed unit disk. For each point of the open disk, there is a circle with coordinate ϕ (defined mod 2π), but when $r = 1$, $\rho = 0$; so the circle collapses to a point over the boundary of the disk. The geometric model of S^3 is two solid cones with points on the boundary cones identified, as shown in Figure 1a. Through each point in the open unit disk with coordinates r, θ there is a line segment (the dashed line) perpendicular to the disk. The angular coordinate ϕ is measured on this segment, $\phi = 0$ is the disk, $\phi = \pi$ is the upper boundary cone, and $\phi = -\pi$ is the lower boundary cone. Each point on the upper boundary cone with coordinates r, θ, $\phi = \pi$ is identified with the point on the lower boundary cone with coordinates r, θ, $\phi = -\pi$.

In this model there are two special orbits where $r = 0$ and $\rho = 0$. Other than these two special circles, on each orbit, as θ increases by 2π, so does ϕ. Thus, each such orbit meets the open disk where $\phi = 0$ (the shaded disk in Figure 1b) in one point. We can identify each such orbit with the unique point where it intersects the disk. One special orbit meets the disk at the center, and so we can identify it with the center. The other is the outer boundary circle, which is a single orbit. When we identify this circle with a point, the closed disk with its outer circle identified with a point becomes a 2-sphere.

Thus, the 3*-sphere* S^3 *is the union of circles. The quotient space obtained by identifying a circle with a point is a* 2*-sphere (the Hopf fibration of* S^3*).*



(a) A model of S^3 (b) An orbit on S^3

Figure 1. *S^3 as a circle bundle over S^2.*

Let D be the open disk $\phi = 0$ (the shaded disk in Figure 1b). The union of all the orbits which meet D is a product of a circle and a 2-disk, so each point not on the special circle $r = 1$ lies in an open set that is the product of a 2-disk and a circle. By reversing r and ρ in the discussion above, the circle where $r = 1$ has a similar neighborhood. So locally the 3-sphere is the product of a disk and a circle, but the sphere is not the product of a 2-manifold and a circle (the sphere has a trivial fundamental group, but such a product would not).

In higher dimensions, consider n harmonic oscillators all with frequency 1; i.e., let $M = \mathbb{R}^{2n}$, $\mathcal{H}_0 = \frac{1}{2}\sum_1^n(x_i^2 + y_i^2)$, and $\mathcal{N} = \mathcal{H}_0^{-1}(h) = S^{2n-1}$ (the sphere of radius $\sqrt{2h}$). Then all solutions are 2π-periodic and B is the complex projective $(n-1)$-space, $\mathbb{CP}^{n-1}$. $\mathbb{CP}^1$ is homeomorphic to the 2-sphere, so when $n = 2$ the reduced space is $B = S^2$ as illustrated above.

Another example is the geodesic flow on the n-sphere S^n; i.e., $M = TS^n$ (the tangent bundle of the sphere), $\mathcal{H}_0 : M \to \mathbb{R} : v_p \mapsto |v_p|$ ($\mathcal{H}_0(v_p)$ is the length of the vector $v_p \in T_pM$), $\mathcal{N} = \{v_p \in TS^n : |v_p| = h\}$ (the h-sphere bundle), and B is $G_{2,n+1}$, the Grassmannian manifold of oriented 2-planes in $\mathbb{R}^{n+1}$ (see, for instance, [38]). If $n = 2$, then B is S^2, whereas it is $S^2 \times S^2$ when $n = 3$.

2.1. Reeb's theorems. Here we state and prove two of Reeb's theorems in more modern terminology. Our proof gives more of the Hamiltonian structure and therefore leads to further applications.

Theorem 2.1. *The base space B inherits a symplectic structure ω from (M, Ω); i.e., (B, ω) is a symplectic manifold.*

This is the original reduction theorem. Now let us look at a perturbation of this situation.

Let ε be a small parameter, $\mathcal{H}_1 : M \to \mathbb{R}$ be smooth, $\mathcal{H}_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$, $Y_\varepsilon = Y_0 + \varepsilon Y_1 = d\mathcal{H}_\varepsilon^{\#}$, $\mathcal{N}_\varepsilon(h) = \mathcal{H}_\varepsilon^{-1}(h)$, and ϕ_ε^t be the flow defined by Y_ε.

Let the average of $\mathcal{H}_1$ be

$$\bar{\mathcal{H}} = \frac{1}{T}\int_0^T \mathcal{H}_1(\phi_0^t)dt,$$

which is a smooth function on $B(h)$, and let $\bar{\phi}^t$ be the flow on $B(h)$ defined by $\bar{Y} = d\bar{\mathcal{H}}^{\#}$.

A critical point of $\bar{\mathcal{H}}$ is *nondegenerate* if the Hessian at the critical point is nonsingular, and the function $\bar{\mathcal{H}}$ is a *Morse function* if all its critical points are nondegenerate. The *index* of a nondegenerate critical point p of $\bar{\mathcal{H}}$ is the dimension of the maximal linear subspace where the Hessian of $\bar{\mathcal{H}}$ at p is negative definite.

Theorem 2.2. *If $\bar{\mathcal{H}}$ has a nondegenerate critical point at $\pi(p) = \bar{p} \in B$ with $p \in \mathcal{N}_0$, then there are smooth functions $p(\varepsilon)$ and $T(\varepsilon)$ for ε small with $p(0) = p$, $T(0) = T$, and $p(\varepsilon) \in \mathcal{N}_\varepsilon$, and the solution of Y_ε through $p(\varepsilon)$ is $T(\varepsilon)$-periodic.*

If $\bar{\mathcal{H}}$ is a Morse function, then Y_ε has at least $\chi(B)$ periodic solutions, where $\chi(B)$ is the Euler–Poincaré characteristic of B.

The proof of Theorem 2.2 yields additional corollaries. The essence of the proof of the local part of Theorem 2.2 is the existence of symplectic coordinates for a tubular neighborhood of the orbit through p. Here we give the proof of the existence of these coordinates.

Lemma 2.1. *Let $p \in \mathcal{N}_0(h)$, with $h \in \mathbb{I}$ fixed. Then there are symplectic coordinates (I, θ, y), valid in a tubular neighborhood of the periodic solution $\phi_0^t(p)$ of $Y_0(h)$, where (I, θ) are action-angle coordinates and $y \in \mathbb{N}$, where $\mathbb{N}$ is an open neighborhood of the origin in $\mathbb{R}^{2n-2}$. The point p corresponds to $(I, \theta, y) = (0, 0, 0)$.*

In these coordinates $\mathcal{H}_0$ is a function of I only; i.e., $\mathcal{H}_0 = \mathcal{H}_0(I)$. A local cross section is $\theta = \alpha$, and a local cross section in an energy level is $\theta = \alpha, I = \beta$, where α, β are constants. In addition, $y \in \mathbb{N}$ are coordinates in the cross section in the energy level.

The Hamiltonian is

$$\mathcal{H}_\varepsilon(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon\mathcal{H}_1(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon\bar{\mathcal{H}}(I, y) + O(\varepsilon^2). \tag{1}$$

Proof. By the Hamiltonian flow box theorem [36, 40] there are local symplectic coordinates $u = (u_1, \dots, u_{2n})$ for M in a neighborhood W of p such that p corresponds to $u_j = 0$, $j = 1, \dots, 2n$ (note that we locate p at the origin). The Hamiltonian is $\mathcal{H}_0 = u_{n+1}$ (so we take $h = 0$), and Y_0 is the differential equation

$$\dot{u}_1 = \partial\mathcal{H}_0/\partial u_{n+1} = 1, \qquad \dot{u}_j = 0, \quad j = 2, \dots, 2n.$$

A local cross section to the flow of Y_0 is $\Sigma = \{u : u_1 = 0\} \cap W$, and a local cross section in an energy level $\mathcal{H}^{-1}(0)$ is $\sigma = \{u : u_1 = u_{n+1} = 0\} \cap W$.

The validity of these coordinates can be extended to a tubular neighborhood $U = \{\phi_0^t(q) : q \in \Sigma,\ t \in \mathbb{R}\}$ of the Y_0-orbit through p. Let $Z = \Sigma \times \mathbb{R}$, and let $\eta : Z \to U : (q, t) \to \phi_0^t(q)$ be a symplectic map. The vector field Y_0 on U lifts to $\dot{u}_1 = 1$, $\dot{u}_j = 0$, $j = 2, \dots, 2n$, on Z.

Recall that we assume that the period T depends smoothly on the value of $\mathcal{H}_0$, which in these coordinates means that the period depends smoothly on u_{n+1}, i.e., that $T(u_{n+1})$ is smooth. Let $F(w)$ satisfy $dF/dw = -2\pi/T(w)$, and let $f = F^{-1}$. Change variables on Z from $\{u_1, u_{n+1}\}$ to $\{I, \theta\}$ by

$$I = f(u_{n+1}), \qquad \theta = -\frac{u_1}{f'(u_{n+1})}.$$

One checks that $dI \wedge d\theta = du_1 \wedge du_{n+1}$, so this is a symplectic change of variables. Since $I = f(u_{n+1})$, we have $\mathcal{H}_0 = u_{n+1} = F(I)$ and $\dot{\theta} = -F'(I) = 2\pi/T(I)$. Thus, when t increases by $T(I)$, the variable θ increases by 2π and so can be considered as an angular variable. Therefore, $(I, u_2, \ldots, u_n, \theta, u_{n+2}, \ldots, u_{2n})$ is a full set of symplectic coordinates for Z and, via η, a full set of symplectic coordinates for U. Let $v = (u_2, \ldots, u_n, u_{n+2}, \ldots, u_{2n})$. So the Hamiltonian is

$$\mathcal{H}_\varepsilon(I, \theta, v) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \theta, v). \tag{2}$$

We use the method of Lie transforms to effect the average. Let $W_1(I, \theta, v)$ be the solution of

$$\mathcal{H}_1 + \{\mathcal{H}_0, W_1\} = \mathcal{H}_1 + \frac{\partial \mathcal{H}_0}{\partial I}\frac{\partial W_1}{\partial \phi} = \bar{\mathcal{H}},$$

that is,

$$W_1 = \int^{\theta} \frac{\partial \mathcal{H}_0}{\partial I}^{-1} \{\mathcal{H}_1 - \bar{\mathcal{H}}\} d\theta.$$

Since $\bar{\mathcal{H}}$ is the mean value of $\mathcal{H}_1$, the function W_1 is 2π-periodic in θ. Let us change variables by $v = V(I, \theta, y, \varepsilon)$, where $V(I, \theta, y, \varepsilon)$ is the solution of

$$\frac{dv}{d\varepsilon} = J\nabla_y W_1(I, \theta, y), \qquad v(0) = y,$$

where J denotes the skew-symmetric matrix

$$J = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}$$

and E stands for the identity matrix. Since V is the solution of a Hamiltonian equation, the change of variables is symplectic, and since W_1 is 2π-periodic in θ, so is V. The resulting Hamiltonian becomes

$$\mathcal{H}_\varepsilon(I, \theta, y) = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, y) + O(\varepsilon^2)$$

by the theory of Lie transforms (cf. [36, p. 168ff]). ■

The proofs of Theorems 2.1 and 2.2 follow from this lemma.

Proof of Reeb's theorems. First, each orbit of $\mathcal{H}_0$ in the level set $\mathcal{H}_0^{-1}(0)$ intersects σ once, so σ can be considered a coordinate patch on the base space B, and y provides symplectic coordinates for σ. Thus, B has an atlas of symplectic charts, and therefore B is a symplectic manifold. This proves Theorem 2.1.

Up to terms of order ε the equations are

$$\dot{I} = O(\varepsilon^2), \quad \dot{\theta} = 2\pi/T(I) + O(\varepsilon^2), \quad \dot{y} = \varepsilon J \nabla_y \bar{\mathcal{H}}(I, y) + O(\varepsilon^2).$$

The return time for θ to increase from 0 to 2π is $T+O(\varepsilon^2)$, and the section map in an energy level $(I=0)$ is $P:\sigma\to\sigma: y\mapsto P(y)$, where $P(y)=y+\varepsilon TJ\nabla_y\bar{\mathcal{H}}(0,y)+O(\varepsilon^2)$. A fixed point of P gives rise to a periodic solution, and so we must solve $P(y)=y$ or, equivalently, $TJ\nabla_y\bar{\mathcal{H}}(0,y)+O(\varepsilon)=0$. By hypothesis $y=0$ is a nondegenerate critical point of $\bar{\mathcal{H}}$ when $I=0$ or $\nabla_y\bar{\mathcal{H}}(0,0)=0$ and $\partial^2\bar{\mathcal{H}}/\partial y^2(0,0)$ is nonsingular. Thus by the implicit function theorem there is a function $\bar{y}(\varepsilon)=O(\varepsilon)$ such that $P(\bar{y}(\varepsilon))=\bar{y}(\varepsilon)$. This fixed point of P is the initial condition for the periodic solution asserted in Theorem 2.2. ■

2.2. Corollaries. Only the last sentence in Theorem 2.2 gives a truly global result. Those conversant with Morse theory [13] will see that there is a sharper global result.

