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STABILITY AND BIFURCATIONS FOR THE $N+1$ VORTEX PROBLEM ON THE SPHERE

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The equations of motion for N vortices on a sphere were derived by V. A. Bogolomov in 1977. References to related work can be found in the book by P. K. Newton. We use the equations of motion found there to discuss the stability of a ring of N vortices of unit strength at the latitude z together with a vortex of strength κ at the north pole. The regions of stability are bounded by curves $\kappa = \kappa(z)$. These curves are computed explicitly for all values of N.

When the stability of a configuration changes, for example by varying the strength of the vortex at the north pole, bifurcations to new configurations are possible. We compute the bifurcation equations explicitly for N=2, 3 and 4. For larger values of N the complexity of the formal computations becomes too great and we use a numerical value for the latitude instead. We thus derive the bifurcation equations in a semi numerical form. As expected the new configurations look very similar to those which had been found previously for the planar case.

1. Introduction

The original interest in the motion of vortices can be traced to the work of Helmholtz [27] in the second half of the nineteenth century. Many of his contemporaries, in particular J. J. Thomson [25, 26], believed that vortex theory could be used to explain the structure of the atom. When it became clear that this approach was wrong the interest in this area diminished, except for those working in fluid mechanics, see for example [9, 11].

It was V. A. Bogomolov in 1977 who derived the equations of motion of point vortices on a sphere [3, 5]. He considered the case of a non rotating and of a rotating sphere. The later case can be used as a simple model for the motion of cyclones and hurricanes in the atmosphere of a planet [4, 23, 15]. Since the three vortex problem on the sphere is integrable it has received special attention [7, 12, 22].

In the last 20 years interest in the motion of point vortices in a plane and on the sphere has increased significantly when it was realized that there are lot of similarities between the N-body problem of celestial mechanics and the N-vortex problem in an ideal fluid, see for example [8, 20]. It was Kirchhoff [13] who formulated the N-vortex problem in the framework of Hamiltonian mechanics. It is this connection to celestial mechanics that gave rise to a large number of papers and even books on this topic in recent years. Of special interest is the book by P. Newton [21] with an extensive list of references to relevant papers.

Mathematics Subject Classification

The problem to be considered here is the motion of N point vortices on a sphere of radius one. Let Γ_j be the strength and $\mathbf{x}_j(t)$ be the position of the j-th vortex. The equation of motion of this vortex under the influence of the other vortices is given by

$$\dot{\mathbf{x}}_j = \sum_{i \neq j}^{N} \frac{\Gamma_i}{2\pi} \frac{\mathbf{x}_i \times \mathbf{x}_j}{|\mathbf{x}_j - \mathbf{x}_i|^2},$$

where $|\mathbf{x}_j - \mathbf{x}_i|$ is the chordal distance between the two vortices. In order to avoid writing the factor 2π we set $\kappa_j = \Gamma_j/2\pi$ and refer to it as the strength of a vortex.

The position of a vortex on the sphere can also be given in cylindrical coordinates, that is, by the distance of the vortex to the equatorial plane and by its longitude. As long as none of the vortices is near a pole of the sphere the equations of motion can be given by a Hamiltonian function in these coordinates. We will use this approach in section 3 in order to determine the stability of a ring of vortices at a fixed latitude.

A vortex at the north pole destroys the Hamiltonian nature of the system of differential equations. Nevertheless, the function remains an integral of motion. It allows us to determine in section 4 how the stability of the ring of vortices changes when there is a vortex at the north pole, whose strength varies. Whenever there is a change in stability, bifurcations to new configurations can be expected. This approach is exploited in sections 6 and 7. Sections 5 and 8 are added for completeness sake, as the problem of three vortices on sphere is integrable and it has been discussed before, see for example [12] and [17].

The stability of $N+1$ vortices on a sphere has been discussed elsewhere from different points of view, see for example [6, 2, 16, 14]. We believe that circulant matrices are the natural tool for the given problem and that they allow for an easier determination of the stability regions and the determination of the bifurcation equations.

2. Preliminaries

Consider n vortices, which are at the vertices of a regular polygon. The Hessian of such a configuration will have the following format

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

This form is known as a circulant matrix, see [1]. Let $\omega = e^{2\pi i/n}$ be the n-th root of unity then a complete set of eigenvectors for such a matrix is given by the columns of

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}.$$

Also the eigenvalues can be computed explicitly and the j-th eigenvalue is found to be

$$\sum_{k=0}^{n-1} a_k \omega^{jk}. \tag{2.1}$$

Since a Hessian is symmetric, we have $a_k = a_{n-k}$ for $k = 1, 2, \ldots, n-1$. Therefore, all eigenvalues will be real and the set of eigenvectors are the real and imaginary parts of the eigenvectors given above.

We will adopt the convention that the set of eigenvectors are in the order as given by the following transformation matrix

$$T = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n/2}} & \frac{1}{\sqrt{n/2}} & \cdots & 0 & 0 \\ \frac{1}{\sqrt{n}} & \frac{\cos 2\pi/n}{\sqrt{n/2}} & \frac{\cos 4\pi/n}{\sqrt{n/2}} & \cdots & \frac{\sin 4\pi/n}{\sqrt{n/2}} & \frac{\sin 2\pi/n}{\sqrt{n/2}} \\ \frac{1}{\sqrt{n}} & \frac{\cos 4\pi/n}{\sqrt{n/2}} & \frac{\cos 8\pi/n}{\sqrt{n/2}} & \cdots & \frac{\sin 8\pi/n}{\sqrt{n/2}} & \frac{\sin 4\pi/n}{\sqrt{n/2}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{\cos (n-1)\pi/n}{\sqrt{n/2}} & \frac{\cos 2(n-1)\pi/n}{\sqrt{n/2}} & \cdots & \frac{\sin 2(n-1)\pi/n}{\sqrt{n/2}} & \frac{\sin (n-1)\pi/n}{\sqrt{n/2}} \end{pmatrix} \tag{2.2}$$

The matrix has been made orthonormal by dividing the column vectors by $\sqrt{n}$ or $\sqrt{n/2}$ respectively. In writing down the above transformation matrix one has to be aware that the case n odd and the case n even are slightly different. The horizontal dots in the above matrix should indicate that on the left additional columns with cosine terms have to be inserted and on the right the same number of columns with sine terms. When n is odd this completes the $n \times n$ transformation matrix. For n even there is one more column in the middle consisting of alternating $+1/\sqrt{n}$ and $-1/\sqrt{n}$. That column comes from the same position in the complex form of the set of eigenvectors.

Other formulas, which will be used in what follows, are

$$\sum_{j=1}^{n-1} \frac{1}{\sin^2 j\pi/n} = \frac{n^2-1}{3}$$

and

$$\sum_{j=1}^{n-1} \frac{\sin^2 kj\pi/n}{\sin^2 j\pi/n} = k(n-k) \quad \text{for} \;\; k = 0, 1, \ldots, n.$$

At that time it has to be remembered that in standard formals like

$$\sum_{j=0}^{n-1} \sin^2 \frac{2\pi j}{n} = \begin{cases} 0 & for\text{n=2} \\ n/2 & for\text{n¿2} \end{cases}$$

and

$$\sum_{j=0}^{n-1} \cos^2 \frac{2\pi j}{n} = \begin{cases} 2 & for\text{n=2} \\ n/2 & for\text{n¿2} \end{cases}$$

the case $n = 2$ is different. Therefore, the case $n = 2$ has to be handled separately. Although it could be done directly without the help of circulant matrices, we will treat it in a later section for completeness sake. For now we will assume that $n > 2$.

3. A ring of vortices at a fixed latitude

In this section we will discuss a ring of n vortices of unit strength at a given latitude on a sphere. This case can be treated completely within the framework of Hamiltonian mechanics. When a vortex near a pole of the sphere is added then the resulting differential equations are no longer Hamiltonian.

