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HOMOCLINIC ORBITS AND BERNOULLI BUNDLES IN ALMOST PERIODIC SYSTEMS

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ABSTRACT. This paper announces the natural generalization to an almost periodic system, of the horseshoe invariant set of Smale and the method of detecting transversal homoclinic orbits of Melnikov. We use the construction to define a skew product flow over the hull of an a.p. function. In this way the non-autonomous equations define a dynamical system whose structure reflects the geometry and spectra of the a.p. forcing term.

Within the context of this skew product dynamical system we provide the natural generalization of a hyperbolic orbit and invariant set, the stable and unstable manifolds, transversal homoclinic orbit and the shift automorphism on a symbol space. This last object we call a Bernoulli bundle because it is a fiber bundle with fiber maps which are Bernoulli automorphisms. We then proceed to prove generalizations of the Melnikov theorem for detecting homoclinic orbits, the shadowing lemma, and Smale's theorem on the existence of the horseshoe invariant set.

I. INTRODUCTION. This paper announces the natural generalization to an almost periodic (a.p.) system, of the horseshoe invariant set of Smale(1963) and the method of detecting transversal homoclinic orbits of Melnikov(1963). Detailed proofs will be given in Meyer and Sell(1986b). The results are natural because we use the Miller(1965) and Sell(1967) construction to define a skew product flow over the hull of an a.p. function. In this way the non-autonomous equations define a dynamical system whose structure reflects the geometry and spectra of the a.p. forcing term.

Within the context of this skew product dynamical system we provide the natural generalization of a hyperbolic orbit and invariant set, the stable and unstable manifolds, transversal homoclinic orbit and the shift automorphism on a symbol space. This last object we call a Bernoulli bundle because it is a fiber bundle with fiber maps which are Bernoulli automorphisms. We then proceed to prove generalizations of the Melnikov theorem for detecting homoclinic orbits, the shadowing lemma, and Smale's theorem on the existence of the horseshoe invariant set.

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Homoclinic orbits and their implications for autonomous and periodic systems have been investigated since their introduction by Poincaré. The classics in the literature are Poincaré (1892), Birkhoff (1932), Cartwright and Littlewood (1945), Melnikov (1963), and Smale (1963). Our work uses many of the ideas and the method of Palmer(1984). The reader is refered to Chow and Hale(1982) and Guckenheimer and Holmes(1983) for a detailed discussion of and historical remarks on this vast literature.

Recently some work has appeared on almost periodic systems. In Wiggens (1986a,1986b) the method of Melnikov and related results are generalized to cover systems which admit an invariant torus. The detection of a transversal intersection of the stable and unstable manifold is based on a generalized Melnikov function and the asymptotic implications are given. He gives a fairly complete picture in the quasi-periodic case. Scheurle (1986) considers a system of a.p. equations slightly more general than ours, but considers only one equation and not a whole class of equations based on the hull of an a.p. function. He uses the theory of exponential dichotomies as extended by Palmer (1984) to find particular solutions which have a somewhat random form.

II. THE HULL, CROSS SECTIONS AND BERNOULLI BUNDLES. Throughout this paper almost periodic (a.p.) will be in the sense of Bohr (1959). Some authors, e.g. Besicovitch(1932), refer to this as uniformly almost periodic. Let $f: R \to R^n$ (or C^n) be an a.p. function. The spectral theory is based on the fact that the mean

$$M\{f\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)\, ds$$

exists and for only a countable number of real numbers ω does $M\{f \exp(-i\omega t)\} \neq 0$. This set $\{\omega_k\}$ is called the set of exponents or frequencies of f. We write

$$f(t) \sim \sum_{-\infty}^{\infty} A_k \exp(i\,\omega_k t) \tag{1}$$

where $A_k = M\{f \exp(-i\,\omega_k t)\}$. This series is called the Fourier Series of f.

Consider the real numbers R as a vector space over the rational numbers. The smallest subspace S R which contains $\{\omega_k\}$ is called the <u>modulus</u> of f. In the case S is one dimensional, i.e. $\omega_k = r_k\,\alpha$, where r_k is rational, the function f is said to be <u>limit periodic</u>, because in this case the function is the uniform limit of periodic functions. The example of a limit periodic function which we shall use throughout this paper is

$$\ell(t) = \sum_{0}^{\infty} a_k \, e^{i2\pi(t/2^k)}$$

(or its real part) where the a_k are choosen so that the series converges rapidly. The N^{th} partial sum of ℓ is 2^N periodic.

If S is a finite module over the integers I, i.e., there exists a finite set $\lambda_1, \ldots, \lambda_l \in R$ such that $\omega_k = \sum \alpha_{ki} \omega_i$ with $\alpha_{ki} \in I$, then f is said to be quasi-periodic. Our standard example of a quasi-periodic function will be

$$q(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

(or its real part) where ω_1/ω_2 is irrational.

Let $\mathbf{C} = \mathbf{C}(R,R^n)$ (or $= \mathbf{C}(R,C^n)$) be the space of continuous functions from R into R^n (or C^n) with the topology of uniform convergence on compact sets (the compact open topology). Translations define a flow on **C** as follows

$$\pi: R \times \mathbf{C} \to \mathbf{C}: (\tau, f) \to f_\tau$$

where $f_\tau(t) = f(t+\tau)$. For any $f \in \mathbf{C}$ the orbit closure of f is called the hull of f and is denoted by H(f). That is $H(f) = cl\{g_\tau : \tau \in R\}$. If f is a.p. then H(f) is a compact minimal set; each element of H(f) is a.p.; $\pi|H(f)$ is equicontinuous, and H(f) can be given a compact, connected, Abelian group structure. We also use the space **AP** of almost periodic functions with sup norm $\|f\|$. The above results hold in this space also.