Corollary 2.1. *Let $\bar{\mathcal{H}}$ be a Morse function, let β_j be the jth Betti number of B, and let C_j be the number of critical points of index j. Then $C_j\geq\beta_j$ or, better yet,*

$$
\begin{aligned}
C_0 &\geq \beta_0,\\
C_1-C_0 &\geq \beta_1-\beta_0,\\
C_2-C_1+C_0 &\geq \beta_2-\beta_1+\beta_0,\\
&\cdots\\
C_k-C_{k-1}+C_{k+2}-\cdots\pm C_0 &\geq \beta_k-\beta_{k-1}+\beta_{k+2}-\cdots\pm\beta_0 \qquad (k<2n-2),\\
C_{2n-2}-C_{2n-3}+C_{2n-4}-\cdots+C_0 &= \beta_{2n-2}-\beta_{2n-3}+\beta_{2n-4}-\cdots+\beta_0=\chi(B).
\end{aligned}
\tag{3}
$$

For these better inequalities on a Morse function, see [37]. The lower estimate on the number of periodic solutions in Theorem 2.2 is $\chi(B)$, the alternating sum of the Betti numbers which could be 0 or negative, whereas the Morse inequalities $C_j\geq\beta_j$ give a lower estimate which is the sum of the Betti numbers. Moreover, the estimates give some information on the number of critical points of various indices. For example, Milnor [37] remarks that if $C_{j+1}=C_{j-1}=0$, then $C_j=\beta_j$.

The nontrivial characteristic multipliers of the periodic solution given in Theorem 2.2 are the eigenvalues of

$$\mathcal{P}=\frac{\partial P}{\partial y}(\bar{y}(\varepsilon))=E+\varepsilon TJ\frac{\partial^2\bar{\mathcal{H}}}{\partial y^2}(0,0)+O(\varepsilon^2),$$

where E is the identity matrix. The eigenvalues of the Hamiltonian matrix

$$A=J\frac{\partial^2\bar{\mathcal{H}}}{\partial y^2}(0,0) \tag{4}$$

are the characteristic exponents of the critical point of $\bar{Y}$ at $\bar{p}$ on B. Thus, the lemma also yields the following corollary.

Corollary 2.2. *Let p be as in Theorem 2.2 and let the characteristic exponents of $\bar{Y}(\bar{p})$ be $\lambda_1,\lambda_2,\ldots,\lambda_{2n-2}$. Then the characteristic multipliers of the periodic solution through $p(\varepsilon)$ are*

$$1,\ 1,\ 1+\varepsilon\lambda_1T+O(\varepsilon^2),\ 1+\varepsilon\lambda_2T+O(\varepsilon^2),\ldots,1+\varepsilon\lambda_{2n-2}T+O(\varepsilon^2).$$

This result was used in [33]. We shall say that a periodic solution is *elliptic* or *linearly stable* if the monodromy matrix is diagonalizable and all its eigenvalues have unit modulus.

One must be careful in applying this corollary, because it gives only an approximation of the characteristic multipliers. Consider the case $2\bar{\mathcal{H}}=(u_1^2+v_1^2)-(u_2^2+v_2^2)$, where $y=$

(u_1, u_2, v_1, v_2), so the eigenvalues are $i, i, -i, -i$. When $T = 1$, Corollary 2.2 says that the multipliers are 1, 1, $1+\varepsilon i+O(\varepsilon^2)$, $1+\varepsilon i+O(\varepsilon^2)$, $1-\varepsilon i+O(\varepsilon^2)$, and $1-\varepsilon i+O(\varepsilon^2)$, which looks like an elliptic periodic solution. But higher-order terms can change the stability. Consider now a perturbation of this example, namely, $2\bar{\mathcal{H}} = (u_1^2 + v_1^2) - (u_2^2 + v_2^2) + 2\varepsilon v_1 v_2$. Now the estimates of the multipliers would be 1, 1, $1 + \varepsilon i + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$, $1 + \varepsilon i - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$, $1 - \varepsilon i + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$, and $1 - \varepsilon i - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$, which gives an unstable periodic solution. The solution of this problem lies in the Krein–Gel'fand concept of parametric stability [47], which we briefly summarize below.

For the moment consider the linear constant coefficient Hamiltonian system

$$\dot{y} = Cy = J\nabla H(y), \qquad H = \frac{1}{2}y^T Sy, \tag{5}$$

where S is a symmetric matrix and $C = JS$ is a Hamiltonian matrix. System (5) (or the Hamiltonian matrix C) is *stable* if all its solutions are bounded for all t, and it is said to be *parametrically stable* or *strongly stable* if it and all sufficiently small linear constant coefficient Hamiltonian perturbations of it are stable. If system (5) is parametrically stable, then it is stable, and it is stable if and only if C is diagonalizable and has only purely imaginary eigenvalues.

Let $\pm\alpha_1 i, \pm\alpha_2 i, \ldots, \pm\alpha_s i$ be the eigenvalues of the stable matrix C, and V_j, $j = 1, \ldots, s$, be the maximal real linear subspace where C has eigenvalues $\pm\alpha_j i$. So V_j is a C-invariant symplectic subspace, C restricted to V_j has eigenvalues $\pm\alpha_j i$, and $\mathbb{R}^{2n} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$. Let H_j be the restriction of H to V_j.

Theorem 2.3 (see [47]). *System (5) is parametrically stable if and only if*

- *all the eigenvalues of C are purely imaginary,*
- *C is nonsingular,*
- *C is diagonalizable over the complex numbers, and*
- *the Hamiltonian H_j is positive or negative definite for each j.*

Thus, $2H = (u_1^2+v_1^2)+(u_2^2+v_2^2)$ is parametrically stable, as the corresponding eigenvalues are $\pm i$ (double); hence $H_1 = H$ is positive definite. The Hamiltonian $2H = (u_1^2+v_1^2)-4(u_2^2+v_2^2)$ has eigenvalues $\pm i$ and $\pm 2i$, so $2H_1 = u_1^2 + v_1^2$ is positive definite and $2H_2 = -4(u_2^2 + v_2^2)$ is negative definite; therefore, H is parametrically stable. However, $2H = (u_1^2 + v_1^2) - (u_2^2 + v_2^2)$ has eigenvalues $\pm i$ (double), and, as $H_1 = H$ is not positive or negative definite, it cannot be parametrically stable.

Now consider the linear T-periodic Hamiltonian system

$$\dot{y} = D(t)y = J\nabla H(y), \qquad H = \frac{1}{2}y^T R(t)y, \tag{6}$$

where $R(t) = R(t+T)$ is symmetric and $D(t) = JR(t)$ is Hamiltonian. The periodic system (6) is *stable* if all its solutions are bounded for all t, and it is said to be *parametrically stable* or *strongly stable* if it and all sufficiently small linear T-periodic Hamiltonian perturbations of it are stable. The monodromy matrix is $M = Z(T)$, where $Z(t)$ is a fundamental matrix solution of (6). If the system is parametrically stable, then it is stable, and (6) is stable if and only if its monodromy matrix is diagonalizable and has only eigenvalues (multipliers) of unit modulus.

Let $\beta_1^{\pm1}, \beta_2^{\pm1}, \dots, \beta_s^{\pm1}$ be the eigenvalues of M and V_j, $j = 1, \dots, s$, be the maximal real linear subspace where M has eigenvalues $\beta_j^{\pm1}$. So V_j is an M-invariant symplectic subspace, M restricted to V_j (denoted by M_j) is symplectic and has eigenvalues $\beta_j^{\pm1}$, and $\mathbb{R}^{2n} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$.

For periodic systems we need to define the analogue of the quadratic form H_j. There are at least three ways to do this: (1) define a bilinear form on the eigenvectors corresponding to $\beta_j^{\pm1}$ [47], (2) use Floquet theory and take logs of M, or (3) use a Cayley transformation. All three ways yield the same result, and we choose the latter because of its simplicity. The particular Möbius transformation

$$\Psi : z \mapsto w = (z-1)(z+1)^{-1}, \qquad \Psi^{-1} : w \mapsto z = (1+w)(1-w)^{-1}$$

is known as the *Cayley transformation*. One checks that $\Psi(1) = 0$, $\Psi(i) = i$, and $\Psi(-1) = \infty$, and so Ψ takes the unit circle in the z-plane to the imaginary axis in the w-plane, the interior of the unit circle in the z-plane to the left half w-plane, etc. Ψ can be applied to any matrix B which does not have -1 as an eigenvalue, and if λ is an eigenvalue of B, then $\Psi(\lambda)$ is an eigenvalue of $\Psi(B)$.

Lemma 2.2. *If M is a symplectic matrix which does not have eigenvalue -1, then $C = \Psi(M)$ is a Hamiltonian matrix. Moreover, if M has only eigenvalues of unit modulus and is diagonalizable, then $C = \Psi(M)$ has only purely imaginary eigenvalues and is diagonalizable.*

Proof. Simply check. ■

M_j is the restriction of M to V_j and is a symplectic matrix, so $C_j = \Psi(M_j)$ is a Hamiltonian matrix and $S_j = JC_j$ is a symmetric matrix.

Theorem 2.4 (see [47]). *System (6) is parametrically stable if and only if*

- *all the eigenvalues of M have unit modulus,*
- *M does not have eigenvalue $+1$ or -1,*
- *M is diagonalizable over the complex numbers, and*
- *the symmetric matrix S_j is positive or negative definite for each j.*

Corollary 2.3. *If one or more of the λ_j of Corollary 2.2 is real or has nonzero real part, then the periodic solution through $p(\varepsilon)$ is unstable.*

If the matrix A in (4) is the coefficient matrix of a parametrically stable system, then the periodic solution through $p(\varepsilon)$ is elliptic. In particular, if $\bar{p}$ is a nondegenerate maximum or minimum of $\bar{\mathcal{H}}$, then the periodic solution through $p(\varepsilon)$ is elliptic. If $\bar{\mathcal{H}}$ is a Morse function, then there are at least two elliptic periodic solutions, since $\bar{\mathcal{H}}$ must have a nondegenerate maximum and minimum.

The authors believe this application of Krein–Gel'fand theory to be new.

Proof. The first sentence is obvious. Recall that the nontrivial multipliers are the eigenvalues of the symplectic matrix $\mathcal{P} = E + \varepsilon TA + O(\varepsilon^2)$. Applying Cayley's transformation to $\mathcal{P}$ yields the Hamiltonian matrix

$$\mathcal{A} = \Psi(\mathcal{P}) = \Psi(E + \varepsilon TA + O(\varepsilon^2)) = \frac{1}{2}\varepsilon TA + O(\varepsilon^2) = \frac{1}{2}\varepsilon T(A + O(\varepsilon)).$$

If A is the matrix of a parametrically stable system, the matrix $A + O(\varepsilon)$ is stable for all small ε, and hence so is $\mathcal{A}$. Thus all eigenvalues of $\mathcal{P} = \Psi^{-1}(\mathcal{A})$ have unit modulus. ■

2.3. KAM tori. One can also detect invariant tori using KAM theory.

Theorem 2.5. *Let p be as in Theorem 2.2 and suppose there are symplectic action-angle variables $(I_1, \ldots, I_{n-1}, \theta_1, \ldots \theta_{n-1})$ at $\bar{p}$ in B such that*

$$\bar{\mathcal{H}} = \sum_{k=1}^{n-1} \omega_k I_k + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} C_{kj} I_k I_j + \mathcal{H}^{\#}, \tag{7}$$

where the ω_k are nonzero, $C_{kj} = C_{jk}$, and $\mathcal{H}^{\#}(I_1, \ldots, I_{n-1}, \theta_1, \ldots \theta_{n-1})$ is at least cubic in $I_1, \ldots, I_{n-1}$.

Assume that $\det C_{kj} \neq 0$. That is, assume the system has been put into Birkhoff normal form and the "twist" condition is satisfied. Furthermore, assume $dT/dh \neq 0$; i.e., assume the period varies with $\mathcal{H}_0$ in a nontrivial way.

Then near the periodic solutions given in Theorem 2.2 there are invariant KAM tori of dimension n. In particular, when $n = 2$, the periodic solution of Theorem 2.2 is orbitally stable.

Proof. In the tubular neighborhood constructed in Lemma 2.1, a full set of symplectic coordinates is $(I, I_1, \ldots, I_{n-1}, \theta, \theta_1, \ldots \theta_{n-1})$ and the Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0(I) + \varepsilon \left\{ \sum_{k=1}^{n-1} \omega_k I_k + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} C_{kj} I_k I_j \right\} + \cdots ,$$

and the theorem follows by Theorem 14 on page 185 of [5]. ■

This is just one of many KAM theorems. We place it here because of its simplicity. We will return to KAM-type results in subsequent papers.