The n vortices on a sphere of radius one can be located by the cylindrical coordinates (z_j, φ_j), $j = 0, 1, \ldots, n-1$, where z_j gives the latitude by measuring the vertical distance from the equatorial plane, and where φ_j gives the longitude. In these coordinates the motion of n vortices of unit strength is given by the Hamiltonian

$$H_1 = \frac{1}{2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \ln\left[1 - z_i z_j - \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\varphi_i - \varphi_j)\right]. \tag{3.1}$$

In our notation the z_j's are the position coordinates and φ_j's are the corresponding momenta coordinates, so that the differential equations are given by

$$\dot{z}_j = \partial H/\partial\varphi_j, \qquad \dot{\varphi}_j = -\partial H/\partial z_j, \qquad j = 0, \ldots, n-1.$$

where for the moment $H = H_1$. It is obvious that the coordinates are singular near the poles and that stationary solutions can only be found when viewed in a rotating coordinate system. Therefore, introduce a coordinate system which rotates uniformly with angular velocity w around the polar axis. We will stay with the same notation for the new coordinates, but the Hamiltonian (3.1) has to be replaced by

$$\begin{aligned} H &= H_1 - w\sum_{i=0}^{n-1} z_i = \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} \ln\left[1 - z_i z_j - \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\varphi_i - \varphi_j)\right] - w\sum_{i=0}^{n-1} z_i. \end{aligned} \tag{3.2}$$

Place n vortices of unit strength at a fixed latitude z with $-1 < z < 1$ and at the vertices of a regular polygon, that is, $z_j = z$ and $\varphi_j = 2\pi j/n$ for $j = 0, 1, \ldots, n-1$. Then this configuration will be stationary in the rotating coordinate system if

$$w = -\frac{(n-1)z}{2(1-z^2)}. \tag{3.3}$$

In order to investigate the stability of this configuration the Hessian of (3.2) or equivalently the Hessian of (3.1) has to be computed and evaluated at the equilibrium. We find with the help of the formulas given in section 2

$$\begin{aligned} \frac{\partial^2 H_1}{\partial z_i \partial z_j} &= \frac{-1}{4(1-z^2)^2\sin^2(j-i)\pi/n} \qquad \text{for } i \neq j, \\ \frac{\partial^2 H_1}{\partial z_i^2} &= \frac{(n-1)(n-5-6z^2)}{12(1-z^2)^2}, \\ \frac{\partial^2 H_1}{\partial\varphi_i \partial\varphi_j} &= \frac{1}{4\sin^2(j-i)\pi/n} \qquad \text{for } i \neq j, \\ \frac{\partial^2 H_1}{\partial\varphi_i^2} &= -\frac{n^2-1}{12}, \\ \frac{\partial^2 H_1}{\partial z_i \partial\varphi_j} &= 0. \end{aligned}$$

In block form the Hessian consists of two circulant matrices along the diagonal. The off-diagonal block matrices are zero. Let $\mathbf{z}$ stand for the vector of the latitudes. The eigenvalues of the circulant matrix $\partial^2 H_1/\partial\mathbf{z}^2$ evaluated at the equilibrium can be computed with the help of (2.1) and they are

$$\lambda_k = \frac{-(n-1)(1+z^2) + k(n-k)}{2(1-z^2)^2} \qquad k = 0, 1, \ldots, n-1.$$

Similarly with φ representing the vector of the longitudes we compute the eigenvalues of $\partial^2 H_1/\partial\varphi^2$ to be

$$\sigma_k = -\frac{k(n-k)}{2} \qquad k = 0, 1, \ldots, n-1.$$

It is seen that many of these eigenvalues are repeated, that is $\lambda_k = \lambda_{n-k}$ and $\sigma_k = \sigma_{n-k}$ for $k = 1, \ldots, \lfloor n/2 \rfloor$. Thus, only when $k = 0$ and when $k = n/2$ with n even are the eigenvalues simple. The ordering of these eigenvalues is due to the use of the transformation matrix T of (2.2). If we denote the vectors of the new variables by ζ and ϕ respectively then the complete transformation is given by

$$\mathbf{z} = \begin{pmatrix} z \\ z \\ \vdots \\ z \end{pmatrix} + T\zeta, \qquad \varphi = \begin{pmatrix} 0 \\ 2\pi/n \\ \vdots \\ 2(n-1)\pi/n \end{pmatrix} + T\phi.$$

The transformation is symplectic. In the new variables the origin is an equilibrium point and the transformed Hamiltonian starts with quadratic terms in normal form:

$$H = \frac{1}{2}\sum_{k=0}^{n-1} \lambda_k \zeta_k^2 + \frac{1}{2}\sum_{k=1}^{n-1} \sigma_k \phi_k^2 + \text{h.o.t.}$$

The stability of the origin is easily deduced. Since $\sigma_k < 0$ for $k = 1, \ldots, n-1$, $\sigma_0 = 0$ and $\lambda_0 < 0$ the stability depends on the λ_k being negative for $k = 1, \ldots, \lfloor n/2 \rfloor$. These eigenvalues are ordered by increasing values so that the largest value is achieved when $k = \lfloor n/2 \rfloor$. In order for that eigenvalue to be negative we find the requirement that $z^2 > (n-2)^2/(4(n-1))$ when n is even and that $z^2 > (n-1)(n-3)/(4(n-1))$ when n is odd. A value of $z^2 < 1$ can only be realized by these formulas for $n < 7$.

Another way of looking at the stability of a ring of vortices at distance z from the equator is to ask where each λ_k changes from a positive to a negative value as z^2 is increased. These values are given in the table below for n up to 12.

$n \backslash k$	1	2	3	4	5	6	7
3	0	0					
4	0	1/3	0				
5	0	1/2	1/2	0			
6	0	3/5	4/5	3/5	0		
7	0	2/3	1	1	2/3	0	
8	0	5/7	8/7	9/7	8/7	5/7	0
9	0	3/4	5/4				
10	0	7/9	4/3				
11	0	4/5	7/5				
12	0	9/11	16/11				

Any value greater than one in the above table should be ignored, as it leads to a configuration, which can not be realized. For that reason not all of these values are listed. The column two of the table shows that the level of stability changes near the physically meaningful value of $z^2 = (n-3)/(n-1)$ due to the change of signs in λ_2 and λ_{n-2}. This happens for all values of n.

4. Ring with vortex at the north pole

If a vortex moves near the north pole on the sphere we can not use cylindrical coordinates and we will have to stay with Cartesian coordinates for that vortex. We will use (x_n, y_n, z_n) to denote the position of this $(n+1)$-st vortex of strength κ near the north pole. The system of differential equations in an inertial coordinate system is now

$$\begin{array}{lll} \dot{\varphi}_k = -\partial H/\partial z_k, & \dot{z}_k = \partial H/\partial \varphi_k, & \text{for } k = 0, 1, \ldots, n-1 \\ \kappa \dot{x}_n = -z_n \partial H/\partial y_n, & \kappa \dot{y}_n = z_n \partial H/\partial x_n, & \end{array}$$

and was used before in [2]. The function H is now $H = H_1 + H_2$ with H_1 given by (3.1) and H_2 by

$$H_2 = \frac{\kappa}{2} \sum_{j=0}^{n-1} \ln \left[1 - z_j z_n - \sqrt{1 - z_j^2} (x_n \cos \varphi_j + y_n \sin \varphi_j) \right]. \tag{4.1}$$