If f is a.p. and $f \sim \sum a_k \exp(i\,\omega_k t)$ then $f_\tau \sim \sum a_k \exp(i\,\omega_k(t+\tau))$. If a sequence of translates $f_{\tau n}$ converge, say $f_{\tau n} \to g$, then one can use the Cantor diagonalization procedure to select a subsequence if necessary such that

$$\tau_n \to \phi_k \pmod{2\pi/\omega_k} \text{ for all } k, \text{ as } n \to \infty$$

Then the Fourier coefficients of f_{τ_n} converge to the Fourier coefficients of g or

$$g \sim \sum A_k \exp(i\,\omega_k\,(t+\phi_k)). \qquad (2)$$

Thus if $g \in H(f)$ there are ϕ_k defined (mod $2\pi/\omega_k$) such that (2) holds.

EXAMPLE 1. Consider $q(t) = a_1 \exp(i\,\omega_1 t) + a_2 \exp(i\,\omega_2 t)$ where ω_1/ω_2 is irrational. In this case

$$H(q) = \{a_1 \exp i(\omega_1 t + \alpha_1) + a_2 \exp i(\omega_2 t + \alpha_2) : \alpha_i \text{ defined } (\text{mod } 2\pi)\}. \qquad (3)$$

The map $h: T^2 \to H(q): (\alpha_1, \alpha_2) \to a_1 \exp i(\omega_1 t + \alpha_1) + a_2 \exp i(\omega_2 t + \alpha_2)$ is continuous, 1-1 and onto and thus a homeomorphism of topological spaces. Also it carries the orbits of the dynamical system

$$T^2 \times R \to T^2: ((\alpha_1, \alpha_2), t) \to (\alpha_1 + \omega_1 t, \alpha_2 + \omega_2 t)$$

onto the orbits of the translation flow and is an isomorphism of the topological groups. The original function q corresponds to $(0,0) \in T^2$ and the orbit of q, $\{q_\tau: \tau \in R\}$ corresponds to the dense line $\{(\omega_1\tau, \omega_2\tau): \tau \in R\}$ on T^2. See Figure 1.

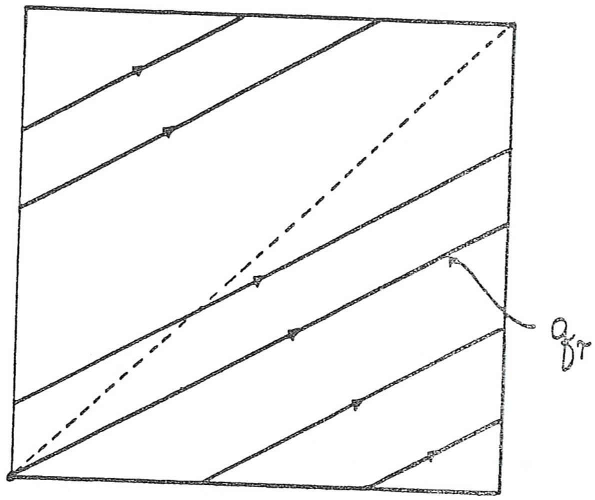

Figure 1: Quasi-periodic hull -- a torus.

EXAMPLE 2: Consider

$$\ell(t) = \sum_0^\infty a_k \exp i\, 2\pi\, (t/2^k) \tag{4}$$

and let τ_n be a sequence such that $\ell_{\tau_n} \to \ell^*$ uniformly where

$$\ell^*(t) = \sum_0^\infty a_k \exp (i\, 2\pi\, (t + \phi_k)/2^k). \tag{5}$$

We assume that $\tau_n \to \phi_k \pmod{2^k}$ as $n \to \infty$ and so $\tau_n \to \phi_{k+1} \pmod{2^{k+1}}$ also, or

$$\phi_k = \phi_{k+1} \pmod{2^k}. \tag{6}$$

It is not hard to see that the hull of ℓ is precisely the set of functions ℓ^* as given in (5) where the ϕ_k are defined (mod 2^k) and satisfy (6).
This suggests a coordinate system for $H(\ell)$ as an infinite product

$$\times_{k=0}^{\infty} S^1$$

where $S^1 = \{z \in C^1: z = \exp i\theta\}$ is the unit circle in the complex plane and the product has the usual product topology. We set $\theta_k = 2\pi\phi_k / 2^k$ so by (6) $\theta_k = 2\theta_{k+1} \pmod{2\pi}$, and we set $z_k = \exp i\theta_k$ so $z_k = z_{k+1}{}^2$. Then ℓ^* is given the coordinate $\{z_0, z_1, \ldots\} \in S$. Since $z_k = z_{k+1}$ the coordinates of ℓ^* are in the inverse limit system

$$S_2 : S^1 \overset{z^2}{\leftarrow} S^1 \overset{z^2}{\leftarrow} S^1 \leftarrow \ldots \tag{7}$$

This is the inverse limit of a solenoid, see Hocking and Young(1961).

Let T be a solid torus in a standard embedding in $\mathbf{R}^3$ as given by rotating a meridianal disk $D(0) = \{(x,0,z): (x-10)^2 + z^2 \leq 1\}$ about the z axis as illustrated in Figure 2. Let ϕ_0 be the polar angle in the (x,y) plane

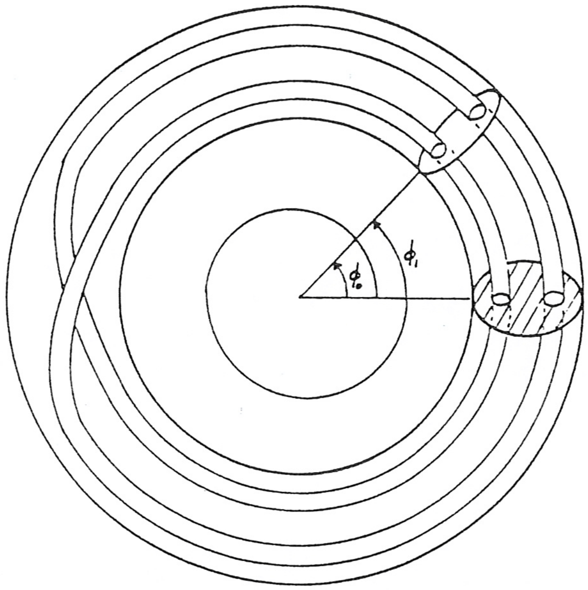

Figure 2: Limit periodic hull -- a solenoid.

normalized so that ϕ_0 is defined (mod 1), and let $D(\phi_0)$ be the image of D(0) after being rotated by ϕ_0. In Figure 2, D(0) is shaded. Let T_1 be a solid torus, lying within the interior of T_0, longitudinally encircling T_0 twice and with medidianal radius 1/4 as illustrated in Figure 2. Let ϕ_1 be an angular variable on T_1 which measures longitudinal displacement in T_1 and is defined (mod 2^1). (See Figure 2). As ϕ_1 traverses [0,2], the meridianal disk $D(\phi_1)$ in T_1 encircles the z-axis twice. Note that T_1 intersects $D(\phi_0)$ in two disks $D(\phi_1')$ and $D(\phi_1'')$ where $\phi_0 \equiv \phi_1' \equiv \phi_1''$ (mod 1).