2.4. Symmetric periodic solutions. In some cases the problem admits a discrete symmetry. Let $R : M \to M$ be an antisymplectic involution; i.e., $R^*\Omega = -\Omega$ and R^2 is the identity map of M. Then $F = \{p \in M : R(p) = p\}$ is a Lagrangian submanifold of M. The system defined by $\mathcal{H}_0$ (or $\mathcal{H}_\varepsilon$) is *reversible* or *admits R as a symmetry* if $\mathcal{H}_0 \circ R = \mathcal{H}_0$ (or $\mathcal{H}_\varepsilon \circ R = \mathcal{H}_\varepsilon$).

Now R maps an orbit of Y_0 into itself and so is well defined on B. Let $\bar{R}$ be R on B, so $\bar{R} : B \to B$, $\bar{R}^*\omega = -\omega$, $\bar{R}^2$ is the identity map on B, and $\bar{\mathcal{H}} \circ \bar{R} = \bar{\mathcal{H}}$. Let $\bar{F} = \{p \in B : \bar{R}(p) = p\}$.

A classical result [8] is the following lemma.

Lemma 2.3. *If a solution of Y_0 (or Y_ε) starts on F at time $t = 0$ and returns to F after time $t = T$, then the solution is $2T$-periodic, and its orbit is mapped onto itself by R.*

Similarly, if a solution of $\bar{Y}$ starts on $\bar{F}$ at time $t = 0$ and returns to $\bar{F}$ after time $t = T$, then the solution is $2T$-periodic and its orbit is mapped onto itself by $\bar{R}$.

These statements follow from the general identities

$$\phi_\varepsilon^t \circ R = R \circ \phi_\varepsilon^{-t}, \qquad \bar{\phi}^t \circ \bar{R} = \bar{R} \circ \bar{\phi}^{-t}. \tag{8}$$

Such periodic solutions are called *symmetric periodic solutions*. Let $\bar{\phi}^t(\bar{p})$ be a symmetric 2τ-periodic solution of $\bar{Y}$ and $\bar{q} = \bar{\phi}^\tau(\bar{p})$; then there are symplectic coordinate systems (ξ, ζ)

and (X, Z) for $\bar{B}$ at $\bar{p}$ and $\bar{q}$ with $(\xi(\bar{p}), \zeta(\bar{p})) = (0, 0)$ and $(X(\bar{q}), Z(\bar{q})) = (0, 0)$ such that

$$\bar{R}(\xi, \zeta) = (\xi, -\zeta) \qquad \text{and} \qquad \bar{R}(X, Z) = (X, -Z)$$

(see [34]). Locally $\bar{F}$ is given by $\zeta = 0$ near $\bar{p}$ and by $Z = 0$ near $\bar{q}$. Let $\bar{\phi}^t(\xi, \zeta) = (X(t, \xi, \zeta), Z(t, \xi, \zeta))$. In these coordinates the solution is a symmetric periodic solution if $Z(\tau, 0, 0) = 0$. Such a periodic solution is called a *nondegenerate symmetric periodic solution* if

$$\det \frac{\partial Z}{\partial \xi}(\tau, 0, 0) \neq 0.$$

In general, a nondegenerate symmetric periodic solution persists under small symmetric perturbations. However, in our case the problem is somewhat degenerate, requiring the use of the method and implicit function theorem of Arenstorf [2, 3, 4]. But first we present another lemma. For simplicity let $n = 2$.

Lemma 2.4. *Let $\mathcal{H}_0$ admit R as a symmetry, and let p and (I, θ, y) be as in Lemma 2.1. If $p \in F$, then $R(I, \theta, y) = (I, -\theta, \bar{R}(y))$ and $R(I, \pi + \theta, y) = (I, \pi - \theta, \bar{R}(y))$.*

Let $n = 2$ and $\bar{\phi}^t(\bar{p})$ be a nondegenerate symmetric 2τ-periodic solution of $\bar{Y}$. There exists a set of symplectic action-angle variables (I_1, θ_1) for $\bar{B}$, valid in a neighborhood of $\{\bar{\phi}^t(\bar{p}) : 0 \leq t \leq \tau\}$, such that $\bar{\mathcal{H}}$ is independent of θ_1. Thus $\bar{\mathcal{H}} = \bar{\mathcal{H}}(I, I_1)$, and in these coordinates $\bar{R}(I_1, \theta_1) = (I_1, -\theta_1)$ and $\bar{R}(I_1, \pi + \theta_1) = (I_1, \pi - \theta_1)$, so $\bar{F} = \{(I_1, \theta_1) : \theta_1 \equiv 0 \mod \pi\}$.

If the periodic solution corresponds to $I = I_1 = 0$, then the solution is nondegenerate if

$$\frac{\partial^2 \bar{\mathcal{H}}}{\partial I_1^2}(0, 0) \neq 0.$$

Since the reduced space B depends on $\mathcal{H}$ or I, we have coordinates such that

$$\mathcal{H}_\varepsilon = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, I_1) + O(\varepsilon^2). \tag{9}$$

Proof. By Lemma 2.3 we have $\phi_0^t(p) = R\phi_0^{-t}(p)$ since $R(p) = p$. By construction, θ is t measured from p, so $R : \theta \mapsto -\theta$. Also, $R(I, \pi + \theta, y) = (I, -\pi - \theta + 2\pi, \bar{R}(y)) = (I, \pi - \theta, \bar{R}(y))$.

The proof of the existence of the action-angle variables (I_1, θ_1) for B follows the proof of Lemma 2.1 and the paragraph above. ■

Theorem 2.6. *Let $\partial\mathcal{H}_0/\partial I$ be nonzero, let $n = 2$, and let $\bar{p} \in \bar{B}$ with $\bar{R}(\bar{p}) = \bar{p}$ be an initial point for a nondegenerate τ-periodic solution of $\bar{Y}$. Let $p \in M$ with $R(p) = p$ be a point on the orbit which projects to $\bar{p}$. Let α, β be positive integers with α fixed, β large, and ε small.*

Then near the initial condition p, the flow of Y_ε has a symmetric periodic solution where $T(\varepsilon) = \alpha\tau + O(\varepsilon) = \beta T + O(\varepsilon)$.

Arenstorf's method of establishing the existence of symmetric periodic solutions has been around for a long time and has been applied to several problems [3, 4, 14, 25]. The authors believe that the above theorem is new at this level of generality.

Proof. Choose coordinates $(I, I_1, \theta, \theta_1)$ by Lemma 2.4 so that

$$\mathcal{H}_\varepsilon = \mathcal{H}_0(I) + \varepsilon \bar{\mathcal{H}}(I, I_1) + O(\varepsilon^2),$$

and the equations of motion are

$$\dot I = O(\varepsilon^2), \qquad \dot\theta = -\frac{\partial \mathcal{H}_0}{\partial I}(I) + O(\varepsilon),$$
$$\dot I_1 = O(\varepsilon^2), \qquad \dot\theta_1 = -\varepsilon\frac{\partial \bar{\mathcal{H}}}{\partial I_1}(I, I_1) + O(\varepsilon^2).$$

Since these equations are autonomous, we may take the fast angle θ as the independent variable so that the equations become

$$\frac{\partial I}{\partial \theta} = 0, \qquad \frac{\partial I_1}{\partial \theta} = 0, \qquad \frac{\partial \theta_1}{\partial \theta} = \varepsilon G(I, I_1) = \varepsilon\left\{\frac{\partial \mathcal{H}_0}{\partial I}(I)\right\}^{-1}\frac{\partial \bar{\mathcal{H}}}{\partial I_1}(I, I_1), \tag{10}$$

plus $O(\varepsilon^2)$ terms. For the moment, ignore the $O(\varepsilon^2)$ terms and seek a symmetric periodic solution of the approximate equations. Let α and β be relatively prime integers, $\nu = G(0,0)^{-1}$, and set $\varepsilon = \nu\alpha/\beta$. Start with initial conditions $I = I_1 = \theta_1 = 0$ and integrate the approximate equations on θ from 0 to $\beta\pi$ to obtain the approximate solution

$$I = 0, \qquad I_1 = 0, \qquad \theta_1 = \alpha\pi.$$

This approximate solution satisfies the symmetry conditions and so to this level of approximation is a symmetric periodic solution.

Fixing α and taking β large, the parameter ε becomes small, and so we might expect that this approximate solution could be continued into the full problem. However, the problem is complicated by the fact that taking β large corresponds to integrating the equations over a large variation of θ. As Arenstorf has observed, the usual implicit function theorem cannot be applied since one cannot set $\varepsilon = 0$ to find an approximate solution. Thus, we must follow Arenstorf and make careful estimates.

First, we fix the integer α and the initial condition $I = 0$ once and for all. Let the superscript f denote the full solution of (10) including $O(\varepsilon^2)$ terms, the superscript a the approximate solution, and the superscript e the error term. Integrate the full equations with initial condition $I_1 = K$, and integrate from $\theta = 0$ to $\theta = \beta\pi$ to obtain

$$\theta_1^f(\beta\pi, \varepsilon, K) = \theta_1^a(\beta\pi, \varepsilon, K) + \theta_1^e(\beta\pi, \varepsilon, K), \tag{11}$$

where $\theta_1^a(\beta\pi, \varepsilon, K) = \varepsilon\beta\pi\, G(0, K)$.

The error term θ_1^e is due to the $O(\varepsilon^2)$ terms appended to (10). Bounding these $O(\varepsilon^2)$ terms by $C\varepsilon^2$, and taking the $O(\varepsilon)$-Lipschitz constant of the θ_1-flow to be $L\varepsilon$, we apply to θ_1^e a standard Gronwall estimate of the form $\{u(0) = 0$ and $d|u|/d\theta \le L\varepsilon|u| + C\varepsilon^2\} \Rightarrow |u(\theta)| \le \varepsilon C(e^{\varepsilon L\theta} - 1)/L$ (see, e.g., Hartman [23]) to conclude that

$$|\theta_1^e| \le \varepsilon C(e^{\varepsilon L\beta\pi} - 1)/L. \tag{12}$$

A similar estimate holds for the first partial derivatives of θ_1^e.

The approximate equation has solution $\theta_1^a(\beta\pi, \varepsilon, K) = \alpha\pi$ by taking $\varepsilon = \nu\alpha/\beta$ and $K = 0$. Also by assumption $\partial\theta_1^a/\partial K$ is nonzero. From the estimate (12) the error term can be made arbitrarily small by taking β large with $\varepsilon = \nu\alpha/\beta$, since in this case the estimate (12) reads

$|\theta_1^e| \leq C\nu\alpha(e^{L\nu\alpha\pi} - 1)/(L\beta)$. Similarly, the derivatives of θ_1^e can be made small by taking β large. These estimates ensure that we remain in a compact neighborhood of the approximate solution. Thus, the implicit function theorem of Arenstorf [2, 3, 4] applies, and there exists β_0 such that if $\beta > \beta_0$, then there is a solution $K_s(\beta)$ such that

$$\theta_1^f(\beta\pi, \nu\alpha/\beta, K_s(\beta)) = \alpha\pi.$$

This gives the initial conditions for a symmetric periodic solution. ■

2.5. Weinstein's theorem. For completeness we add this much deeper global result on the existence of periodic solutions which is not a corollary of Reeb's theorems. Let X be a topological space; then the category of X in the sense of Lusternik–Schnirelmann, $\text{cat}(X)$, is the smallest number of open sets that are contractible in X and that cover X [26, 31]. One of the main uses of this concept is in the theorem that says that every smooth function on a compact manifold M has at least $\text{cat}(M)$ critical points. Weinstein extended the connection between critical points of functions and periodic solutions of Hamiltonian systems to prove the following theorem.

Theorem 2.7. *Assume B is compact and simply connected in the sense that $H^1(B, \mathbb{R}) = 0$, where $H^1(B, \mathbb{R})$ is the one-dimensional cohomology group of B over the real numbers, and let $\ell = cat(B)$ be the Lusternik–Schnirelmann category of B. Then for small ε the flow of Y_ε has at least ℓ periodic solutions with periods near T (there is no nondegeneracy assumption)* [45, 46].

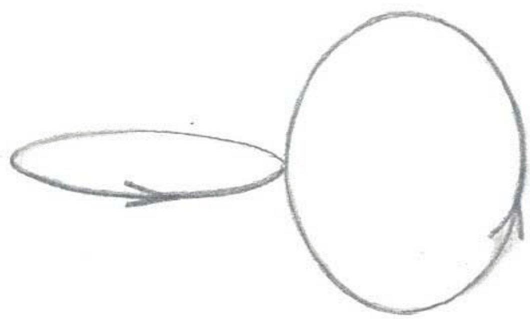
The n-sphere S^n has category 2 and all other compact manifolds have category greater than 2. For $X = S^n \times S^n$, we produce three contractible open sets that cover X, so $\text{cat}(S^n \times S^n) = 3$. (We illustrate this in Figure 2 for the case $n = 1$.)