Thus the problem is no longer Hamiltonian, but H still serves as an integral of motion. If the problem is to be considered in a uniformly rotating coordinate system, then H has to be replaced by

$$H = H_1 + H_2 - w\left(\sum_{j=0}^{n-1} z_j + \kappa z_n\right). \tag{4.2}$$

A polygonal ring of n unit vortices at latitude z and a vortex of strength κ at the north pole (i.e. $x_n = 0$, $y_n = 0$) is at an equilibrium if $\nabla H = 0$. This happens when

$$w = -\frac{(n-1)z}{2(1-z^2)} - \frac{\kappa}{2(1-z)}.$$

We extend the vectors $\mathbf{z}$ and φ to be $\tilde{\mathbf{z}} = (z_0, z_1, \ldots, z_{n-1}, y_n)$ and $\tilde{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}, x_n)$. In order to investigate the stability of this configuration we again compute the Hessian of H at the given configuration. The new terms arising from H_2 are

$$\begin{aligned}
\frac{\partial^2 H_2}{\partial z_i \partial z_j} &= 0 && \text{for } i \neq j, \\
\frac{\partial^2 H_2}{\partial z_i^2} &= \frac{-\kappa}{2(1-z)^2} && \\
\frac{\partial^2 H_2}{\partial \varphi_i \partial \varphi_j} &= 0 && \text{for } 0 \leqslant i, j \leqslant n-1, \\
\frac{\partial^2 H_2}{\partial \varphi_j \partial x_n} &= \frac{\kappa r \sin 2j\pi/n}{2(1-z)} && 0 \leqslant j \leqslant n-1, \\
\frac{\partial^2 H_2}{\partial \varphi_j \partial y_n} &= -\frac{\kappa r \cos 2j\pi/n}{2(1-z)} && 0 \leqslant j \leqslant n-1, \\
\frac{\partial^2 H_2}{\partial z_j \partial x_n} &= -\frac{\kappa \cos 2j\pi/n}{2r(1-z)} && 0 \leqslant j \leqslant n-1, \\
\frac{\partial^2 H_2}{\partial z_j \partial y_n} &= -\frac{\kappa \sin 2j\pi/n}{2r(1-z)} && 0 \leqslant j \leqslant n-1, \\
\frac{\partial^2 H_2}{\partial x_n^2} &= \frac{\partial^2 H_2}{\partial y_n^2} = \frac{\kappa n}{4} && \\
\frac{\partial^2 H_2}{\partial x_n \partial y_n} &= 0. &&
\end{aligned}$$

In the above formulas we have used the abbreviation $r=+\sqrt{1-z^2}$. In (4.2) the last term of H will also contribute to the second order derivatives, since $z_n=\sqrt{1-x_n^2-y_n^2}$ and therefore

$$\frac{\partial^2 z_n}{\partial x_n^2}=\frac{\partial^2 z_n}{\partial y_n^2}=-1$$

when $x_n=y_n=0$.

In order to use the normal form of D^2H_1 we have to transform the new terms in the same way as those of D^2H_1. For that purpose we extend the matrix T of (2.2) in the natural way, so that the full transformation matrix in block form is

$$\tilde{T}=\begin{pmatrix} T & 0 & 0 & 0\\ 0^t & 1 & 0^t & 0\\ 0 & 0 & T & 0\\ 0^t & 0 & 0^t & 1\end{pmatrix}$$

with 0 representing an $n\times n$ matrix, an n dimensional vector or even a scalar where appropriate. In order to see what happens to the terms of D^2H_2 we can ignore the terms on the diagonal for a moment. Without them the Hessian of H_2 in block form is

$$C=\begin{pmatrix} 0 & u_{\varphi x} & 0 & u_{\varphi y}\\ u_{\varphi x}^t & 0 & u_{zx}^t & 0\\ 0 & u_{zx} & 0 & u_{zy}\\ u_{\varphi y}^t & 0 & u_{zy}^t & 0\end{pmatrix}$$

where $u_{\varphi x}$ is the column vector made up of $\partial^2H_2/\partial\varphi_j\partial x_n$, $j=0,1,\dots,n-1$, and similarly for the other vectors. From

$$\tilde{T}^tC\tilde{T}=\begin{pmatrix} 0 & T^tu_{\varphi x} & 0 & T^tu_{\varphi y}\\ u_{\varphi x}^tT & 0 & u_{zx}^tT & 0\\ 0 & T^tu_{zx} & 0 & T^tu_{zy}\\ u_{\varphi y}^tT & 0 & u_{zy}^tT & 0\end{pmatrix}$$

we see that we have to find the product $T^tu_{\varphi x}$ and three other products. In each case the vector is orthogonal to all but one column of T. Thus $T^tu_{\varphi x}$ has only one nonzero entry in position $n-1$ of this vector and we call this nonzero term α. The value $-\alpha$ occurs in position 1 in $T^tu_{\varphi y}$. A nonzero value denoted by β occurs in position 1 of T^tu_{zx} and in position $n-1$ of T^tu_{zy}. The two nonzero values are

$$\alpha=\frac{\kappa r\sqrt{n/2}}{2(1-z)}\qquad\qquad \beta=-\frac{\kappa\sqrt{n/2}}{2r(1-z)}.$$

Combining everything, the Hessian of H in (4.2) evaluated at the stationary solution and given in accordance with the vector $(\varphi,x_n,\mathbf{z},y_n)$ is

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \sigma_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha\\
0 & 0 & \sigma_2 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & \sigma_{n-1} & \alpha & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & \alpha & \sigma_n & 0 & \beta & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & \beta & 0 & \lambda_1 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \lambda_{n-1} & \beta\\
0 & -\alpha & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \beta & \lambda_n
\end{pmatrix}. \tag{4.3}$$

The terms along the diagonal have the values

$$\begin{aligned}
\sigma_k &= -\frac{k(n-k)}{2} && \text{for } k=0,1,\cdots,(n-1)\\
\lambda_k &= \frac{-(n-1)(1+z^2)+k(n-k)-\kappa(1+z)^2}{2(1-z^2)^2} && \text{for } k=0,1,\cdots,(n-1)\\
\sigma_n = \lambda_n &= -\frac{\kappa n}{4}+\kappa w.
\end{aligned}$$

Since a portion of the above matrix is already in diagonal form, several eigenvalues can be read off at once. They are

$$\begin{aligned}
\sigma_0 &= 0\\
\lambda_0 &= -\frac{(n-1)(1+z^2)+\kappa(1+z)^2}{2(1-z^2)^2}\\
\sigma_k &= -\frac{(n-k)k}{2} && \text{for } k=2,\ldots,n-2\\
\lambda_k &= -\frac{(n-1)(1+z^2)-(n-k)k+\kappa(1+z)^2}{2(1-z^2)^2} && \text{for } k=2,\ldots,n-2
\end{aligned}$$

Since $\sigma_k<0$ for $k=2,...,n-2$ we need only to check that $\lambda_k<0$ for the same range of indices. Again we have $\lambda_k=\lambda_{n-k}$ and the largest of these eigenvalues occurs for $k=\lfloor n/2\rfloor$. Six eigenvalues of the Hessian are more difficult to determine. On closer inspection these eigenvalues follow from the two submatrices

$$\begin{pmatrix}\sigma_1 & 0 & -\alpha\\ 0 & \lambda_{n-1} & \beta\\ -\alpha & \beta & \lambda_n\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}\sigma_{n-1} & \alpha & 0\\ \alpha & \sigma_n & \beta\\ 0 & \beta & \lambda_1\end{pmatrix}.$$

Since $\lambda_{n-1}=\lambda_1$, $\sigma_{n-1}=\sigma_1$ and $\lambda_n=\sigma_n$ the two submatrices have the same set of real eigenvalues and it suffices to look just at the first submatrix. It reads in details

$$m_3=\begin{pmatrix}
-\frac{n-1}{2} & 0 & -\frac{\kappa\sqrt{n/2}\sqrt{1-z^2}}{2(1-z)}\\
0 & \frac{-(n-1)z^2-\kappa(1+z)^2}{2(1-z^2)^2} & -\frac{\kappa\sqrt{n/2}}{2(1-z)\sqrt{1-z^2}}\\
-\frac{\kappa\sqrt{n/2}\sqrt{1-z^2}}{2(1-z)} & -\frac{\kappa\sqrt{n/2}}{2(1-z)\sqrt{1-z^2}} & -\kappa\left(\frac{n}{4}+\frac{(n-1)z+\kappa(1+z)}{2(1-z^2)}\right)
\end{pmatrix}.$$