Continue in this fashion to define T_{k+1} encircling the torus T_k twice with meridianal diameter $1/4^{k+1}$ with longitudinal angle coordinate ϕ_{k+1} defined (mod 2^{k+1}). The 2-solenoid Σ_2 is simply the intersection $\Sigma_2 = \bigcap_{k=0}^{\infty} T_k$ which is a non-empty, connected, compact, one-dimensional subspace of R^3 and so is a "Klosed Kurve" in the sense of Menger. However, Σ_2 is not locally connected and hence cannot be a topological manifold. It is clear from the above construction that $H(\ell)$, the inverse limit system S_2 in (7) and $\sum_2$ are all homeomorphic.

There is a standard minimal flow on $\sum_2$ which corresponds to the translation flow restricted to $H(\ell)$. The flow is defined by

$$P_t(\ldots, z_k, \ldots) = (\ldots, z_k \exp (i2\pi t/2^k), \ldots)$$

which corresponds to uniform rotation about the z-axis in the solenoid $\sum_2$ in Figure 2. The solenoid obtains a continuous Abelian group or structure by component multiplication in the inverse limit representation and this corresponds to the general Abelian group structure on the hull of any a.p. function.

A flow π: $X \times R \to X$, X a compact metric space, admits a (global) cross section Z if i) Z is a closed subset of X ii) all trajectories meet Z and iii) there is a continuous function $T: Z \to R$ such that $T(z) > 0$, $\pi(z,T(z)) \in Z$ and $\pi(z,\tau) \notin Z$ for all $z \in Z$ and $0 < \tau < T(z)$. The function T is called the first return time. The Poincaré map (or section map) is the map

$$P: Z \to Z: z \to \pi(z,T(z)).$$

P is a homeomorphism of Z and defines a discrete dynamical system associated with the flow π. Flows which admit global cross sections are precisely supensions of discrete dynamical systems. It should be noted that global cross sections are not unique.

The translation flow on the hull of an almost periodic function always admits a cross section. Let f be a.p. and have a Fourier series as in (1) the $g \in H(f)$ has a Fourier series (3) and

$$g_\tau(t) \sim \sum A_k \exp (i\ \omega_k\ (t + \phi_k + \tau))$$
$$\sim \sum A_k \exp (i\ \omega_k(\phi_k + \tau)) \exp (i\ \omega_k\ t)$$

Thus the Fourier coefficient of g_τ corresponding to the frequency ω_k is

$$s(g_\tau) = M(g_\tau(t)\exp(-i\omega_k t)) = A_k \exp i\ \omega_k(\phi_k + \tau),$$

which has a constantly changing argument as τ varies provided $\omega_k \neq 0$ and $A_k \neq$ Thus a cross section to this translation flow is given by

$$Z = \{g \in H(f): \arg s(g) = 0 \text{ or } \phi_k \equiv 0 \pmod{2\pi/\omega_k}\}.$$

In this case the first return time is $T = 2\pi/\omega_k$. Thus cross sections and their first return times are intimately connected with the Fourier spectrum of

EXAMPLE 1: $q(t) = a_1 \exp (i\ \omega_1\ t) + a_2 \exp (i\ \omega_2\ t)$ and $q^*(t) \in H(q)$: If $q^*(t) = a_1 \exp i(\omega_1 t + \alpha_1) + a_2 \exp i\ (\omega_2 + \alpha_2)$ so one cross section is $\alpha_1 = 0$ and the first return time is $2\pi/\omega_1$. The angle α_2 is a coordinate on Z so Z is circle in the torus. The Poincaré map in this coordinate is $P: \alpha_2 \to \alpha_2 + (\omega_2/\omega_1)2\pi$ which is an irrational map of the circle.

EXAMPLE 2: Let $\ell(t)$ and $\ell^*(t)$ be as given previously in (4) and (5). As in the previous example we can define a cross section by requiring that the argument of one of the Fourier coefficients of ℓ^* be zero, or equivalently that $\phi_k \equiv 0 \pmod{2^k}$ for some fixed k. To be specific consider the cross section, Z, defined by $\phi_0 \equiv 0 \pmod 1$ (the shaded disk in Figure 1)), so the first return time is 1. The intersection of this disk and Σ_2 is a Cantor set, and the associated Poincaré map is the classical adding machine of dynamical systems. See Meyer and Sell (1986b).

Let $Q = Q_n = \times_{-\infty}^{\infty} \{1, \ldots, n\}$, i.e., Q is the set of bi-infinite sequences on the symbols 1, 2, ..., n with the usual product topology. So if $q \in Q$ then $q = (\ldots q_{-1}, q_0, q_1 \ldots)$ or more simply written $q = \ldots q_{-1}q_0.q_1 \ldots$ with a decimal point to the right of q_0. Let $A: Q \to Q$ be the <u>shift map</u> or <u>shift automorphism</u> defined by $A(q)_i = q_{i+1}$, i.e., A shifts the "decimal point" one unit to the right. This is the classical Bernoulli dynamical system which has found many applications in contemporary dynamical systems.