It is helpful to think of X as a cellular complex. Starting with two n-cells, identify one point in one cell with one point in the other cell to form the wedge product (or wedge sum) of two spheres $S^n \vee S^n$. (The wedge product of two circles is thus a figure eight, as in Figure 2a; for a precise definition of $S^n \vee S^n$, see [24, p. 10].) Now attach a $2n$-cell to $S^n \vee S^n$ to form X (Figure 2b). Take the $2n$-cell to be the first contractible set. For the second set, delete one point from each of the two spheres in $S^n \vee S^n$. This set is $D^n \vee D^n$ and can easily be "fattened up" to an open set in $S^n \times S^n$. These two sets cover all but the two points deleted from $S^n \times S^n$. For the third set, choose any contractible open set in X that covers these two points. (Figure 2c shows sets that could be fattened up to form the second and third contractible open sets for $n = 1$.)

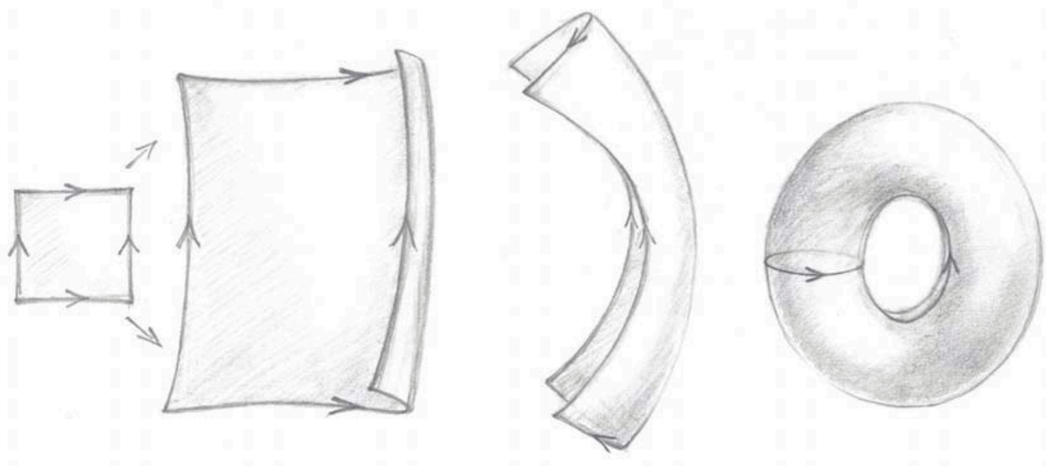
3. The planar lunar problem.

3.1. The Hamiltonians. For us the lunar problem is the restricted three-body problem where the infinitesimal particle is close to one of the primaries [35, 36]. Note that, in this context, the terminology "lunar problem" means that the zero mass point can move about either primary, which is more general than the way it is historically defined, where the infinitesimal mass point moves only about the smaller primary (or secondary).

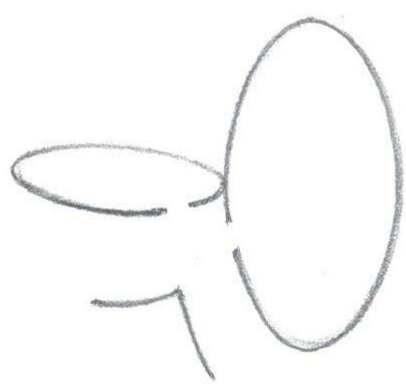
Here we summarize the normalization and reduction as given in [42] and then apply the general theorems from section 2. Figure 3 is a sketch of the planar lunar problem in the rotating frame—the projection on the $x_1 x_2$-plane. The primary bodies are point particles



(a) $S^1 \vee S^1$



(b) Attaching the 2-cell to $S^1 \vee S^1$ to form $S^1 \times S^1$



(c) The second and third contractible sets (before "fattening")

Figure 2. *The Lusternik–Schnirelmann category of* $S^1 \times S^1$.

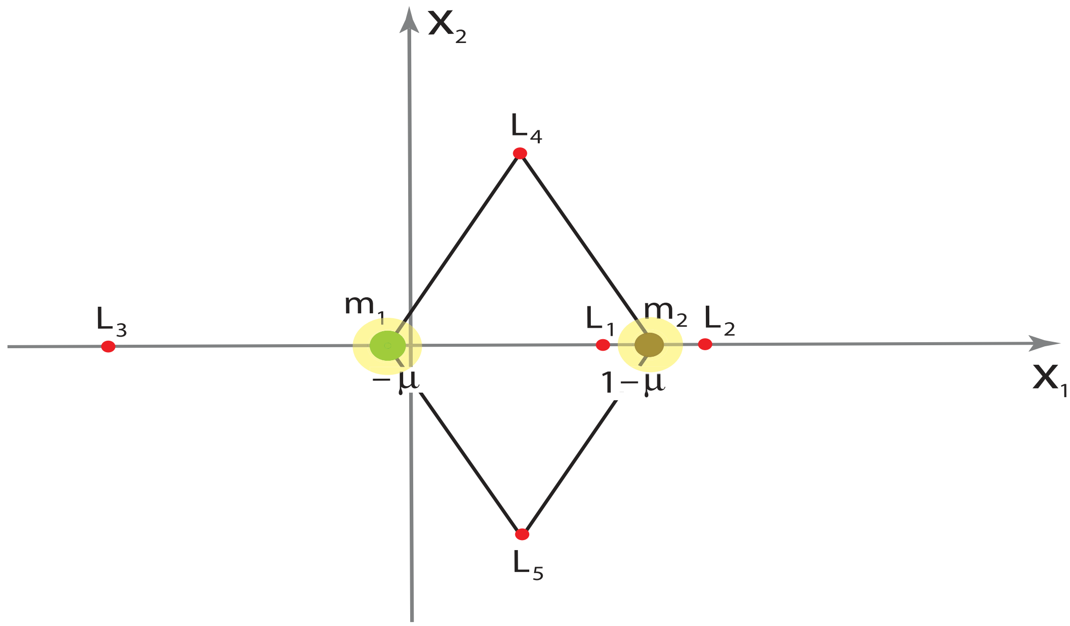


Figure 3. *The lunar problem.*

with masses m_1 and m_2 and are located at the points $(-\mu, 0)$ and $(1-\mu, 0)$, respectively. Parameter $\mu = m_1/(m_1+m_2)$ (it is assumed that $m_1 \geq m_2$). The motion of the infinitesimal particle is confined to either one of the yellow regions around the primaries. The points $L_1, \ldots, L_5$ are the equilibria of the restricted three-body problem in the rotating frame. The infinitesimal particle touches neither L_1 nor L_2.

We start with the Hamiltonian of the planar circular restricted three-body problem in rotating coordinates given by

$$\mathcal{H} = \frac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2}} - \frac{1-\mu}{\sqrt{(x_1+\mu)^2 + x_2^2}}. \tag{13}$$

We now change coordinates in order to bring $\mathcal{H}$ into suitable form. First we perform the linear change from y_2 and x_1 to $y_2 - \mu$ and $x_1 - \mu$, respectively, to bring one primary to the origin. Then, we introduce a small parameter ε by replacing $y = (y_1, y_2)$ by $\varepsilon^{-1}(1-\mu)^{1/3} y$ and $x = (x_1, x_2)$ by $\varepsilon^2(1-\mu)^{1/3} x$. By doing so we restrict $\mathcal{H}$ to a particular case where the infinitesimal particle is moving around one of the primaries. This change is symplectic with multiplier $\varepsilon^{-1}(1-\mu)^{-2/3}$; thus $\mathcal{H}$ must be replaced by $\varepsilon^{-1}(1-\mu)^{-2/3}\mathcal{H}$.

In the next step, we scale time by dividing t by ε^3 and multiplying $\mathcal{H}$ by ε^3. Then we expand the resulting Hamiltonian in powers of ε to get

$$\mathcal{H}_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2) - \frac{1}{\sqrt{x_1^2 + x_2^2}} - \varepsilon^3(x_1 y_2 - x_2 y_1) + \frac{1}{2}\varepsilon^6 \mu(-2x_1^2 + x_2^2) + \cdots . \tag{14}$$

The zeroth-order term is the Hamiltonian of the Kepler problem and the $O(\varepsilon^3)$ term is due to the rotating coordinates. It is not until $O(\varepsilon^6)$ that the second primary influences the motion.

Moser has shown [39] that the n-dimensional Kepler problem can be regularized and the regularized flow is equivalent to the geodesic flow on S^n. Let us be more specific for our case. Let $\mathcal{K} = \mathcal{H}_0$ be the Hamiltonian of the planar Kepler problem defined on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$, $K_0 = \{(x, y) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 : \mathcal{K}(x, y) = -\frac{1}{2}\}$. Let S^2 be the unit sphere, $\hat{S}^2$ the unit sphere punctured at the north pole, TS^2 ($T\hat{S}^2$) the tangent bundle of the (punctured) 2-sphere, and $T_0S^2 = \{v \in TS^2 : \|v\| = 1\}$ ($T_0\hat{S}^2 = \{v \in T\hat{S}^2 : \|v\| = 1\}$) the unit (punctured) sphere bundle.

The elliptic domain $\mathcal{E}$ is the set of points in K_0 which gives rise to elliptic orbits. All the solutions of the Kepler problem in $\mathcal{E}$ are periodic with the same period. Thus $\mathcal{E}$ is a circle bundle but is not compact. The base is two punctured disks, as we show below.

Moser constructs a symplectic diffeomorphism from $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ onto $T\hat{S}^2$ which, when restricted to K_0, maps onto $T_0\hat{S}^2$. After changing the time variable for the Kepler problem, the diffeomorphism takes Kepler flow on K_0 to the geodesic flow on $T_0\hat{S}^2$. The geodesic flow on $T_0\hat{S}^2$ obviously extends to all of T_0S^2 and is considered the regularized Kepler problem.

All the geodesics on T_0S^2 are periodic, so T_0S^2 is a circle bundle with base S^2. Moser shows that a small perturbation of the Kepler problem can be carried over as a small perturbation of the geodesic problem. He then shows that the average of the perturbation over the geodesic flow defines a smooth flow on the base. We next proceed to construct this flow on the base.

Now express (14) in mixed polar and Delaunay coordinates (see, for instance, [9, 18]) so that the Hamiltonian becomes

$$\mathcal{H}_\varepsilon = -\frac{1}{2\,L^2} - \varepsilon^3 G - \frac{1}{4}\varepsilon^6 \mu r^2 \big(1 + 3\cos(2\vartheta)\big) + \cdots .$$

Here, (ℓ, g, L, G) are the usual Delaunay variables, ℓ the mean anomaly, g the argument of the pericenter, and L the square of the semimajor axis. G is the third component of the angular momentum vector $\mathbf{G} = (0, 0, G)$; thus $0 \le |G| \le L$ and G can be positive, negative, or zero. This is a coordinate system on $\mathcal{E}$. Finally, (r, ϑ) are the usual polar coordinates.

We eliminate the mean anomaly ℓ to a certain order by means of a special Lie transformation well suited for perturbed Kepler problems, the so-called normalization of Delaunay [18, 19]. We arrive at

$$\mathcal{H}_\varepsilon = -\frac{1}{2\,L^2} - \varepsilon^3 G + \frac{1}{8}\varepsilon^6\, \mu\, L^2 \left(3\,G^2 - 5\,L^2 + 15\,(G^2 - L^2)\cos(2g)\right) + \cdots . \tag{15}$$

Here only finitely many terms have been put into normal form. This normalization is effectively the average of the perturbations over the periodic orbits of the Kepler problem in $\mathcal{E}$.

The base space (or reduced space or orbit space) for the regularized Kepler problem is a 2-sphere S^2 [39]. Figure 4 may be helpful in visualizing this base space. The flow is given by the circles around the poles on the sphere. A coordinate system for the reduced space is $\mathbf{a} = \mathbf{G} + LA$, where A is the Laplace–Runge–Lenz vector. One has $A = e\,(\cos g, \sin g, 0)$ and then $a_1 = e\cos g$, $a_2 = e\sin g$, and $a_3 = G$ on $\mathcal{E}$, where $e = \sqrt{1 - G^2/L^2}$ is the eccentricity. One can check that $|\mathbf{a}| = L$ and the vector $\mathbf{a}$ uniquely determines an orbit of the Kepler problem on the energy level $h = -1/(2L^2)$. Each point on the sphere $a_1^2 + a_2^2 + a_3^2 = L^2$ corresponds to a bounded orbit of the Kepler problem. Points $(0, 0, \pm L)$ correspond to the circular orbits, the circle $a_3 = 0$ (the equator, or the green circle in Figure 4) corresponds to collision orbits, and the other points on the sphere correspond to elliptic orbits. The

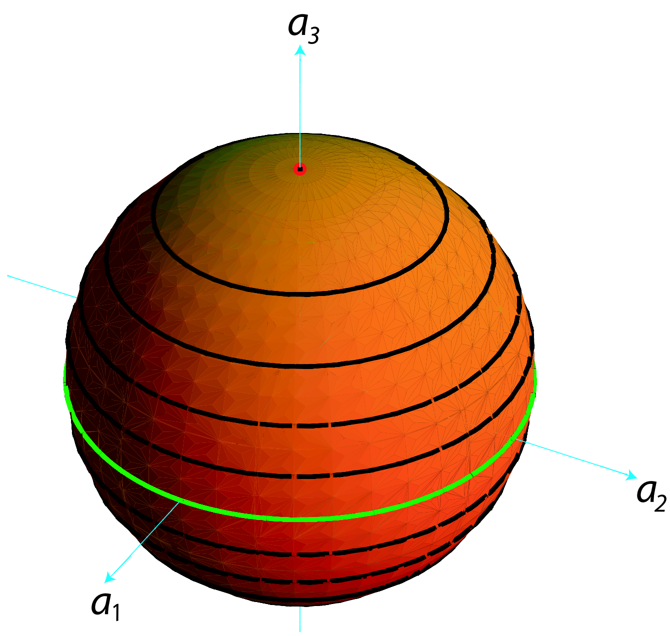


Figure 4. *The base space for the regularized Kepler problem.*

complement of $(0, 0, \pm L) \cup \{a_3 = 0\}$ is the reduced space of the elliptic domain $\mathcal{E}$.