Instead of computing the eigenvalues of m_3 directly, it is easier to decide if all eigenvalues are negative by looking at the determinants of the principal minors of the main diagonal. Since the first element on the diagonal of m_3 is already negative, the next minor will be positive if

$$F_1(z,\kappa)=(n-1)z^2+\kappa(1+z)^2>0 \tag{4.4}$$

and finally we need $|m_3|<0$. It turns out that the expression for $|m_3|$ factors nicely into

$$|m_3|=\frac{\kappa((n-1)z+\kappa(1+z))((n-1)z(2z-n-2nz+nz^2)+\kappa(2-n+nz)(1+z)^2)}{16(1-z^2)^3}.$$

For the numerator we require

$$\kappa F_2(z,\kappa)F_3(z,\kappa)<0 \tag{4.5}$$

where

$$F_2(z,\kappa) = (n-1)z + \kappa(1+z), \tag{4.6}$$
$$F_3(z,\kappa) = (n-1)z(2z-n-2nz+nz^2) + \kappa(2-n+nz)(1+z)^2. \tag{4.7}$$

Since the functions in (4.4) to (4.7) are all linear in κ, we can plot the curves in the z-κ plane where the functions F_1, F_2 and F_3 are zero and thus find the regions where the three eigenvalues will be negative. These curves found from (4.4) to (4.7) respectively are

$$g_1 = -\frac{(n-1)z^2}{(1+z)^2} \tag{4.8}$$
$$g_2 = -\frac{(n-1)z}{1+z} \tag{4.9}$$
$$g_3 = -\frac{(n-1)z(2z-n-2nz+nz^2)}{(2-n+nz)(1+z)^2}. \tag{4.10}$$

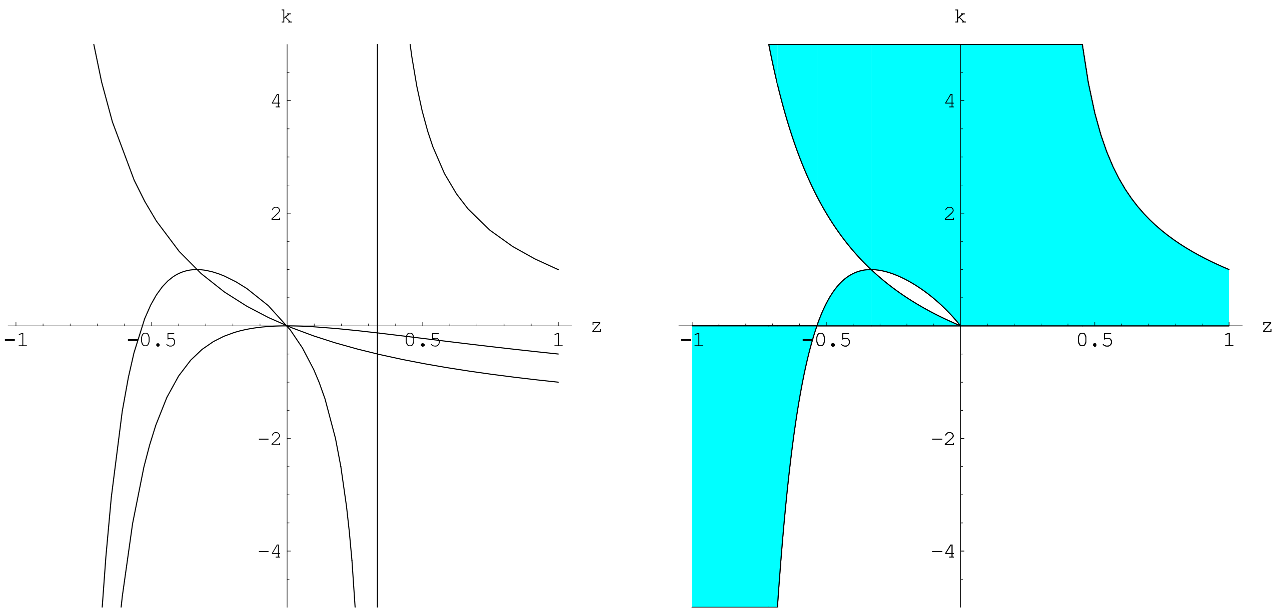

Fig. 1. Case $n=3$. The three curves $g_1 = -\dfrac{2z^2}{(1+z)^2}$, $g_2 = -\dfrac{2z}{1+z}$ and $g_3 = -\dfrac{2z(3+4z-3z^2)}{(1+z)^2(1-3z)}$ defining the regions of stability for m_3. On the right the stability regions are shaded in gray.

For $n=3$ the three curves are depicted on the left in figure 1, including an asymptotic line for g_3 at $z=1/3$. In the same figure on the right are the regions where all eigenvalues of m_3 are negative. The regions can be found more easily by considering the condition (4.5) first. After plotting the curves g_2 and g_3 the regions where (4.5) is satisfied can be broken down into $F_2(z,\kappa)F_3(z,\kappa) < 0$ when $\kappa > 0$ and $F_2(z,\kappa)F_3(z,\kappa) > 0$ when $\kappa < 0$. It will then be seen that the condition (4.4) imposes no additional constraints, since $g_1(z) \leqslant 0$ for all z and furthermore $g_1(z) \leqslant g_3(z)$ in $-1 < z \leqslant 0$. It means that $F_1(z,\kappa) = 0$ and with it $\lambda_1 = 0$ is not one of the places where the stability of a stationary solution can change. This remark will be important when we consider bifurcations in Section 6. The condition that $\lambda_0 < 0$ imposes no additional constraints on the stability regions found so far.

When $n=3$ there are no other constraints to be considered. When $n>3$ other eigenvalues exist and all of them need to be negative. We restrict ourselves to the largest of these eigenvalues, that is

$\lambda_{\lfloor n/2 \rfloor}$ and insist that it is negative. The condition $\lambda_{\lfloor n/2 \rfloor} = 0$ is linear in κ, so that from it we can easily plot κ as a function of z. It is given by

$$g_0 = \begin{cases} \dfrac{((n-2)/2)^2 - (n-1)z^2}{(1+z)^2} & \text{for } n \text{ even} \\ \dfrac{(n-1)(n-3)/4 - (n-1)z^2}{(1+z)^2} & \text{for } n \text{ odd} \end{cases}$$

.