Given the flow π on X with section Z, and Poincaré map $P: Z \to Z$ and first return time $T: Z \to R$ we define a <u>Bernoulli product</u> as the suspension of the product of A and P. The product of A and P is simply

$$A*P: Q \times Z \to Q \times Z: (q,z) \to (A(q), P(z)),$$

and so the Bernoulli product flow is the projection of the parallel flow

$$\Psi: (Q \times Z \times R) \times R \to Q \times Z \times R$$

$$: ((q,z,\tau),t) \to (q,z,\tau+t)$$

onto the quotient space

$$Q**Z = (Q \times Z \times R)/\sim$$

where $(q,z,t+\tau(z)) \sim (A(q),P(z),t)$. We called the space Q**Z a <u>Bernoulli bundle</u>.

The shift automorphism, A, has n fixed points $(p_1, \ldots, p_n))$ where $p_i = \ldots iii.iii\ldots$ for $1 \leq i \leq n$. One can see that $\{p_i\}**Z$ is an invariant subset of Q**Z which is globally flow equivalent to the original flow π on X. In other words, the original flow π on X lifts to n copies in Q**Z.

III. THE MELNIKOV METHOD FOR ALMOST PERIODIC SYSTEMS. The Melnikov method for detecting transverse homoclinic orbits in periodic systems has been discussed by many authors since the landmark paper of Melnikov (1963). Therefore, we shall not develop this theory in great detail nor seek great generality, but simply give an outline the salient features. We follow the more leisure development in Guckenheimer and Holmes (1983) and supplement with results on a.p. differential equations as found in Coppel(1978), Hale(1969), Sell(1978) and Sacker and Sell(1974).

Consider the differential equation

$$\dot{x} = F(x) + \varepsilon f(t) \tag{8}$$

where $x \in R^2$ and $F: R^2 \to R^2$ and $f: R \to R^2$ are smooth. Actually we shall consider (8) as a whole class of differential equations since we shall take f from a subset of **AP**, say the hull of another a.p. function.

While the theory we will describe below extends readily to the more genera equation

$$\dot{x} = F(x) + \varepsilon f(x,t),$$

all of the essential features appear in the simpler equation (8). Therefore we shall restrict our attention to (8) in this note. The general case will be treated in a forthcoming paper, Meyer and Sell (1986b).

The unperturbed system (when $\varepsilon = 0$) is assumed to be Hamiltonian so, in particular, the trace of the Jacobian of F is identically zero. Furthermore, the unperturbed system is assumed to have a non-degenerate saddle point at $v_0 \in R^2$ and an orbit $u^0(t)$ homoclinic to v_0, i.e. $u^0(t) \to v_0$ as $t \to \pm\infty$. Thus the stabl and unstable manifold of v_0 intersect along the orbit of u^0. See Figure 3.

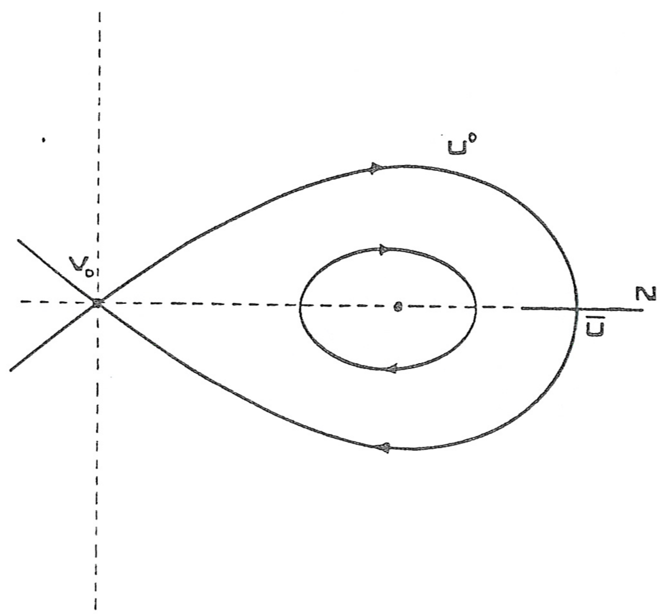

Figure 3: Unperturbed phase portrait.

Let K be a large closed disk in $\mathbf{R}^2$ which contains v_0 and the orbit $u^0(t)$ in its interior, $\varepsilon_0 > 0$ and $\ell_0 > 0$ constants and $L = \{f \in \mathbf{AP} : \|f\| < \ell_0\}$. Let $\phi(t,x_0,f,\varepsilon)$ be the solution of (8) which satisfies $\phi(0,x_0,f,\varepsilon) = x_0$. We consider F as fixed and so suppress the dependence of the solution ϕ on F for simplicity, however f will be taken from a subset of the class of functions L.

In order to understand the underlying geometry of (8) it is useful to recall the concept of a skew product flow. By modifying F outside the large disk K, if necessary, we may assume that the solution $\phi(t,x_0,f,\varepsilon)$ is defined for all t when $|\varepsilon| < \varepsilon_0$ and $f \in L$. Let H be a subset of L invariant under the translation flow. The <u>skew product flow</u> on $R^2 \times H$ is defined by

$$\Pi : R \times (R^2 \times H) \to R^2 \times H : (t,x_0,f) \to (\phi(t,x_0,f,\varepsilon),f_t)$$

see Sell(1971) for the verification that Π defines a flow and for a background discussion.

When $\varepsilon = 0$ the autonomous system has a non-degenerate saddle point at v_0 and a standard theorem on nonlinear a.p. systems, as found for example in Hale(1969) Chapter 4, applies to (8). For some ε_0, equation (8) has a unique a.p. solution $v(t,f,\varepsilon) = v_0 + O(\varepsilon)$ for $|\varepsilon| < \varepsilon_0$ and $f \in L$ and the $O(\varepsilon)$ is uniform in $t \in \mathbf{R}^1$ and $f \in L$. From the proofs found in Hale(1969), one finds that v satisfies

$$v(t,f_\tau,\varepsilon) = v(t+\tau,f,\varepsilon) = v_\tau(t,f,\varepsilon) \ . \qquad (9)$$

Define $x(f,\varepsilon) = v(0,f,\varepsilon)$ so $\phi(t,x(f,\varepsilon),f,\varepsilon) = v(t,f,\varepsilon)$ and define $\mathbf{V} = \{(x(f,\varepsilon),f) \in R^2 \times H : f \in H\}$. The identity (9) shows that $\mathbf{V}$ is an invariant set for the Π flow since

$$\phi(t,x(f,\varepsilon),f,\varepsilon) = v(t,f,\varepsilon) = v(0,f_t,\varepsilon) = x(f_t,\varepsilon).$$

The mapping $\Psi : H \to \mathbf{V} : f \mapsto (x(f,\varepsilon),f)$ is a flow equivalence by (9) also. Thus a classical theorem from differential equations says that the skew product flow defined by (8) has, for small ε, an invariant set near $\{v_0\} \times H$, which is flow equivalent to the translation flow on H.