Now compute from (15) the Hamiltonian on the reduced space of $\mathcal{E}$. Use $\cos(2g) = (a_1^2 - a_2^2)/(L^2 - G^2)$, so

$$\mathcal{H}_\varepsilon = -\frac{1}{2L^2} - \varepsilon^3 a_3 + \frac{1}{8}\varepsilon^6 \mu L^2 \left(3a_3^2 - 5L^2 - 15(a_1^2 - a_2^2)\right) + \cdots .$$

We first drop the higher-order nonnormalized terms and then use $a_3^2 = L^2 - a_1^2 - a_2^2$, dropping additive constants and dividing by ε^3 to get the Hamiltonian

$$\bar{\mathcal{H}} = -a_3 - \frac{3}{4}\varepsilon^3 \mu L^2 \left(3a_1^2 - 2a_2^2\right) + \cdots . \tag{16}$$

We note that this Hamiltonian is well defined and smooth on the exceptional set $(0, 0, \pm L) \cup \{a_3 = 0\}$. Since Moser proved that the averaged (normalized) Hamiltonian of the perturbation is defined and smooth on all of S^2, (16) is the Hamiltonian on the full reduced space S^2.

To obtain the equations of motion, note that $\{a_1, a_2\} = a_3$, $\{a_2, a_3\} = a_1$, $\{a_3, a_1\} = a_2$, and $\dot{a}_j = \sum_l \{a_j, a_l\}\partial\bar{\mathcal{H}}/\partial a_l$. So the equations of motion become

$$\begin{aligned} \dot{a}_1 &= a_2 + 3\varepsilon^3 \mu L^2 a_2 a_3 + \cdots , \\ \dot{a}_2 &= -a_1 + \tfrac{9}{2}\varepsilon^3 \mu L^2 a_3 a_1 + \cdots , \\ \dot{a}_3 &= -\tfrac{15}{2}\varepsilon^3 \mu L^2 a_1 a_2 + \cdots . \end{aligned} \tag{17}$$

3.2. Analysis of equilibria. We now apply the results of section 2 to the Hamiltonians for the planar lunar problem. Just from the facts that $B = S^2 = \{a_1^2 + a_2^2 + a_3^2 = L^2\}$, $H^1(S^2) = 0$, and the Lusternik–Schnirelmann category of S^2 is 2, by Weinstein's theorem, Theorem 2.7, we conclude that there are at least two periodic solutions of the corresponding flow defined by Y_ε with period near $T = 2\pi L^3$. This applies to any (small) perturbation of the planar Kepler problem.

Looking at the Hamiltonian on B yields more information about these periodic solutions. The Hamiltonian (16) has two nondegenerate critical points, a maximum at $\mathbf{a} = (0, 0, -L)$ and a minimum at $\mathbf{a} = (0, 0, L)$, which by Reeb's theorem, Theorem 2.2, and Corollaries 2.2 and 2.3 correspond to elliptic periodic solutions of the planar restricted three-body problem of period $T(\varepsilon) = T + O(\varepsilon^3)$. (Note that $(0, 0, -L)$ and $(0, 0, L)$ are parametrically stable points according to Corollary 2.3, as they are respectively a minimum and a maximum.) These are the classical Hill's orbits of the restricted problem, which are the continuation of the circular solutions of the Kepler problem (see [12, 36] and the references therein). The maximum gives the prograde orbit, which is located at the north pole of the sphere in Figure 4 (it is represented by a red point), and the minimum provides the retrograde orbit (the south pole in Figure 4). The index of $(0, 0, -L)$ is 2, whereas the index of $(0, 0, L)$ is 0. Hence $C_0 = C_2 = 1$ and $C_j = 0$ for $j \notin \{0, 2\}$. The Betti numbers of S^2 are $\beta_0 = \beta_2 = 1$, and the others are zero. Moreover, the Euler–Poincaré characteristic of S^2 is 2, which is consistent with the Betti and C_j numbers. Thus, for all j, $C_j = \beta_j$, and the Morse inequalities (given in Corollary 2.1) become equalities. Note that in this case the Lusternik–Schnirelmann category and the Euler–Poincaré characteristic of S^2 yield the same estimate, which coincides with the number of critical points of the Hamiltonian $\bar{\mathcal{H}}$.

Since $\bar{\mathcal{H}} = -a_3 + \cdots$ the linearized equations about $(0, 0, \pm L)$ are

$$\dot{a}_1 = a_2, \qquad \dot{a}_2 = -a_1,$$

and so the characteristic exponents at these critical points are $\pm i$ (see Figure 4).

Thus, these near-circular periodic solutions are elliptic with characteristic multipliers 1, 1, $1 + \varepsilon^3 T i + O(\varepsilon^6)$, *and* $1 - \varepsilon^3 T i + O(\varepsilon^6)$.

As a last step, we have to undo the initial scalings and the shift to return to the Hamiltonian $\mathcal{H}$. Taking into account that the periodic solutions are near-circular, they have approximate radii $|x| \approx L^2$ and periods near $2\pi L^3$. Hence, because of the scalings, we conclude that the periodic solutions of $\mathcal{H}$ have radii $|x| \approx \varepsilon^2 L^2$ and periods $T(\varepsilon) \approx 2\pi\varepsilon^3 L^3$.

3.3. A twist condition. To see if Theorem 2.5 applies at $(0, 0, \pm L)$ we need several changes of variables. We start by moving the equilibria $(0, 0, \pm L)$ to the origin of a coordinate system. Therefore, we define

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \bar{a}_3 = a_3 \mp L,$$

and then we introduce (local) symplectic coordinates Q and P as

$$\begin{aligned} Q &= \sqrt{2}\frac{L\,\bar{a}_1}{\sqrt{2\,L \pm \bar{a}_3}} = \sqrt{2}\sqrt{L \mp G}\cos g, \\ P &= \pm\sqrt{2}\frac{L\,\bar{a}_2}{\sqrt{2\,L \pm \bar{a}_3}} = \pm\sqrt{2}\sqrt{L \mp G}\sin g. \end{aligned}$$

By recalling that (ℓ, g, L, G) are symplectic variables, it is almost straightforward to check that $\{Q, P\} = 1$; thus Q has the role of a coordinate, whereas P corresponds to its conjugate momentum. These coordinates are valid in the hemispheres $\pm a_3 > 0$ (i.e., $\pm G < L$).

Now, to write $\bar{\mathcal{H}}$ in these coordinates, first note that

$$\frac{1}{2}(Q^2 + P^2) = L \mp G = L \mp a_3,$$

and also

$$a_1^2 = \frac{Q^2}{2L^2}(L \pm a_3), \qquad a_2^2 = \frac{P^2}{2L^2}(L \pm a_3).$$

Making this change of variables and dropping additive constants gives

$$\bar{\mathcal{H}} = \pm\frac{1}{2}(Q^2 + P^2) - \frac{3}{16}\varepsilon^3\mu(2P^2 - 3Q^2)(P^2 + Q^2 - 4L) + \cdots .$$

Change to action-angle variables by

$$Q = \sqrt{2I_1}\cos\theta_1, \qquad P = \sqrt{2I_1}\sin\theta_1$$

(note that $dQ \wedge dP = dI_1 \wedge d\theta_1$) to get

$$\bar{\mathcal{H}} = \pm I_1 - \frac{3}{4}\varepsilon^3\mu\, I_1\,(2L - I_1)\,(-2 + 5\cos^2\theta_1) + \cdots ,$$

and then average over θ_1 to get

$$\bar{\mathcal{H}} = \pm I_1 - \frac{3}{8}\varepsilon^3\mu\, I_1\,(2L - I_1) + \cdots .$$

Note that the second derivative of $\bar{\mathcal{H}}$ with respect to I_1 is

$$\frac{\partial^2\bar{\mathcal{H}}}{\partial I_1^2} = \frac{3}{4}\varepsilon^3\,\mu, \tag{18}$$

and it does not vanish. Thus there is a twist term, but the hypothesis of Theorem 2.5 does not hold, as there is an additional ε^3 in front of the twist term.

This suggests, but does not prove, that these near-circular periodic solutions are stable and enclosed by invariant KAM tori.

We push the normalization up to order ε^8 in order to prove that the periodic solutions associated with the equilibria $(0, 0, \pm L)$ are not circular but have a small eccentricity. The terms factorized by ε^8 are

$$\tfrac{5}{32}\varepsilon^8\,\mu\,(1-\mu)^{1/3}\,e\,L^4\cos g\left(13\,G^2 - 7\,L^2 - 35\,(G^2 - L^2)\cos(2\,g)\right). \tag{19}$$

Now, after incorporating these terms into the Hamiltonian $\mathcal{H}_\varepsilon$ given by (15), the equilibria $(0, 0, \pm L)$ are transformed to

$$\left(\mp\frac{15}{16}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6,\quad 0,\quad \pm\frac{\sqrt{256L^2 - 225\varepsilon^{10}\mu^2(1-\mu)^{2/3}L^{12}}}{16}\right).$$

Now the above equilibria do not correspond to circular solutions because their eccentricity is given by $e = \frac{15}{16}\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^5+\cdots$. The magnitude of their angular momentum vector is $G = \pm L \mp \frac{225}{512}\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{11}+\cdots$. This implies that the periodic solutions associated with $(0,0,\pm L)$ are indeed elliptic periodic solutions whose projections onto configuration space yield elliptic orbits with eccentricity close to zero. The inclination is zero for the periodic solution related to $(0,0,L)$, while it is π for the periodic solution related to $(0,0,-L)$. This proves that up to terms of order ε^8 the periodic solutions are near-circular periodic solutions.

Thus, these equilibria correspond to near-circular elliptic periodic orbits.

3.4. Continuation of elliptic orbits.

The planar restricted three-body problem is symmetric in the line of syzygy; i.e., $R : (x_1, x_2, y_1, y_2) \to (x_1, -x_2, -y_1, y_2)$ is an antisymplectic involution that leaves the Hamiltonian (13) invariant. The Lagrangian subspace $F = \{(x_1, 0, 0, y_2)\}$ corresponds to orthogonal crossings of the line of syzygy. In Delaunay variables $R : (\ell, g, L, G) \to (-\ell, -g, L, G)$ and $F = \{(0, 0, L, G)\}$.

On the reduced space $\bar{R} : (a_1, a_2, a_3) \to (a_1, -a_2, a_3)$ or $\bar{R} : (Q, P) \to (Q, -P)$. The Lagrangian subspace $\bar{F}$ is the meridian circle $\{(a_1, 0, a_3)\}$ or $\{(Q, 0)\}$. A point on $\bar{F}$ corresponds to a symmetric elliptic orbit of the Kepler problem and the periodic solution on $\bar{\mathcal{H}} =$ constant corresponds to a family of precessing Keplerian ellipses which start and end at a symmetric ellipse.

By Theorem 2.6 the existence of symmetric periodic solutions which are the continuation of this family of precessing Keplerian ellipses is ensured because, according to (18), the condition $\frac{\partial^2\bar{\mathcal{H}}}{\partial I_1^2} \neq 0$ holds. These are the periodic solutions obtained by Arenstorf in [3].

4. The spatial lunar problem.

4.1. The Hamiltonians.

The Hamiltonian of the spatial problem is given in the rotating frame by

$$\mathcal{H} = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - (x_1 y_2 - x_2 y_1) - \frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2}} - \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}}.$$

We change variables, scale time, and scale the Hamiltonian in the same way as in the planar case in order to arrive to the lunar case of the spatial restricted circular three-body problem (see [25]). After expanding in powers of the small parameter, we end up with the system

$$\mathcal{H}_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} - \varepsilon^3\,(x_1 y_2 - x_2 y_1) + \frac{1}{2}\,\varepsilon^6\,\mu\,(-2x_1^2 + x_2^2 + x_3^2) + \cdots .$$

Now we have a perturbation of the spatial Kepler problem. Moser has shown that the three-dimensional Kepler problem can be regularized and the regularized flow is equivalent to the geodesic flow on S^3. We proceed just as in the planar problem to find and analyze the averaged equations on the reduced space.