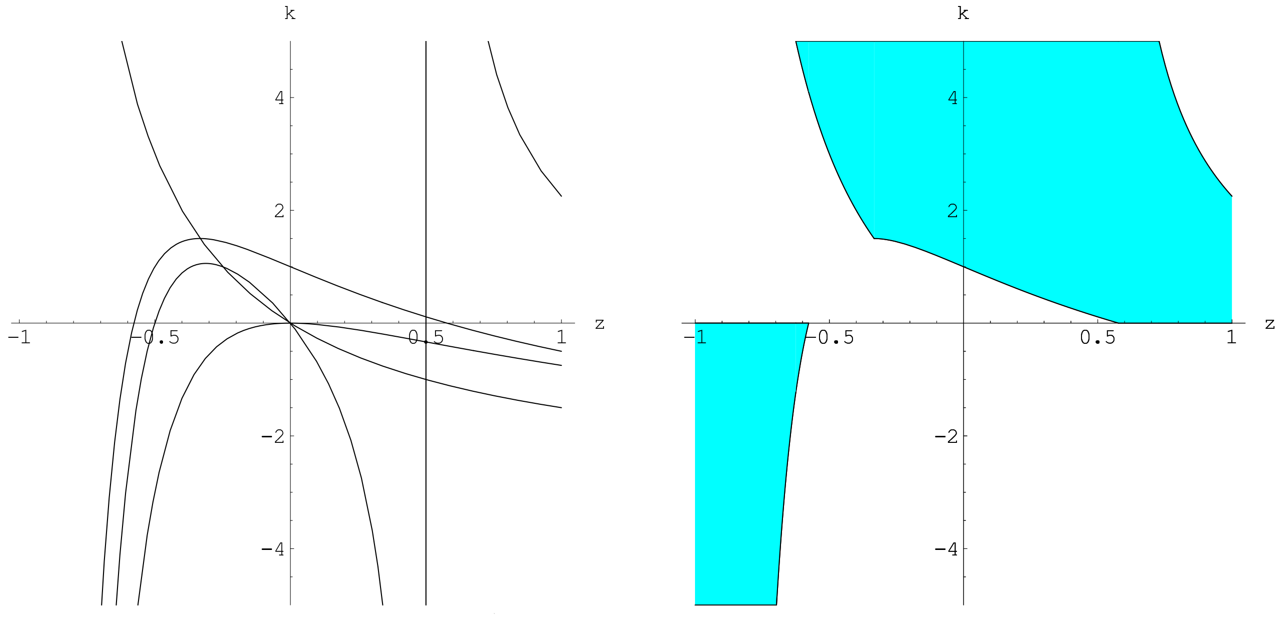

Fig. 2. The case $n = 4$. The curves g_0, g_1, g_2 and g_3 in the z–κ plane defining the stability regions. The stable regions are shaded on the right.

For $4 \leqslant n < 7$ the function g_0 has two zeros in $-1 < z < 1$ and also a maximum value in this interval. This means that g_0 has an asymptote to $-\infty$ as $z \to -1$. It allows a region of stability for negative κ near $z = -1$. When $n > 7$ g_0 tends to $+\infty$ as $z \to -1$ and therefore this region disappears. For $n = 4$, 5, and 6 the curves g_0 and g_2 intersect, and to the left of that intersection point g_2 is the lower limit for the region of stability. For $n > 7$ the function g_0 is the lower limit for all $-1 < z < 1$.

The case $n = 7$ is the dividing one, and it is here where g_0 simplifies to $g_0 = 6(1-z)/(1+z)$. From its asymptote at $z = -1$ to its zero at $z = 1$ this function is the lower limit for κ in the entire interval $-1 < z < 1$. For all cases of n the function g_3 from the determinant of m_3 imposes an upper limit on κ. This upper limit exists from the asymptote of g_3 at $z = (n-2)/n$ to $z = 1$.

In the following theorem our findings are summarized. The intervals in κ where a configuration is stable changes at certain values of z. These values are indexed by n and they are α_n the positive zero of $g_0(z) = 0$, β_n the intersection of $g_0(z)$ and $g_2(z)$ which occurs at a maximum of $g_0(z)$, and γ_n, which is an asymptote of $g_3(z)$. Values of interest in $-1 < z < 1$ are

$$\alpha_4 = \frac{\sqrt{3}}{3}, \qquad \alpha_5 = \frac{\sqrt{2}}{2}, \qquad \alpha_6 = \frac{2\sqrt{5}}{5},$$
$$\beta_4 = -\frac{1}{3}, \qquad \beta_5 = -\frac{1}{2}, \qquad \beta_6 = -\frac{4}{5},$$
$$\gamma_n = \frac{n-2}{n} \qquad \text{for all } n \geqslant 3.$$

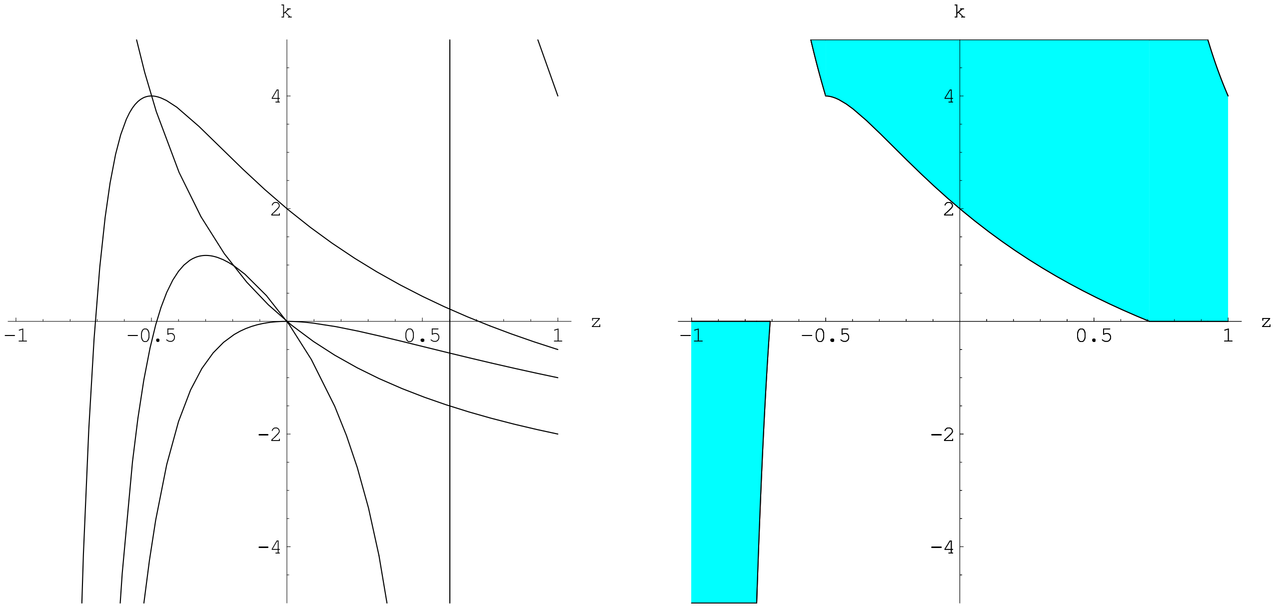

Fig. 3. The case $n = 5$. The curves g_0, g_1, g_2 and g_3 in the z–κ plane defining the stability regions. The stable regions are shaded on the right.

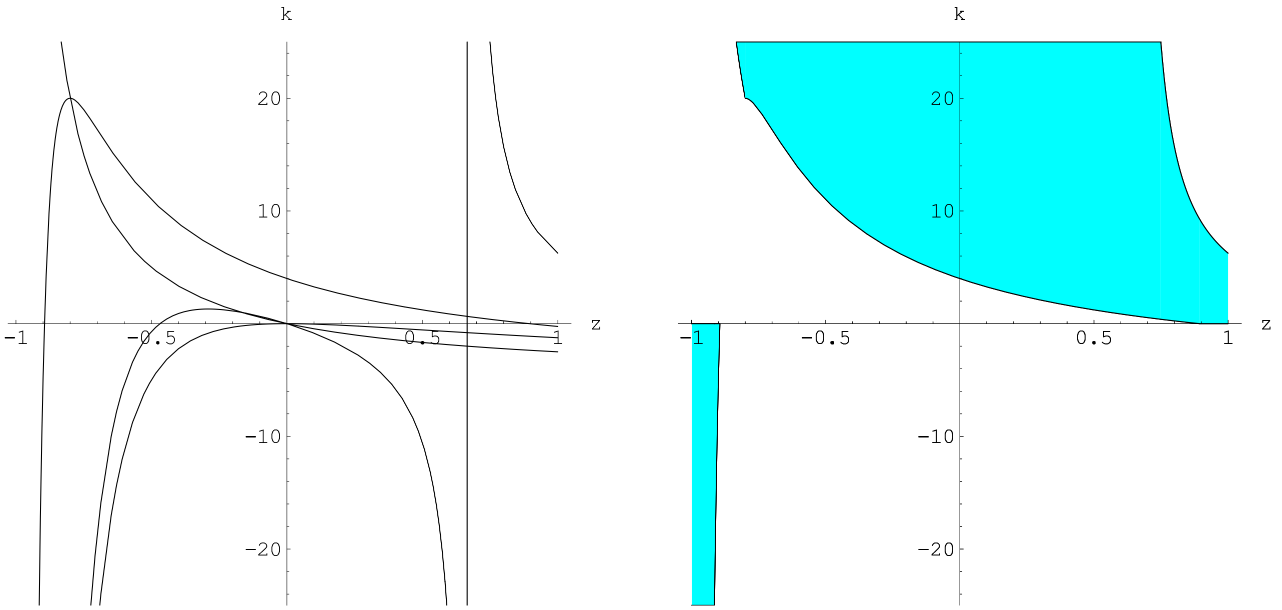

Fig. 4. The case $n = 6$. The curves g_0, g_1, g_2 and g_3 in the z–κ plane defining the stability regions. The stable regions are shaded on the right.