The local stable manifold theorem states that there is a sufficiently small $\delta > 0$ such that

$$W^s_{loc}(f,\varepsilon) = \{x_0 : \|\phi(t,x_0,f,\varepsilon) - v(t,f,\varepsilon)\| < \delta \text{ for } t \geq 0\} \qquad (10)$$

is a smooth manifold. In particular, there is a smooth function

$$w^s_{loc} : (-\delta,\delta) \times H \times [0,\varepsilon_0) \to R^2$$

such that

$$W^s_{loc}(f,\varepsilon) = \{w^s_{loc}(\sigma,f,\varepsilon) : \sigma \in (-\delta,\delta)\},$$

and for fixed (f,ε) the map

$$w^s_{loc}(\cdot,f,\varepsilon) : (-\delta,\delta) \to R^2$$

is an embedding of the interval $(-\delta,\delta)$. The function w^s_{loc} is smooth in all it arguments when H is considered as a subset of **AP**, the space of a.p. functions with sup norm. By backward integration w^s_{loc} can be extended to a mapping

$$w^s : R \times H \times [0,\varepsilon_0) \to R^2 ,$$

where $w^s|(-\delta,\delta) \times H \times [0,\varepsilon_0) = w^s_{loc}$, with the property that for fixed (f,ε) t map

$$w^s(\cdot,f,\varepsilon): R \to R^2$$

is an immersion of the line R into R^2. We define the set $W^s(f,\varepsilon)$ by $W^s(f,\varepsilon) = \{w^s(\sigma,f,\varepsilon): \sigma \in R\}$. One then has the characterization:

$$W^s(f,\varepsilon) = \{x_0 : \|\phi(t,x_0,f,\varepsilon)-v(t,f,\varepsilon)\| \to 0 \text{ as } t \to +\infty\}. \tag{11}$$

For the skew product flow we define

$$\mathbf{W}^s(\varepsilon) = \{(W^s(f,\varepsilon),f):f \in H\} .$$

By using (9) one can see that

$$\phi(\tau,W^s(f,\varepsilon),f,\varepsilon) = W^s(f_\tau,\varepsilon).$$

Hence $\mathbf{W}^s(\varepsilon)$ is an invariant set for the skew product flow and is characterized by the formula

$$\mathbf{W}^s(\varepsilon) = \{p \in R^2 \times H : \pi(t,p) \to \mathbf{V} \text{ as } t \to +\infty\}. \tag{12}$$

W^s_{loc}, W^s and $\mathbf{W}^s$ are called, respectively, the <u>local stable manifold</u>, the <u>stable manifold</u> and the <u>skew stable manifold</u>. By replacing $t \geq 0$ by $t \leq 0$ in (10) and $t \to +\infty$ by $t \to -\infty$ in (11) and (12) one defines the corresponding unstable manifolds $W^u_{loc}, W^u, \mathbf{W}^u$.

Let $p = (x_0,f) \in \mathbf{W}^s(\varepsilon)$. We define the (partial) tangent space by

$$T_p\mathbf{W}^s(\varepsilon) = \text{span} \left\{\frac{\partial w^s}{\partial \sigma}(\sigma_0,f,\varepsilon)\right\}$$

where $x_0 = w^s(\sigma_0,f,\varepsilon)$. That is we consider only the component of the tangent space which lies in the phase plane R^2.

If $p \in \mathbf{W}^s(\varepsilon)$ $\mathbf{W}^u(\varepsilon)$ then we say p is a <u>homoclinic point</u> (homoclinic to V) and $\{\pi(t,p):t \in R\}$ is a homoclinic orbit. Thus $\pi(t,p) \to \mathbf{V}$ as $t \to \pm\infty$. If $p \in \mathbf{W}^s(\varepsilon) \cap \mathbf{W}^u(\varepsilon)$ and $R^2 = T_p\mathbf{W}^s(\varepsilon) + T_p\mathbf{W}^u(\varepsilon)$ then we say $\mathbf{W}^s(\varepsilon)$ and $\mathbf{W}^u(\varepsilon)$ <u>intersect transversally</u> at p. If at each point $p \in \mathbf{W}^s(\varepsilon) \cap \mathbf{W}^u(\varepsilon)$, the sets $\mathbf{W}^s(\varepsilon)$ and $\mathbf{W}^u(\varepsilon)$ intersect transversally then we say $\mathbf{W}^s(\varepsilon)$ and $\mathbf{W}^u(\varepsilon)$ intersect transversally and write $\mathbf{W}^s(\varepsilon) \pitchfork \mathbf{W}^u(\varepsilon)$.

The Melnikov function gives a criteria for the existence of transversal homoclinic orbits. At a point $\overline{u} = u^0(0)$ on the homoclinic orbit of the unperturbed system we construct a normal line n perpendicular to $F(\bar{u})$ and measure the separation of the stable and unstable manifolds on this normal, see Figure 3.