The following step consists in expressing $\mathcal{H}_\varepsilon$ in such a way that we can perform Lie transformations conveniently (see [17]). We use polar-nodal coordinates $(r, \vartheta, \nu, R, G, N)$ and Delaunay coordinates (ℓ, g, ν, L, G, N). The angle ϑ is the argument of the latitude, and ν is the argument of the node. The coordinate R is the momentum conjugate to the radial variable r, $G = |\mathbf{G}|$ is the magnitude of angular momentum, and N is the third component of the angular momentum $\mathbf{G}$, so $0 \le |N| \le G \le L$. Expressing $\mathcal{H}_\varepsilon$ in these variables, we get

$$\begin{aligned}\mathcal{H}_\varepsilon = -\frac{1}{2\,L^2} - \varepsilon^3\,N + \frac{1}{8}\,\varepsilon^6\,\mu\,r^2\,\Big(1 - 3\,c^2 - 3\,(1-c^2)\cos(2\,\vartheta)\\ - 3\,\big(1 - c^2 + (1+c^2)\cos(2\,\vartheta)\big)\cos(2\,\nu) + 6\,c\sin(2\,\nu)\sin(2\,\vartheta)\Big) + \cdots\,,\end{aligned}$$

where $c = N/G$. After performing the normalization of Delaunay to a fixed finite order, we arrive at the Hamiltonian

$$\begin{aligned}\mathcal{H}_\varepsilon = -\frac{1}{2\,L^2} - \varepsilon^3 N + \frac{1}{16}\varepsilon^6\,\mu\,L^4\,\Big((2+3\,e^2)\,\big(1 - 3\,c^2 - 3\,(1-c^2)\cos(2\,\nu)\big)\\ - 15\,e^2\cos(2\,g)\big(1 - c^2 + (1+c^2)\cos(2\,\nu)\big) + 30\,c\,e^2\sin(2\,g)\sin(2\,\nu)\Big) + \cdots\,,\end{aligned} \tag{20}$$

where $e = \sqrt{1 - G^2/L^2}$. This normal form Hamiltonian was calculated previously in [42]. The transformed Hamiltonian, after truncating higher-order terms, depends on the two angles g and ν and their associated momenta G and N, respectively, whereas L is an integral of motion. Applying reduction theory, once higher-order terms have been dropped, $\mathcal{H}_\varepsilon$ is defined on the orbit space, or base space, which is the four-dimensional space $S^2 \times S^2$ [39].

We can use the set of variables given by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with the constraints $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$ to parameterize $S^2 \times S^2$, where $\mathbf{a} = \mathbf{G} + LA$ and $\mathbf{b} = \mathbf{G} - LA$. We recall that $\mathbf{G}$ is the angular momentum vector and A is the Laplace–Runge–Lenz vector; moreover, $|\mathbf{a}| = |\mathbf{b}| = L$. Notice that the a_i and b_i belong to the interval $[-L, L]$. The explicit expressions for $\mathbf{a}$ and $\mathbf{b}$ in terms of Delaunay variables are found in Coffey, Deprit, and Miller [11] and in Cushman [15].

In particular, $2G = ((a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2)^{1/2}$, so $G = 0$ in $S^2 \times S^2$ if and only if $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 \equiv 0$, $a_1^2 + a_2^2 + a_3^2 = L^2$, and $b_1^2 + b_2^2 + b_3^2 = L^2$. Thus, the subset of $S^2 \times S^2$ given by $\mathcal{R} = \{(\mathbf{a}, -\mathbf{a}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}$ is a two-dimensional set homeomorphic to S^2 consisting of the rectilinear trajectories. In Delaunay elements the circular orbits satisfy the condition $G = L$, and in terms of $\mathbf{a}$ and $\mathbf{b}$ this implies that $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. So the circular orbits define the two-dimensional set homeomorphic to S^2 given by $\mathcal{C} = \{(\mathbf{a}, \mathbf{a}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}$. Similarly, equatorial trajectories satisfy $G = |N|$ and are given by the two-dimensional set $\mathcal{E} = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2,$ $b_1 = -a_1,\ b_2 = -a_2,\ b_3 = a_3\}$, which is again homeomorphic to S^2. Just as in the planar case, the introduction of these invariants extends the use of the Delaunay variables, as we can include equatorial, circular, and rectilinear solutions [41]. The other points on $S^2 \times S^2$ correspond to elliptic orbits of the Kepler problem.

After several simplifications and manipulations over $\mathcal{H}_\varepsilon$, including the dropping of the constant term $-1/(2L^2)$ and division by ε^3, we arrive at

$$\begin{aligned}\bar{\mathcal{H}} = -\frac{1}{2}\,(a_3 + b_3) - \frac{1}{8}\,\varepsilon^3\,\mu\,L^2\,\big(3\,a_1^2 - 3\,a_2^2 - 3\,a_3^2 - 12\,a_1\,b_1 + 3\,b_1^2\\ + 6\,a_2\,b_2 - 3\,b_2^2 + 6\,a_3\,b_3 - 3\,b_3^2\big) + \cdots\,.\end{aligned} \tag{21}$$

The Poisson structure on $S^2 \times S^2$ in these coordinates is

$$\begin{aligned} &\{a_1, a_2\} = 2\,a_3, \qquad \{a_2, a_3\} = 2\,a_1, \qquad \{a_3, a_1\} = 2\,a_2, \\ &\{b_1, b_2\} = 2\,b_3, \qquad \{b_2, b_3\} = 2\,b_1, \qquad \{b_3, b_1\} = 2\,b_2, \qquad \{a_i, b_j\} = 0. \end{aligned}$$

The corresponding equations of motion are

$$\begin{aligned} \dot{a_1} &= a_2 - \tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(a_3\,b_2 - a_2\,b_3) + \cdots\,, \\ \dot{a_2} &= -a_1 + \tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(2a_1\,a_3 - 2\,a_3\,b_1 - a_1\,b_3) + \cdots\,, \\ \dot{a_3} &= -\tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(2\,a_1\,a_2 - 2\,a_2\,b_1 - a_1\,b_2) + \cdots\,, \\ \dot{b_1} &= b_2 + \tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(a_3\,b_2 - a_2\,b_3) + \cdots\,, \\ \dot{b_2} &= -b_1 - \tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(a_3\,b_1 + 2\,a_1\,b_3 - 2\,b_1\,b_3) + \cdots\,, \\ \dot{b_3} &= \tfrac{3}{2}\,\varepsilon^3\,\mu\,L^2\,(a_2\,b_1 + 2\,a_1\,b_2 - 2\,b_1\,b_2) + \cdots\,. \end{aligned} \tag{22}$$

We stress that the equations of motion are global in the whole base space B. Including terms of order ε^3 is enough to determine the relative equilibria of $\bar{\mathcal{H}}$.

4.2. Analysis of equilibria. Let us now turn to the application of the results of section 2 to the spatial lunar problem. Just from the facts that $B = S^2 \times S^2 = \{a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2\}$, $H^1(S^2 \times S^2) = 0$, and the Lusternik–Schnirelmann category of $S^2 \times S^2$ is 3, by Weinstein's theorem, Theorem 2.7, we can conclude that there are at least three periodic solutions of the corresponding flow defined by Y_ε with period near $T = 2\pi L^3$. This holds for any perturbation of the spatial Kepler problem.

Looking at the Hamiltonian on B yields more information about these periodic solutions. The Hamiltonian (21) starts as $\bar{\mathcal{H}} = -\frac{1}{2}(a_3 + b_3) + \cdots$, so it has a nondegenerate maximum at $(\mathbf{a}, \mathbf{b}) = (0, 0, -L, 0, 0, -L)$ and a nondegenerate minimum at $(\mathbf{a}, \mathbf{b}) = (0, 0, L, 0, 0, L)$, which by Reeb's theorem, Theorem 2.2, and Corollary 2.2 correspond to elliptic periodic solutions of the spatial restricted three-body problem of period $T(\varepsilon) = T + O(\varepsilon^3)$. These are the circular equatorial motions already encountered in the planar case. It also has two nondegenerate critical points of index 2 at $(\mathbf{a}, \mathbf{b}) = (0, 0, \pm L, 0, 0, \mp L)$ which correspond to rectilinear motions whose projection in the coordinate space leads to periodic orbits in the vertical axis x_3. They correspond to the rectilinear trajectories found by Belbruno [6] for small μ.

The Betti numbers of $S^2 \times S^2$ are $\beta_0 = \beta_4 = 1$, $\beta_2 = 2$, and all the others are zero. As we have seen, $\bar{\mathcal{H}}$ is a Morse function and has the minimum number of critical points consistent with the Morse inequalities found in Corollary 2.1.

Near the critical points we can use (a_1, a_2, b_1, b_2) as coordinates on $B = S^2 \times S^2$. From the equations (22) one sees that the characteristic exponents of all four critical points of Y_ε at the four equilibria are $\pm i$ (double). Thus, by Corollary 2.2, the characteristic multipliers of the corresponding periodic solutions are 1, 1, $1 + \varepsilon^3\,T\,i$, $1 + \varepsilon^3\,T\,i$, $1 - \varepsilon^3\,T\,i$, and $1 - \varepsilon^3\,T\,i$ plus terms of order ε^6. As we have said, the maxima and minima at $(0, 0, \pm L, 0, 0, \pm L)$ give rise to elliptic periodic solutions, but since the minimax critical points at $(0, 0, \pm L, 0, 0, \mp L)$ have not been shown to be parametrically stable, we cannot conclude at this point that they give rise to elliptic periodic solutions. The deeper analysis of the next subsection is needed to decide the stability of those periodic solutions arising from the minimax critical points.

4.3. Linear stability and the twist condition. The aim of this section is the analysis of the linear stability of the families of periodic solutions established before, using the methods given in section 2. We also check that the twist condition needed for the possible existence of invariant tori is too degenerate. Finally, we also deal with the nonlinear stability of the four critical points of $S^2 \times S^2$. We start with the points related to the periodic near-rectilinear solutions.

4.3.1. Points $(0, 0, \pm L, 0, 0, \mp L)$. After moving the origin to the point of interest through

$$a_1 = \bar{a}_1, \quad a_2 = \bar{a}_2, \quad a_3 = \bar{a}_3 \pm L, \quad b_1 = \bar{b}_1, \quad b_2 = \bar{b}_2, \quad b_3 = \bar{b}_3 \mp L,$$

we introduce the local transformation

$$\begin{aligned} Q_1 &= \frac{\bar{a}_2}{\sqrt{\pm\bar{a}_3 + 2L}}, & Q_2 &= \frac{\bar{b}_2}{\sqrt{\mp\bar{b}_3 + 2L}}, \\ P_1 &= \mp\frac{\bar{a}_1}{\sqrt{\pm\bar{a}_3 + 2L}}, & P_2 &= \pm\frac{\bar{b}_1}{\sqrt{\mp\bar{b}_3 + 2L}}, \end{aligned}$$

with inverse

$$\begin{aligned} \bar{a}_1 &= \mp P_1\sqrt{2L - P_1^2 - Q_1^2}, & \bar{a}_2 &= Q_1\sqrt{2L - P_1^2 - Q_1^2}, & \bar{a}_3 &= \mp(P_1^2 + Q_1^2), \\ \bar{b}_1 &= \pm P_2\sqrt{2L - P_2^2 - Q_2^2}, & \bar{b}_2 &= Q_2\sqrt{2L - P_2^2 - Q_2^2}, & \bar{b}_3 &= \pm(P_2^2 + Q_2^2). \end{aligned}$$

The variables (Q_1, Q_2, P_1, P_2) are a canonical set for which Q_1, Q_2 can be interpreted as coordinates, whereas P_1 and P_2 represent their associated momenta, respectively.

The resulting Hamiltonian is obtained after putting $\bar{\mathcal{H}}$ in terms of Q_i and P_i and dropping constant terms. We get

$$\begin{aligned} \bar{\mathcal{H}} \;=\; & \pm\tfrac{1}{2}(P_1^2 + Q_1^2) \mp \tfrac{1}{2}(P_2^2 + Q_2^2) - \tfrac{3}{4}\varepsilon^3\,\mu\,L^2\,\big(3L\,(P_1^2 + P_2^2) + L\,(Q_1^2 + Q_2^2) \\ & + (2P_1\,P_2 + Q_1\,Q_2)\sqrt{2\,L - P_1^2 - Q_1^2}\,\sqrt{2\,L - P_2^2 - Q_2^2} \\ & - (P_2^2 + Q_1^2)\,(P_2^2 + Q_2^2) - P_1^2\,(P_1^2 + P_2^2 + Q_1^2 + Q_2^2)\big) + \cdots . \end{aligned}$$

The Hamiltonian $\bar{\mathcal{H}}$ is valid in a neighborhood of the points $(0, 0, \pm L, 0, 0, \mp L)$.