Theorem 1. *A ring of n unit vortices at the latitude z and a vortex of strength κ at the north pole of the sphere is stable in the following regions*

Case $n = 3$:

for $-1 < z \leqslant (2-\sqrt{13})/3$ *when* $g_3(z) < \kappa < 0$ *or* $g_2(z) < \kappa$,
for $(2-\sqrt{13})/3 \leqslant z < -1/3$ *when* $0 < \kappa < g_3(z)$ *or* $g_2(z) < \kappa$,
for $-1/3 \leqslant z \leqslant 0$ *when* $0 < \kappa < g_2(z)$ *or* $g_3(z) < \kappa$,

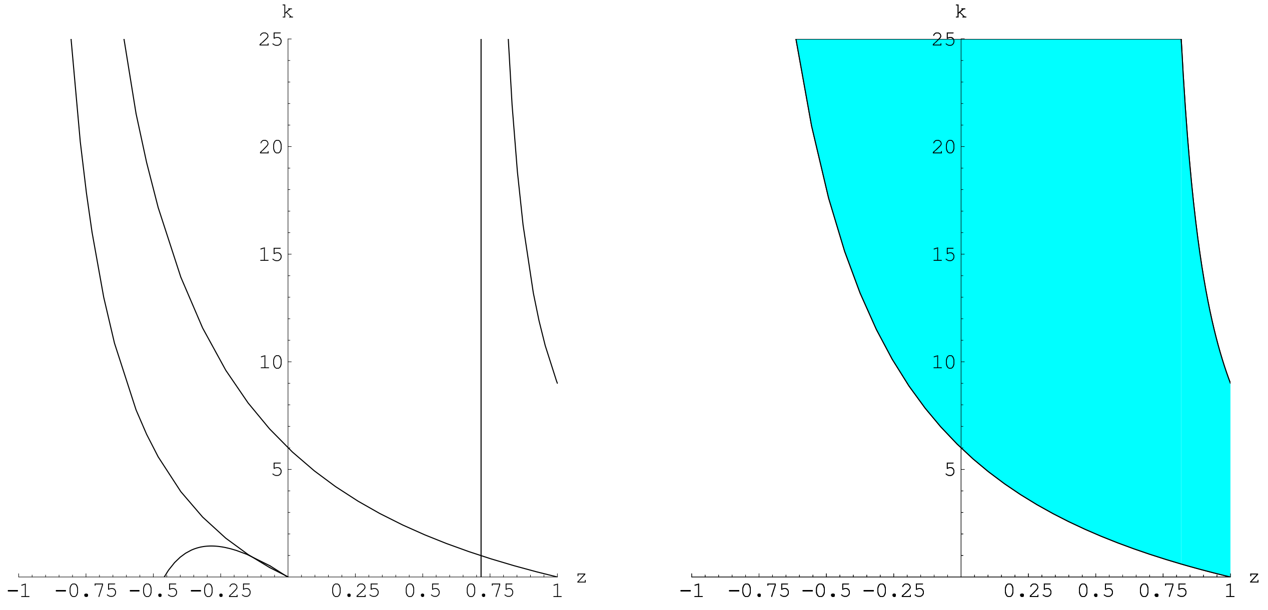

Fig. 5. The case $n = 7$. The curves g_0, g_2 and g_3 in the z–κ plane defining the stability regions. The stable regions are shaded on the right. Stability is only possible for $\kappa > 0$.

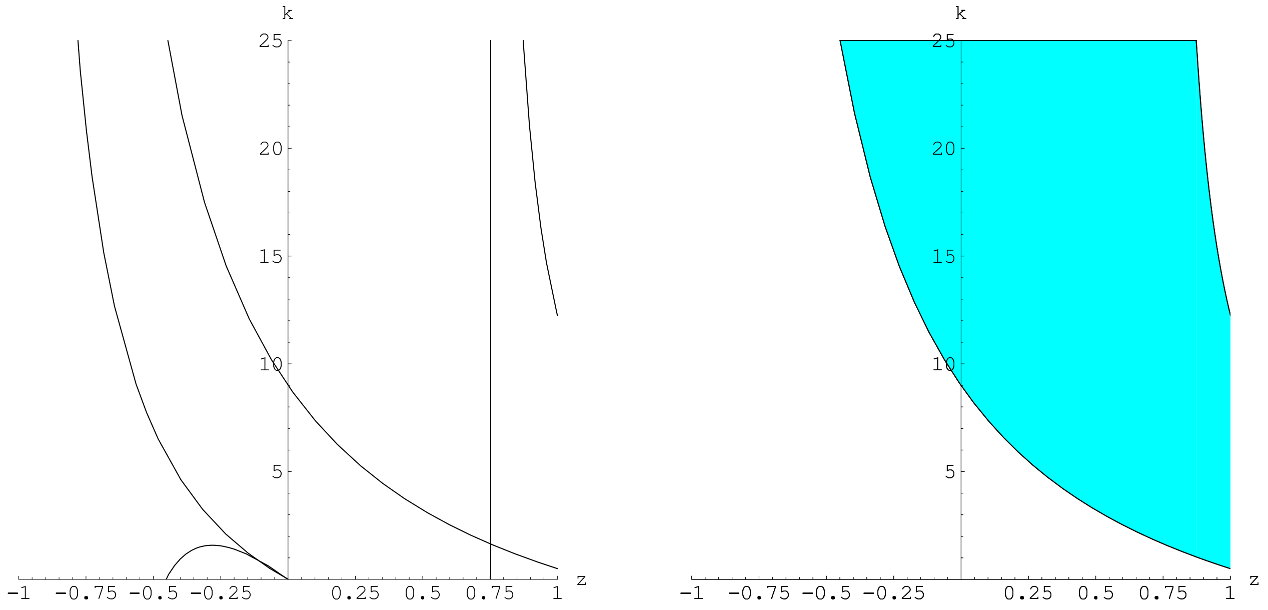

Fig. 6. The case $n = 8$. The curves g_0, g_2 and g_3 in the z–κ plane defining the stability regions. The stable regions are shaded on the right. Stability is only possible for $\kappa > 0$.

for $0 \leqslant z \leqslant 1/3$ *when* $0 < \kappa$,
for $1/3 < z < 1$ *when* $0 < \kappa < g_3(z)$.

Case $n = 4,\ 5,\ or\ 6$:

for $-1 < z \leqslant -\alpha_n$ *when* $g_0(z) < \kappa < 0$ *or* $g_2(z) < \kappa$,
for $-\alpha_n \leqslant z \leqslant \beta_n$ *when* $g_2(z) < \kappa$,
for $\beta_n \leqslant z \leqslant \gamma_n$ *when* $g_0(z) < \kappa$,
for $\gamma_n < z \leqslant \alpha_n$ *when* $g_0(z) < \kappa < g_3(z)$,

for $\alpha_n \leqslant z < 1$ *when* $0 < \kappa < g_3(z)$.

Case $n \geqslant 7$:

for $-1 < z \leqslant \gamma_n$ *when* $g_0(z) < \kappa$,

for $\gamma_n < z < 1$ *when* $g_0(z) < \kappa < g_3(z)$.