For the perturbed system we let $u_s(0,f,\varepsilon)$ and $u_u(0,f,\varepsilon)$ denote the point on the normal line n where, respectively, the stable and unstable manifolds meet n. We define the separation of the stable and unstable manifold on n to be

$$d(f,\varepsilon) = \| u_s(0,f,\varepsilon) - u_u(0,f,\varepsilon) \| .$$

By a standard argument one finds that

$$d(f,\varepsilon) = \varepsilon \, \nu \, M(f) + O(\varepsilon^2)$$

where $\nu = \| F(\bar{u}) \|^{-1}$ is constant, and M is the Melnikov function

$$M(f) = \int_{-\infty}^{\infty} F(u^0(t)) \quad f(t)dt .$$

Define the zero set of M to be

$$Z = \{ f \in H : M(f) = 0 \} .$$

We say Z is a simple zero set if

$$\frac{d}{dt} M(f_t) |_{t=0} \neq 0$$

for all $f \in Z$. If Z is a non-empty simple zero set of M, then Z is a cross section for the restriction of the translation flow to H. See Meyer and Sell(1986b).

If $Z \subset H$ is a cross section for the translation flow π with first return time $T: Z \to R$, then $R^2 \times Z$ is a cross section for the skew product flow Π with the same first return time. Define $\psi(x_0,f,\varepsilon) = \phi(T(f),x_0,f,\varepsilon)$ and $\eta(f) = f_{T(f)}$ so the Poincaré map is

$$\Psi = (\psi,\eta) : R^2 \times Z \to R^2 \times Z : (x_0,f) \to (\psi(x_0,f), \eta(f)),$$

which is a discrete skew product dynamical system.

When $\varepsilon = 0$ the system decouples and the skew product becomes an ordinary product. Thus if f = g of example 1 then Π is the product of the irrational flow on the torus as pictured in Figure 1 and flow pictured in Figure 3. Similarly if $f = \ell$ of example 2, then Π is the product of the solenoidal flow and the flow of Figure 3.

It is easier to visualize the Poincaré map. First consider the case when f = q of Example 1 with $\varepsilon = 0$. The cross section of the translation flow is a circle, the diagonal of Figure 1, and the Poincaré map is an irrational rotation of the circle with return time T. Integrate the autonomous equations when $\varepsilon = 0$ for a time T to obtain a map of the plane as shown in Figure 3. Figure 4

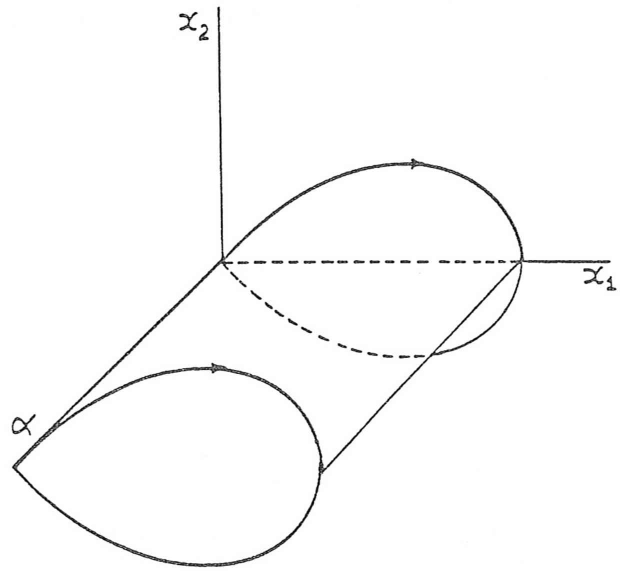

Figure 4: Unperturbed Poincare map.

tries to illustrate the product map. The α axis coming out of the plane of the paper is an angular variable and should be identified (mod 2π) since the space is $R^2 \times S^1$. The map carries a plane perpendicular to the α axis into another such plane. The α axis is an invariant circle for the map, and its stable and unstable manifolds are the products of the loop and a circle.

In the case when $f = \ell$ of example 2 with $\varepsilon = 0$ then one must use some imagination. Think of a Cantor set along the α axis in Figure 4. The Poincaré map is similar to the above except the Cantor set of planes perpendicular to the α-axis are shuffled by the adding machine map.

The main result of this section is:

THEOREM 1: *Let the Melnikov function define a non-trivial, simple zero set* Z H. *Then for each* $f \in Z$ *there is a unique* $\xi(f,\varepsilon) = \bar{u} + O(\varepsilon) \in R^2$ *such that* $(\xi(f,\varepsilon),f) \in W^s(\varepsilon) \cap W^u(\varepsilon)$ *for* $0 < \varepsilon < \varepsilon_0$. *The function* ξ *is continuous. Moreover, if* $\Xi^0 = \{ (\xi(f,\varepsilon),f) : f \in Z \}$, $\Xi^k = \Psi^k(\Xi^0)$ *then* $\Lambda = V \cup \{ \bigcup_{-\infty}^{\infty} \Xi^k \}$ *is a compact invariant set for the Poincaré map* Ψ.

Outline of proof: The separation of the stable and unstable manifold in the normal direction n is

$$d(f,\varepsilon) = \| u_s(0,f,\varepsilon) - u_u(0,f,\varepsilon) \|$$
$$= e\nu\, M(f) + O(\varepsilon)$$

where ν is a non-zero constant. Define a coordinate system near $\bar{u}$ in R^2 as follows: Consider the map $(\alpha,\beta) \to u_s(\alpha,f,\varepsilon) + \beta n$, where n is now a unit normal vector to $F(\bar{u})$. This map takes a neighborhood of the origin in R^2 onto a neighborhood of $\bar{u}$. Note that α is a coordinate along the stable manifold essentially the time parameter), and β is a coordinate in the unit normal direction, see Figure 5. By the fact that n is normal to $F(\bar{u})$, (α,β) constitutes