Next we scale variables through the change $\bar{Q}_j = \varepsilon^{-3/2}\,Q_j$ and $\bar{P}_j = \varepsilon^{-3/2}\,P_j$ for $j \in \{1, 2\}$. To make the change canonical we must divide $\bar{\mathcal{H}}$ by ε^3. Expanding this Hamiltonian in powers of ε (and keeping the same name for it), we arrive at the Hamiltonian

$$\begin{aligned} \bar{\mathcal{H}} \;=\; & \pm\tfrac{1}{2}(\bar{P}_1^2 + \bar{Q}_1^2) \mp \tfrac{1}{2}(\bar{P}_2^2 + \bar{Q}_2^2) - \tfrac{3}{4}\varepsilon^3\,\mu\,L^3\,\big(3(\bar{P}_1^2 + \bar{P}_2^2) + 4\bar{P}_1\,\bar{P}_2 + \bar{Q}_1^2 + \bar{Q}_2^2 + 2\bar{Q}_1\,\bar{Q}_2\big) \\ & + \tfrac{3}{8}\varepsilon^6\,\mu\,L^2\,\big(2(\bar{P}_1^4 + \bar{P}_1^3\,\bar{P}_2 + \bar{P}_1^2\,\bar{P}_2^2 + \bar{P}_1\,\bar{P}_2^3 + \bar{P}_2^4) + 2\bar{P}_2\,(\bar{P}_1 + \bar{P}_2)\,\bar{Q}_1^2 \\ & + (\bar{P}_1^2 + \bar{P}_2^2)\,\bar{Q}_1\,\bar{Q}_2 + 2(\bar{P}_1^2 + \bar{P}_1\,\bar{P}_2 + \bar{P}_2^2)\,\bar{Q}_2^2 + \bar{Q}_1\,\bar{Q}_2\,(\bar{Q}_1 + \bar{Q}_2)^2\big) + \cdots . \end{aligned}$$

The eigenvalues associated with the linear differential equation given through the quadratic part of $\bar{\mathcal{H}}$ are the expressions

$$\pm\sqrt{1 + 20\bar{\varepsilon}^2 + 2\sqrt{5}\bar{\varepsilon}\sqrt{3 + 20\bar{\varepsilon}^2}}\;i = \pm\omega_1\,i, \quad \pm\sqrt{1 + 20\bar{\varepsilon}^2 - 2\sqrt{5}\bar{\varepsilon}\sqrt{3 + 20\bar{\varepsilon}^2}}\;i = \pm\omega_2\,i, \tag{23}$$

where $\bar{\varepsilon}$ stands for $\frac{3}{4}\varepsilon^3\,\mu\,L^3$ and $\omega_1 > 1 > \omega_2 > 0$. Note that $\omega_1 = \omega_2 = 1$ when $\varepsilon = 0$, and the quadratic part of $\bar{\mathcal{H}}$ is in 1-1 resonance. However, we now see that when $\varepsilon \neq 0$ the eigenvalues are distinct.

These equilibria are parametrically stable and correspond to elliptic periodic solutions.

We keep ε small but positive so that we may perform further normalization. By doing so, both ω_1 and ω_2 remain close to 1 but different from it. As the corresponding set of eigenvectors forms a basis of $\mathbb{R}^4$, the quadratic part of $\bar{\mathcal{H}}$ may be brought into normal form through a canonical change of variables. This linear change has to be applied to $\bar{\mathcal{H}}$. The columns of the transformation matrix are the eigenvectors related to $\pm\omega_1 i$ and $\pm\omega_2 i$ multiplied by scale constants chosen to make the change symplectic. We do not give the explicit expression for this change because it is lengthy and the procedure is standard; see, for instance, [10, 30]. Defining the new variables by (q_1, q_2, p_1, p_2) and using the same name for the Hamiltonian, its quadratic part becomes

$$\pm\omega_1\, i\, q_1\, p_1 \mp \omega_2\, i\, q_2\, p_2.$$

Next we introduce action-angle variables $(I_1, I_2, \varphi_1, \varphi_2)$ by means of

$$\begin{aligned} q_1 &= \sqrt{I_1/\omega_1}\,(\cos\varphi_1 - i\sin\varphi_1), \qquad & q_2 &= \sqrt{I_2/\omega_2}\,(\cos\varphi_2 - i\sin\varphi_2),\\ p_1 &= \sqrt{\omega_1\, I_1}\,(\sin\varphi_1 - i\cos\varphi_1), \qquad & p_2 &= \sqrt{\omega_2\, I_2}\,(\sin\varphi_2 - i\cos\varphi_2). \end{aligned}$$

It is easy to check that $dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = dI_1 \wedge d\varphi_1 + dI_2 \wedge d\varphi_2$. This transformation brings the quadratic terms of $\bar{\mathcal{H}}$ to $\pm\omega_1\, I_1 \mp \omega_2\, I_2$, while its quartic terms are converted into a finite Fourier series in φ_1 and φ_2 whose coefficients are homogeneous quadratic polynomials in I_1 and I_2. We do not give the Hamiltonian because it is enormous.

Now we average $\bar{\mathcal{H}}$ over φ_1 and φ_2, arriving in both cases at

$$\begin{aligned} \bar{\mathcal{H}} \quad = \quad & \pm\omega_1\, I_1 \mp \omega_2\, I_2 + \frac{(7\omega_1^6 + 13\omega_1^4 + 13\omega_1^2 + 3)(\omega_1^2 - 1)^2}{30\mu\, L^4\, \omega_1^2\, (\omega_1^2 + 2)^2\, (2\omega_1^2 + 1)}\, I_1^2 \\ & + \frac{2(\omega_1^2 - 1)^2\, (\omega_1^4 - 14\omega_1^2 - 5)\, (2\omega_2^2 + 1)}{135\mu\, L^4\, \omega_1\, (\omega_1^2 + 2)^2\, \omega_2}\, I_1\, I_2 \\ & + \frac{(7\omega_2^6 + 13\omega_2^4 + 13\omega_2^2 + 3)(\omega_2^2 - 1)^2}{30\mu\, L^4\, \omega_2^2\, (\omega_2^2 + 2)^2\, (2\omega_2^2 + 1)}\, I_2^2 + \cdots . \end{aligned}$$

The coefficients of I_1^2, I_2^2 and I_1, I_2 may be expressed in terms of $\bar{\varepsilon}$, and, after expanding them in powers of $\bar{\varepsilon}$ about 0, one obtains a formula starting in $\bar{\varepsilon}^2$. The generating function responsible for this averaging step is too big to be reproduced here, but it is a finite Fourier series in the angles φ_1 and φ_2.

Now we can compute the determinant of the Hessian associated with $\bar{\mathcal{H}}$. Using the constraint which relates ω_1 and ω_2 through (23) given by

$$\omega_2 = \sqrt{\frac{4 - \omega_1^2}{2\omega_1^2 + 1}},$$

we get

$$\det\begin{bmatrix} \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_1^2} & \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_1\partial I_2} \\ \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_2\partial I_1} & \dfrac{\partial^2\bar{\mathcal{H}}}{\partial I_2^2} \end{bmatrix} = \frac{(\omega_1^2-1)^6\,(7\omega_1^8-28\omega_1^6-534\omega_1^4-604\omega_1^2-137)}{225\mu^2\,L^8\,\omega_1^2\,(\omega_1^2-4)\,(\omega_1^2+2)^4\,(2\omega_1^2+1)^2}+\cdots,$$

which does not vanish since the (positive) real roots of the determinant occur for $\omega_1 = 1$ or $\omega_1 = 3.37369\ldots$, but as ε does not vanish, ω_1 remains greater than 1. Unfortunately, Theorem 2.5 does not apply since the twist condition is at a higher order in ε.

This suggests, but does not prove, that there are families of invariant 3*-tori around these periodic solutions.*

We leave the question of the existence of invariant KAM tori about these periodic solutions to a future paper. However, we can say something about the stability of the equilibria $(0,0,\pm L,0,0,\mp L)$ of the reduced system on the base space.

For the analysis of the stability of these equilibria we use Arnold's theorem [44]. We fix ε small and positive for this analysis. We need to find $\bar{\mathcal{H}}_4$, i.e., the quartic terms of $\bar{\mathcal{H}}$, and then compute

$$\begin{aligned} \bar{\mathcal{H}}_4(-\omega_2,\omega_1) &= \det\begin{bmatrix} \dfrac{\partial^2\bar{\mathcal{H}}_4}{\partial I_1^2} & \dfrac{\partial^2\bar{\mathcal{H}}_4}{\partial I_1\partial I_2} & \omega_1 \\ \dfrac{\partial^2\bar{\mathcal{H}}_4}{\partial I_2\partial I_1} & \dfrac{\partial^2\bar{\mathcal{H}}_4}{\partial I_2^2} & \omega_2 \\ \omega_1 & \omega_2 & 0 \end{bmatrix} \\ &= \frac{(\omega_1^2-1)^2\,(\omega_1^{12}-16\,\omega_1^{10}+66\,\omega_1^8-268\,\omega_1^6-275\,\omega_1^4-132\,\omega_1^2-24)}{15\,\mu\,L^4\,\omega_1^2\,(\omega_1^2-4)\,(2\,\omega_1^4+5\,\omega_1^2+2)^2}. \end{aligned}$$

Since this term does not vanish for ω_1 close to (but larger than) 1, Arnold's theorem applies, and so the following statement holds.

The equilibrium points $(0,0,\pm L,0,0,\mp L)$ *are stable on the reduced space* $S^2\times S^2$*.*

Now, if higher-order terms are included in the Hamiltonian (20), we see that the equilibria $(0,0,\pm L,0,0,\mp L)$ are distorted a bit. Specifically, the terms factorized by ε^8 are

$$\begin{aligned} &\tfrac{5}{64}\,\varepsilon^8\,\mu\,(1-\mu)^{1/3}\,e^2\,L^6\,(\cos g\cos h - c\sin g\sin h) \\ &\quad\times\Big(-18-31\,e^2+5\,c^2\,(6+e^2)+5\,(1-c^2)\,(6+e^2)\cos(2\,h) \\ &\qquad+35\,e^2\cos(2\,g)\,(1-c^2+(1+c^2)\cos(2\,h))-70\,c\,e^2\sin(2\,g)\sin(2\,h)\Big). \end{aligned} \tag{24}$$

Thus, after incorporating terms of order ε^8, the equilibria are transformed to

$$\begin{aligned} &\Big(\pm\tfrac{105}{16}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6,\quad 0,\quad \pm\tfrac{\sqrt{256\,L^2-11025\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{12}}}{16}, \\ &\ \pm\tfrac{105}{16}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6,\quad 0,\quad \mp\tfrac{\sqrt{256\,L^2-11025\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{12}}}{16}\Big). \end{aligned}$$

Hence, it is not difficult to deduce that these equilibria correspond to near-rectilinear solutions whose eccentricity is given by $e = 1-\frac{11025}{512}\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{10}+\cdots$. The magnitude of their angular momentum vector is $G = \frac{105}{16}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6 + \cdots$, and its third component is $N = -\frac{33075}{512}\,\varepsilon^{13}\,\mu^3\,(1-\mu)^{2/3}\,L^{14} + \cdots$. This implies that the periodic solutions associated with $(0, 0, \pm L, 0, 0, \mp L)$ are indeed elliptic periodic solutions such that their projections in configuration space yield elliptic orbits with eccentricity close to 1 and inclination angles given by $\pm\cos^{-1}(-\frac{315}{32}\,\varepsilon^8\,\mu^2\,(1-\mu)^{1/3}\,L^8 + \cdots)$.

Thus, these equilibria correspond to elliptic periodic orbits close to rectilinear orbits.

Moreover, it can be proved that the projection of the periodic orbits onto configuration space lies in the plane defined by x_2 and x_3. More precisely, in (the averaged) Cartesian variables $x_1, x_2, x_3, y_1, y_2, y_3$, the coordinates of these periodic orbits up to terms of order ε^{10} are

$$\left(0, \pm\frac{105}{8y_3}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6, \mp\frac{2}{y_3^2} \pm \frac{33075}{512}\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\,L^{12}, 0, -\frac{105}{32}\,\varepsilon^5\,\mu\,(1-\mu)^{1/3}\,L^6\,y_3^2, y_3\right).$$

We remark that y_3 acts as the parameter of the periodic solution.

Next, the points related to the periodic near-circular equatorial solutions are analyzed.