5. Two vortices at a fixed latitude and a vortex at the north pole

As was mentioned in Section 2 the case $n = 2$ requires slightly different formulas. Although a direct treatment of this case can be found in the literature, we will use the same method as given above for $n > 2$. The transformation $\tilde{T}$ will deliver the Hessian with invariant subspaces, which are then analyzed more easily.

The calculations for the Hessian of H_1 remain unchanged, and its eigenvalues are

$$\sigma_0 = 0 \qquad \sigma_1 = -\frac{1}{2}, \qquad \lambda_0 = -\frac{1+z^2}{2(1-z^2)^2} \qquad \lambda_1 = -\frac{z^2}{2(1-z^2)^2}$$

On the other hand some of the second order derivatives of H_2 are different and the modified set is listed below with i and j taking on the values 0 or 1:

$$\frac{\partial^2 H_2}{\partial z_i \partial z_j} = 0 \qquad\qquad \text{for } i \neq j,$$

$$\frac{\partial^2 H_2}{\partial z_j^2} = \frac{-\kappa}{2(1-z)^2}$$

$$\frac{\partial^2 H_2}{\partial \varphi_i \partial \varphi_j} = 0$$

$$\frac{\partial^2 H_2}{\partial \varphi_j \partial x_2} = \frac{\partial^2 H_2}{\partial z_j \partial y_2} = 0$$

$$\frac{\partial^2 H_2}{\partial \varphi_j \partial y_2} = -\frac{\kappa r(-1)^j}{2(1-z)}$$

$$\frac{\partial^2 H_2}{\partial z_j \partial x_2} = -\frac{\kappa(-1)^j}{2r(1-z)}$$

$$\frac{\partial^2 H_2}{\partial x_2^2} = -\frac{\kappa}{1-z}$$

$$\frac{\partial^2 H_2}{\partial y_2^2} = \frac{\kappa z}{1-z}$$

$$\frac{\partial^2 H_2}{\partial x_2 \partial y_2} = 0.$$

Also different than before is the outcome of $T^t u_{\varphi x}$ and the other products. When these calculations are performed the Hessian of H at the stationary solution will be

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \dfrac{-r\kappa}{\sqrt{2}(1-z)} \\ 0 & 0 & -\dfrac{\kappa(2+3z+\kappa(1+z))}{2(1-z^2)} & 0 & \dfrac{-\kappa}{r(1-z)\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\dfrac{1+z^2+\kappa(1+z)^2}{2(1-z^2)^2} & 0 & 0 \\ 0 & 0 & \dfrac{-\kappa}{r(1-z)\sqrt{2}} & 0 & -\dfrac{z^2+\kappa(1+z)^2}{2(1-z^2)^2} & 0 \\ 0 & \dfrac{-\kappa r}{(1-z)\sqrt{2}} & 0 & 0 & 0 & \dfrac{\kappa(z+2z^2-\kappa(1+z))}{2(1-z^2)} \end{pmatrix}. \tag{5.1}$$

As before we have the eigenvalue $\sigma_0 = 0$ and the eigenvalue

$$\lambda_0 = -\frac{1+z^2+\kappa(1+z)^2}{2(1-z^2)^2}. \tag{5.2}$$

Different are the two invariant subspaces of dimension 2 given by the matrices

$$m_1 = \begin{pmatrix} \dfrac{-1}{2} & \dfrac{-r\kappa}{\sqrt{2}(1-z)} \\ \dfrac{-\kappa r}{(1-z)\sqrt{2}} & \dfrac{\kappa(z+2z^2-\kappa(1+z))}{2(1-z^2)} \end{pmatrix} \text{ and } m_2 = \begin{pmatrix} -\dfrac{\kappa(2+3z+\kappa(1+z))}{2(1-z^2)} & \dfrac{-\kappa}{r(1-z)\sqrt{2}} \\ \dfrac{-\kappa}{r(1-z)\sqrt{2}} & -\dfrac{z^2+\kappa(1+z)^2}{2(1-z^2)^2} \end{pmatrix}.$$

It turns out that the eigenvalues of m_2 are reasonably simple expressions and are given by

$$\lambda_1 = -\frac{z^2+\kappa(1+z)^2}{2(1-z^2)^2} \tag{5.3}$$

$$\sigma_2 = -\frac{\kappa(2+3z+\kappa(1+z))}{2(1-z^2)} \tag{5.4}$$

We have selected this notation to indicate that the eigenvalue in (5.3) reduces to the λ_1 of the Hessian of H_1, that is without a vortex at the north pole.

The eigenvalues λ_2 and σ_1 are then those of m_1. Unfortunately their expressions are somewhat complicated. For this reason we limit ourselves to give them here in form of a series expansion in κ:

$$\sigma_1 = -\frac{1}{2} - \frac{1+z}{1-z}\kappa^2 + \cdots, \tag{5.5}$$

$$\lambda_2 = \frac{z(1+2z)}{2(1-z)}\kappa + \frac{1+2z}{2(1-z)}\kappa^2 + \cdots. \tag{5.6}$$

On the other hand the condition that both eigenvalues are negative reduces to the condition that the determinant of m_1 is positive. This determinant is

$$|m_1| = -\kappa(1+2z)(z+\kappa(1+z))/(4(1-z^2)),$$

so that the condition for both eigenvalues to be negative becomes

$$F(z,\kappa) = \kappa(1+2z)(z+\kappa(1+z)) < 0.$$

In order to check that all eigenvalues are negative we have to add to this $\lambda_0 < 0$, $\sigma_2 < 0$, and $\lambda_1 < 0$. From (5.2) and (5.3) it is obvious that $\lambda_0 < \lambda_1$, so that the condition for λ_0 can be ignored. The boundaries for these conditions can be found by plotting the curves for $F(z,\kappa) = 0$, $\sigma_2 = 0$ and $\lambda_1 = 0$. These equations give rise to the five curves or straight lines:

$$\kappa = 0, \qquad z = -\frac{1}{2}, \qquad \kappa = \frac{-z}{1+z}, \qquad \kappa = \frac{-z^2}{(1+z)^2}, \qquad \kappa = -\frac{2+3z}{1+z}.$$

These curves and lines are drawn on the left in figure 7, and the regions of stability are given on the right.

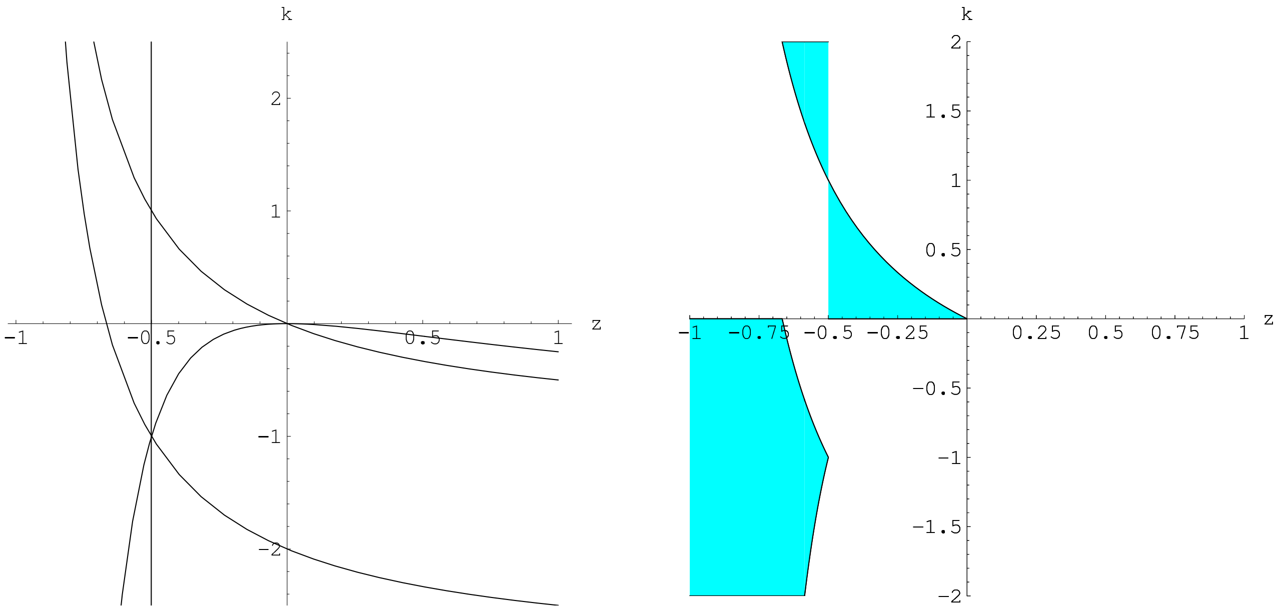

Fig. 7. The curves defining the regions of stability for $n = 2$. The regions of stability are shaded on the right