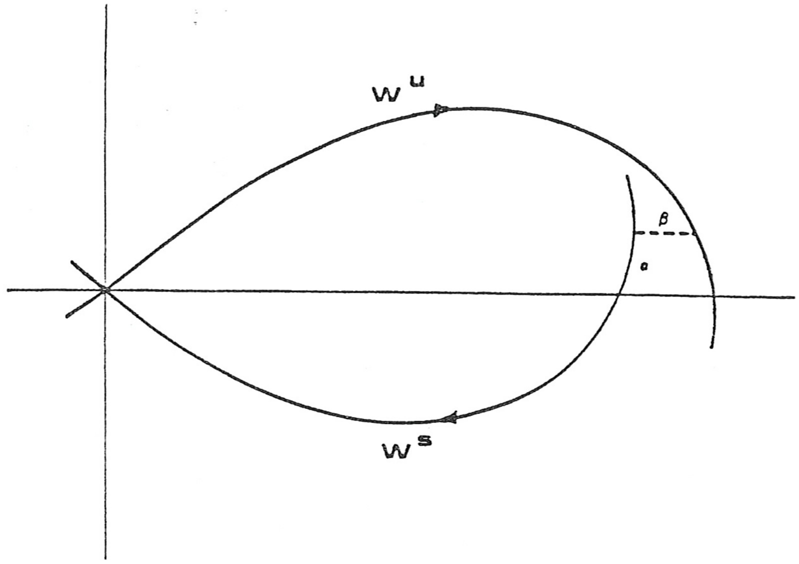

Figure 5: Stable manifold coordinates.

direction, see Figure 5. By the fact that n is normal to $F(\bar{u})$, (α,β) constitutes
a valid coordinate system when ε is small.

In this coordinate system the stable manifold has coordinates $(\alpha,0)$. The unstable manifold

$$u_u(\alpha,f,\varepsilon) = u_s(\alpha,f,\varepsilon) + d(f_\alpha,\alpha)n$$
$$= u_s(\alpha,f,\varepsilon) + \varepsilon\nu\, M(f_\alpha) + O(\varepsilon^2)$$

has coordinates $(\alpha, \varepsilon\nu\, M(f_\alpha) + O(\varepsilon^2))$, and so in these coordinates an intersection of the stable and unstable manifold is obtained when $\varepsilon\nu M(f_\alpha) + O(\varepsilon^2) = 0$. In words $G(\alpha,f,\varepsilon) = 0$ where

$$G(\alpha,f,\varepsilon) = M(f_\alpha) + O(\varepsilon).$$

By assumption

$$G(0,f,0) = M(f) = 0\ ,\ D_1G(0,f,0) \neq 0$$

when $f \varepsilon Z$. Thus by the implicited function theorem we can solve for $\alpha = \overline{\alpha}(f,\varepsilon)$. In these coordinates the intersection of the stable and unstable manifold is $(\alpha, \overline{\alpha}(f,\varepsilon))$, which is just $\xi(f,\varepsilon)$.

Under the Poincaré Ψ map the limit of $(\xi(f,\varepsilon),f)$ lies in V under forward ar backward iteration since it lies on both the stable and unstable manifolds. Thu Λ is closed and hence compact.

IV. HYPERBOLIC SETS, THE SHADOWING LEMMA AND BERNOULLI BUNDLES. Here we shall concentrate on the Poincaré map considered as a discrete skew product dynamical system. In this section the notation of the last section will be used in a slightly more general setting. Let

$$\Psi = (\psi,\eta): R^n \times Z \to R^n \times Z: (x,f) \to (\psi(x,f), \eta(f))$$

define a discrete skew product dynamical system where Z is an arbitrary compact Hausdorff space. Thus Ψ and η are homeomorphisms. Furthermore assume Ψ and ψ ar smooth in their first argument, i.e. $D_1{}^k\psi(x,f)$ is defined and continuous for all $(x,f) \varepsilon R^n \times Z$.

A compact, invariant set $\Lambda \subset R^n \times Z$ is called (skew) hyperbolic if there exist a constant μ, $0 < \mu < 1$, and a continuous splitting of R^n into $E_p{}^s + E_p{}^u$, $p \varepsilon \Lambda$, such that

$$D_1\psi(p): E_p{}^s \to E_q{}^s \qquad \text{and} \qquad D_1\psi(p): E_p{}^u \to E_q{}^u$$

where $q = \Psi(p)$ and

$$\| D_1\psi(p)(u) \| < \mu \| u \| \quad \text{for } u \varepsilon E_p{}^s$$
$$\| D_1\psi^{-1}(p)(v) \| < \mu \| v \| \quad \text{for } v \varepsilon E_p{}^u .$$

THEOREM 2: The invariant set Λ of Theorem 1 is skew hyperbolic.

The proof follows by standard results as found in Coppel(1978) and Palmer(1986).

For $\alpha > 0$ a (skew) α-pseudo-orbit for $\Psi|\Lambda$ is a bisequence $\{p_i=(x_i,f_i)\}$, i ranging from $-\infty$ to $+\infty$, $p_i \varepsilon \Lambda$, such that $f_{i+1} = \eta(f_i)$ and $d(\psi(x_i,f_i),x_{i+1}) < \alpha$ for all i. Thus the sequence $\{f_i\}$ is an η-orbit but the components in R^n may jump by as much as α.

One says a Ψ orbit $\{ \Psi^i(y)=(y_i,f_i) \}$ β-shadows an α-pseudo-orbit $\{(x_i,f_i)\}$ provided $d(x_i,y_i) < \beta$ and, of course, $f_{i+1} = \eta(f_i)$. Note that the base orbits are the same. In this context we have:

THEOREM 3 (The Shadowing Lemma): If Λ is a compact, skew hyperbolic invariant s for $\Psi: R^n \times Z \to R^n \times Z$, then for every $\beta > 0$ there is a $\alpha > 0$ such that every α-pseudo-orbit for $\Psi|\Lambda$ is β-shadowed by some Ψ orbit $\{\Psi^i(q)\}$. Moreover, there is a $\beta_0 > 0$ such that if $0 < \beta < \beta_0$ then the Ψ-orbit given above is uniquely determined by the α-psuedo-orbit.

Proof: A slight variation of the proof given in Meyer and Sell(1986a).

Henceforth, assume that the Melnikov function defines a non-trivial, simple zero set Z H. Let $\Lambda = V \cup \{\bigcup \Xi^k\}$ be the hyperbolic set for the Poincare map $\Psi = (\psi, \eta)$ as given in Theorems 1 and 2. Let $A: Q \to Q$ be the shift automorphism on n symbols as discussed in section II.