4.3.2. Points $(0, 0, \pm L, 0, 0, \pm L)$. We first move the Hamiltonian to the origin by

$$a_1 = \bar{a}_1, \quad a_2 = \bar{a}_2, \quad a_3 = \bar{a}_3 \pm L, \quad b_1 = \bar{b}_1, \quad b_2 = \bar{b}_2, \quad b_3 = \bar{b}_3 \pm L;$$

then we change variables by

$$\begin{aligned} Q_1 &= \frac{\bar{a}_2}{\sqrt{\pm\bar{a}_3 + 2L}}, & Q_2 &= \frac{\bar{b}_2}{\sqrt{\pm\bar{b}_3 + 2L}}, \\ P_1 &= \mp\frac{\bar{a}_1}{\sqrt{\pm\bar{a}_3 + 2L}}, & P_2 &= \mp\frac{\bar{b}_1}{\sqrt{\pm\bar{b}_3 + 2L}}, \end{aligned}$$

with inverse

$$\begin{aligned} \bar{a}_1 &= \mp P_1\sqrt{2L - P_1^2 - Q_1^2}, & \bar{a}_2 &= Q_1\sqrt{2L - P_1^2 - Q_1^2}, & \bar{a}_3 &= \mp(P_1^2 + Q_1^2), \\ \bar{b}_1 &= \mp P_2\sqrt{2L - P_2^2 - Q_2^2}, & \bar{b}_2 &= Q_2\sqrt{2L - P_2^2 - Q_2^2}, & \bar{b}_3 &= \mp(P_2^2 + Q_2^2). \end{aligned}$$

The change of variables is canonical, with Q_1 and Q_2 as coordinates and P_1 and P_2 as their associated momenta.

The resulting Hamiltonian is obtained after writing $\bar{\mathcal{H}}$ in terms of Q_i and P_i and dropping constant terms, so

$$\begin{aligned} \bar{\mathcal{H}} \;=\; & \pm\tfrac{1}{2}(P_1^2 + Q_1^2) \pm \tfrac{1}{2}(P_2^2 + Q_2^2) - \tfrac{3}{4}\varepsilon^3\,\mu\,L^2\,\big(L\,(P_1^2 + P_2^2) - L(Q_1^2 + Q_2^2) \\ & - (2P_1\,P_2 - Q_1\,Q_2)\,\sqrt{2\,L - P_1^2 - Q_1^2}\,\sqrt{2\,L - P_2^2 - Q_2^2} \\ & - (P_2^2 - Q_1^2)\,(P_2^2 + Q_2^2) - P_1^2\,(P_1^2 - P_2^2 + Q_1^2 - Q_2^2)\big) + \cdots . \end{aligned}$$

The Hamiltonian $\bar{\mathcal{H}}$ is valid in a neighborhood of $(0, 0, \pm L, 0, 0, \mp L)$.

Now we scale by $\bar{Q}_j = \varepsilon^{-3/2}\, Q_j$ and $\bar{P}_j = \varepsilon^{-3/2}\, P_j$ for $j \in \{1,2\}$. The canonical structure is preserved by dividing $\bar{\mathcal{H}}$ by ε^3. After expansion of this Hamiltonian in powers of ε we obtain

$$\begin{aligned}\bar{\mathcal{H}} \;=\; & \pm\tfrac{1}{2}(\bar{P}_1^2+\bar{Q}_1^2) \pm \tfrac{1}{2}(\bar{P}_2^2+\bar{Q}_2^2) - \tfrac{3}{4}\varepsilon^3\,\mu\, L^3\left(\bar{P}_1^2+\bar{P}_2^2-4\bar{P}_1\,\bar{P}_2-\bar{Q}_1^2-\bar{Q}_2^2+2\bar{Q}_1\,\bar{Q}_2\right)\\ & + \tfrac{3}{8}\varepsilon^6\,\mu\, L^2\left(2(\bar{P}_1^4-\bar{P}_1^3\,\bar{P}_2-\bar{P}_1^2\,\bar{P}_2^2-\bar{P}_1\,\bar{P}_2^3+\bar{P}_2^4)+(\bar{P}_1^2+\bar{P}_2^2)\,\bar{Q}_1\,\bar{Q}_2\right.\\ & \left. + 2(\bar{P}_1^2-\bar{P}_1\,\bar{P}_2+\bar{P}_2^2)\,(\bar{Q}_1^2-\bar{Q}_2^2)+\bar{Q}_1\,\bar{Q}_2\,(\bar{Q}_1-\bar{Q}_2)^2\right)+\cdots\,.\end{aligned}$$

For $(0,0,L,0,0,L)$ the eigenvalues associated with the linear differential equation given through the quadratic part of $\bar{\mathcal{H}}$ are

$$\pm\sqrt{1+2\bar{\varepsilon}}\, i = \pm\omega_1\, i, \quad \pm\sqrt{1-2\bar{\varepsilon}-24\bar{\varepsilon}^2}\, i = \pm\omega_2\, i \tag{25}$$

with $\bar{\varepsilon} = \frac{3}{4}\varepsilon^3\,\mu\, L^3$ and $\omega_1 > 1 > \omega_2 > 0$. For the point $(0,0,-L,0,0,-L)$ the eigenvalues are

$$\pm\sqrt{1-2\bar{\varepsilon}}\, i = \pm\omega_1\, i, \quad \pm\sqrt{1+2\bar{\varepsilon}-24\bar{\varepsilon}^2}\, i = \pm\omega_2\, i. \tag{26}$$

In this case $\omega_2 > 1 > \omega_1 > 0$. We remark that if $\varepsilon = 0$, then $\omega_1 = \omega_2 = 1$; thus the quadratic part of $\bar{\mathcal{H}}$ is in 1-1 resonance. So we keep ε small but positive so that we can apply KAM theory. As a consequence, ω_1 and ω_2 are close to 1 but different from it.

The eigenvectors related to ω_1 and ω_2 form a basis of $\mathbb{R}^4$; thus the quadratic part of $\bar{\mathcal{H}}$ is brought into normal form through a canonical change of variables. This linear change must be applied to $\bar{\mathcal{H}}$. The columns of the matrix are the eigenvectors scaled so that the matrix is symplectic. After defining the new variables by (q_1, q_2, p_1, p_2), the quadratic part of $\bar{\mathcal{H}}$ becomes

$$\pm\omega_1\, i\, q_1\, p_1 \pm \omega_2\, i\, q_2\, p_2.$$

The values of the frequencies ω_1 and ω_2 are given in (25) if the quadratic part is $\omega_1\, i\, q_1\, p_1 + \omega_2\, i\, q_2\, p_2$, whereas if the quadratic part is $-\omega_1\, i\, q_1\, p_1 - \omega_2\, i\, q_2\, p_2$, we take the frequencies from (26). From now on when we refer to $(0,0,L,0,0,L)$ we assume that ω_1 and ω_2 are as in (25), and when we study the point $(0,0,-L,0,0,-L)$ we take the frequencies from (26).

We have

$$\begin{aligned} q_1 &= \sqrt{I_1/\omega_1}\,(\cos\varphi_1 - i\sin\varphi_1), & q_2 &= \sqrt{I_2/\omega_2}\,(\cos\varphi_2 - i\sin\varphi_2),\\ p_1 &= \sqrt{\omega_1\, I_1}\,(\sin\varphi_1 - i\cos\varphi_1), & p_2 &= \sqrt{\omega_2\, I_2}\,(\sin\varphi_2 - i\cos\varphi_2),\end{aligned}$$

and the change satisfies $dq_1\wedge dp_1 + dq_2\wedge dp_2 = dI_1\wedge d\varphi_1 + dI_2\wedge d\varphi_2$. This transforms the quadratic terms of $\bar{\mathcal{H}}$ into $\pm\omega_1\, I_1 \pm \omega_2\, I_2$, while the quartic terms are converted into a finite Fourier series in φ_1 and φ_2 whose coefficients are homogeneous quadratic polynomials in I_1 and I_2.

Now we average $\bar{\mathcal{H}}$ over φ_1 and φ_2. For the two equilibria we obtain

$$\begin{aligned}\bar{\mathcal{H}} \;=\; & \omega_1\, I_1 + \omega_2\, I_2 - \frac{(\omega_1^2-1)^2\,(\omega_1^2+3)}{24\mu\, L^4\,\omega_1^2}\, I_1^2 - \frac{(\omega_1^2-1)^2\,(21\omega_1^2-13)}{6\mu\, L^4\,\omega_1\,\omega_2}\, I_1\, I_2\\ & - \frac{(6\omega_1^2-5)^2\,(48\omega_1^4+62\omega_1^2-93)}{1728\mu\, L^4\,\omega_2^2}\, I_2^2 + \cdots\,.\end{aligned}$$

In both cases the coefficients of I_1^2, I_2^2 and I_1, I_2 may be expressed in terms of $\bar{\varepsilon}$, and expanding them in powers of $\bar{\varepsilon}$ around 0 yields expressions starting in $\bar{\varepsilon}^2$. The generating functions computed in the averaging process in the two cases are enormous, but they are finite Fourier series in the angles φ_1 and φ_2.

At this point we can compute the determinants of the Hessian associated with $\bar{\mathcal{H}}$. First we calculate the constraint relating ω_1 to ω_2 through $\bar{\varepsilon}$ using (25) or (26), obtaining in both situations

$$\omega_2 = \sqrt{(2\omega_1^2 - 1)\,(-3\omega_1^2 + 4)}.$$

We end up with the same expression for the points $(0, 0, L, 0, 0, L)$ and $(0, 0, -L, 0, 0, -L)$, which is

$$\det \begin{bmatrix} \dfrac{\partial^2 \bar{\mathcal{H}}}{\partial I_1^2} & \dfrac{\partial^2 \bar{\mathcal{H}}}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 \bar{\mathcal{H}}}{\partial I_2 \partial I_1} & \dfrac{\partial^2 \bar{\mathcal{H}}}{\partial I_2^2} \end{bmatrix} = \frac{(\omega_1^2 - 1)^4\,(24\omega_1^6 - 1811\omega_1^4 + 1918\omega_1^2 - 403)}{144\mu^2\, L^8\, \omega_1^2\, \omega_2^2} + \cdots .$$

The determinant vanishes when $\omega_1 \in \{0.536925\ldots, 0.88488\ldots, 1, 8.62479\ldots\}$. However, ω_1 is near 1 (either above or below, but it never reaches this value as ε cannot be zero). Again Theorem 2.5 does not apply since the twist occurs at too high an order in ε.

This suggests, but does not prove, that there are families of invariant 3*-tori around these periodic solutions.*

Again, we leave the question of the existence of invariant KAM tori about these periodic solutions to a future paper. However, we can easily say something about the stability of the equilibria $(0, 0, \pm L, 0, 0, \pm L)$ of the reduced system on the base space. Since the Hamiltonian $\bar{\mathcal{H}}$ is positive or negative definite at these points, the classical theorem already known to Dirichlet [20, 36] applies.

The equilibrium points $(0, 0, \pm L, 0, 0, \pm L)$ *are stable on the reduced space* $S^2 \times S^2$.

Finally, we prove that the near-circular equatorial periodic solutions are indeed equatorial but not circular periodic solutions. We start by taking into account the terms of the averaged Hamiltonian given through (24). Hence, the equilibria $(0, 0, \pm L, 0, 0, \pm L)$ are refined, yielding

$$\Big(\mp \tfrac{15}{16}\varepsilon^5\, \mu\, (1-\mu)^{1/3}\, L^6, \quad 0, \quad \pm \tfrac{\sqrt{256\, L^2 - 225\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\, L^{12}}}{16},$$
$$\pm \tfrac{15}{16}\,\varepsilon^5\, \mu\, (1-\mu)^{1/3}\, L^6, \quad 0, \quad \pm \tfrac{\sqrt{256\, L^2 - 225\,\varepsilon^{10}\,\mu^2\,(1-\mu)^{2/3}\, L^{12}}}{16}\Big).$$

As a consequence of the above, the given equilibria no longer correspond to circular solutions, because their eccentricity is $e = \frac{15}{16}\varepsilon^5\, \mu\, (1-\mu)^{1/3}\, L^5 + \cdots$. The magnitude of their angular momentum vector is $G = L - \frac{225}{512}\varepsilon^{10}\, \mu^2\, (1-\mu)^{2/3}\, L^{11} + \cdots$, and the third component of angular momentum is $N = \pm L \mp \frac{225}{512}\varepsilon^{10}\, \mu^2\, (1-\mu)^{2/3}\, L^{11} + \cdots$. This means that the periodic solutions associated with $(0, 0, \pm L, 0, 0, \pm L)$ are indeed elliptic periodic solutions whose projections in configuration space yield elliptic orbits with eccentricity close to zero; the inclination for the solution related to $(0, 0, L, 0, 0, L)$ is zero, whereas it is π for the periodic solution related to $(0, 0, -L, 0, 0, -L)$. This proves that up to terms of order ε^8 the periodic solutions are near-circular periodic solutions of equatorial type.

Thus, these equilibria correspond to elliptic periodic orbits remaining in the same plane as the two primaries.

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