THEOREM 4: Under the above assumptions and for small ε, there is an $\iota \in I$ and a compact invariant set Ω $R^2 \times Z$ for the Poincare map Ψ such that $\Psi^\iota|\Omega$ is equivalent to the product map $A \times \eta$: $Q \times Z \to Q \times Z$: $(q,z) \to (A(q), z(\eta))$

Outline of the proof: Assume that $\varepsilon > 0$ is so small that the conclusions of Theorems 1 and 2 hold, so that $\Lambda = V \cup \{\bigcup \Xi^k\}$ is a compact, skew hyperbolic invariant set for the Poincaré map Ψ: $R^2 \times Z \to R^2 \times Z$. Fix ε and henceforth the ε dependence will not be displayed. Let $\beta = \beta_0$ be as given in the uniqueness part of the shadowing lemma and α corresponding to this β. Assume also the β is so small that $\text{dist}(\Xi^0, \Xi^1) > 4\beta$. Since $\Xi^k \to V$ as $k \to \pm\infty$ there is a $\kappa > 0$ such that $\text{dist}(\Xi^k, V) < \alpha/4$ when $|k| \geq \kappa$. Let N be this $\alpha/2$ neighborhood of V.

Our proof is based on Palmer's (1984) construction of an invariant Cantor set for the Poincaré map of a periodic system. His idea is to use the shadowing lemma instead of constructing a horseshoe a lá Smale. Let us relabel some of the Ξ^k as follows: Set $\Xi_0 = \Xi^k$ and $\Xi_\ell = \Psi^{-(2\kappa+1)\ell}(\Xi_0)$ for $1 \leq \ell \leq n$, see Figure 6.

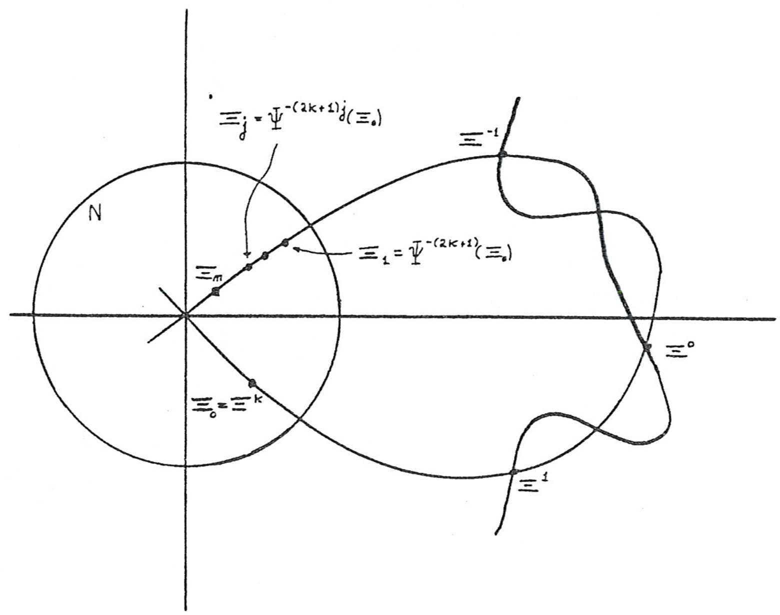

Figure 6: Homoclinic Orbit

Note that Ξ_j, $0 \leq j \leq n$, lie in N. Moreover the Poincaré mapping satisfies $\Psi^{(2\kappa+1)}(\Xi_j) = \Xi_{j-1}$ for $1 \leq j \leq n$. Furthermore for each $z \in Z$ and $0 \leq \ell$, there is a unique point $\xi_\ell(z)$ in R^2 with $(\xi_\ell(z),z) \in \Xi_\ell$, and one has

$$\Psi^{2\kappa+1}(\xi_\ell(z),z) = (\xi_{\ell-1}(\eta^{2\kappa+1}(z)),\ \eta^{2\kappa+1}(z)),\ 1 \leq \ell \leq n.$$

Given a point $(q,z) \in Q \times Z$ we construct an α-pseudo orbit as follows: We begin with $(\xi_0(z),z)$ in Ξ_0. With $q_0 = j$, $1 \leq j \leq n$, we jump to the point $(\xi_j(\eta^{2\kappa+1}(z)),\ \eta^{2\kappa+1}(z))$ in Ξ_j. Next we follow the orbit through the latter point by repeated applications of $\Psi^{2\kappa+1}$ until we return to Ξ_0. Since $\Psi^{(2\kappa+1)j}(\Xi_j) = \Xi_0$, this occurs after j applications of $\Psi^{2\kappa+1}$. We have thus returned to the point $(\xi_0(\eta^{(2\kappa+1)j}(z)),\ \eta^{(2\kappa+1)j}(z))$ in Ξ_0. With $q_1 = \ell$, we then jump to the point $(\zeta_\ell(\eta^{(2\kappa+1)(j+1)}(z)), \eta^{(2\kappa+1)(j+1)}(z))$ in Ξ_ℓ and repeat the above. By using $\Psi^{-(2\kappa+1)}$ with appropriate jumps from Ξ_ℓ to Ξ_0 (when $q_{-n} = \ell$), we construct the α-pseudo orbit for negative time. In this way we obtain an α-pseudo orbit Γ over the trajectory $z^k = \eta^k(z)$ in Z.

Let $\{\Psi^k(P)\}$ be the unique orbit which β-shadows Γ and set $G(q,z) = P$. Thus one has

$$G\colon Q\times Z \to R^2\times Z.$$

By the shadowing lemma, G is continuous and one-to-one and therefore, since $Q\times Z$ compact, G is a homeomorphism onto a subset Ω $R^2\times Z$.

It is clear that $\Psi^\iota|\Omega$ where $\iota = 2\kappa+1$ is conjugate to

$$A*\eta\colon Q\times Z \to Q\times Z\colon (q,z) \to (A(q),\eta(z)).$$

We thus obtain our final result:

THEOREM 5: *Under the above assumptions the skew product flow for small ε admits an invariant set which is flow equivalent to the flow on a Bernoulli bundle.*

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