Surprising is that there is no region of stability for $z > 0$, although the two vortices without a vortex at the north pole are stable for all values of z. How this can happen can be seen by looking at the expressions for the eigenvalues λ_2 and σ_2 in (5.4) and (5.6) for small κ. It is seen from there that λ_2 and σ_2 always have different signs for small values of κ when z is in the interval $-2/3 < z < -1/2$ or in $0 < z < 1$.

6. Bifurcations of new configurations

In Section 4 we determined the stability of an equilibrium solution of (4.2). The solution consisted of a ring of n unit vortices at a latitude z and a vortex of strength κ at the north pole. The stability depends on the two parameters z and κ. The regions of stability were enclosed by curves in the z–κ plane. It is natural to ask if additional configurations can bifurcate at the curves where the stability changes.

The method to be used is the following. Given a fixed value z for the latitude find the values of κ where the stability changes. Let κ_c be one of these values and set $\kappa = \kappa_c + \mu$ with μ a small perturbation parameter. Put the system of equations into normal form and see if different configurations can be found for $\mu \neq 0$.

If $u = (\phi, x_n, \zeta, y_n)$ is the vector made up of the $2n+2$ coordinates then an equilibrium solution of (4.2) is found by solving $DH(u) = 0$. The rank of the Hessian $D^2H(u)$ is usually $2n+1$ as seen from (4.3), but it is diminished, whenever the stability changes. The places where this happens is determined by the eigenvalues of the given matrix. Except for two 3×3 submatrices the Hessian is already in diagonal form. The Hessian (4.3) can be put into diagonal form with the help of an additional noncanonical transformation.

The Hamiltonian function corresponding to the Hessian is

$$K = \frac{1}{2}\sum_{k=1}^{n-1}\sigma_k\phi_k^2 + \frac{1}{2}\sum_{k=0}^{n-1}\lambda_k\zeta_k^2 + \frac{\lambda_n}{2}(x_n^2+y_n^2) + \alpha(\phi_{n-1}x_n - \phi_1 y_n) + \beta(\zeta_{n-1}y_n + \zeta_1 x_n).$$

Recall that $\sigma_n = \lambda_n$, $\sigma_{n-k} = \sigma_k$ and $\lambda_k = \lambda_{n-k}$ for $k = 1, \ldots, n-1$. By completing squares we get

$$\begin{aligned} K = &\frac{1}{2}\left(\sum_{k=2}^{n-2}\sigma_k\phi_k^2 + \lambda_0\zeta_0^2 + \sum_{k=2}^{n-2}\lambda_k\zeta_k^2 + (\lambda_n - \frac{\alpha^2}{\sigma_1} - \frac{\beta^2}{\lambda_1})(x_n^2+y_n^2)\right) \\ &\frac{\sigma_1}{2}(\phi_1 - \frac{\alpha}{\sigma_1}y_n)^2 + \frac{\sigma_{n-1}}{2}(\phi_{n-1} + \frac{\alpha}{\sigma_1}x_n)^2 + \frac{\lambda_1}{2}(\zeta_1 + \frac{\beta}{\lambda_1}x_n)^2 + \frac{\lambda_{n-1}}{2}(\zeta_{n-1} + \frac{\beta}{\lambda_1}y_n)^2. \end{aligned}$$

Since $\sigma_1 = -(n-1)/2 \neq 0$ dividing by σ_1 is not a problem. But $\lambda_1 = -\dfrac{(n-1)z^2 + \kappa(1+z)^2}{2(1-z^2)^2}$ can be zero and it happens at the values $\kappa = g_1(z)$ with g_1 given by (4.8). It was shown there that the stability of an equilibrium solution can not change at $\kappa = g_1(z)$. Thus, we can assume that $\lambda_1 \neq 0$ for the values of z and κ which we will investigate. With this in mind we can use the following noncanonical transformation to a new set of variables (ψ, x_n, ξ, y_n) given by

$$\begin{aligned} \phi_k &= \psi_k & \zeta_k &= \xi_k & &\text{for } k = 0, 2\ldots, n-2, \\ \phi_1 &= \psi_1 + \frac{\alpha}{\sigma_1}y_n, & \zeta_1 &= \xi_1 - \frac{\beta}{\lambda_1}x_n, \\ \phi_{n-1} &= \psi_{n-1} - \frac{\alpha}{\sigma_1}x_n, & \zeta_{n-1} &= \xi_{n-1} - \frac{\beta}{\lambda_1}y_n. \end{aligned}$$

This transformation brings the quadratic terms into the form

$$K = \frac{1}{2}\left(\sum_{k=1}^{n-1}\sigma_k\psi_k^2 + \sum_{k=0}^{n-1}\lambda_k\xi_k^2 + \tilde{\lambda}(x_n^2+y_n^2)\right) \tag{6.1}$$

with

$$\tilde{\lambda} = \kappa F_2(z,\kappa)F_3(z,\kappa),$$

that is $\tilde{\lambda} = |m_3|$ with the functions F_2 and F_3 defined in (4.6) and (4.7). From (6.1) it is seen that the rank of the Hessian changes when $\kappa = 0$, or $F_2(z,\kappa) = 0$ or $F_3(z,\kappa) = 0$, or $\lambda_k = 0$ for $k = 0, 2, 3, \ldots, n-2$. As mentioned previously the case $\lambda_1 = 0$ has to be excluded. Also $\kappa = 0$ has to be excluded, since $\dot{x}_n$ and $\dot{y}_n$ are multiplied with κ and $\kappa = 0$ is a singularity of the differential equations. Since $\lambda_{n-k} = \lambda_k$ for $k = 2, \ldots, n-2$ it suffices to look at just half of these values. It also shows that the rank of the Hessian is reduced by 2, except for $k = 0$. When n is even then for $k = n/2$ the rank of the Hessian is also reduced only by one. Thus, the values of κ where a bifurcation is possible is given by

$$\begin{aligned} \kappa_0 &= -\frac{(n-1)(1+z^2)}{(1+z)^2} & &\text{originating from } \lambda_0 = 0, \\ \kappa_1 &= -\frac{(n-1)z(2z-n-2nz+nz^2)}{(2-n+nz)(1+z)^2} & &\text{originating from } F_3 = 0, \\ \kappa_k &= \frac{-(n-1)(1+z^2)+k(n-k)}{(1+z)^2} & &\text{originating from } \lambda_k = 0, \end{aligned}$$

with $k = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$ the range of indices to be considered. There is one more value

$$\tilde{\kappa} = -\frac{(n-1)z}{1+z} \qquad \text{originating from } F_2 = 0,$$

where the Hessian is degenerate. For this value we only found the trivial solution to the bifurcation equations, that is, no new configuration nearby. Therefore, we exclude it as an uninteresting case from what follows.

The transformation used above destroyed the Hamiltonian nature of the given system. Since we are looking for equilibrium solutions, that is, critical points of the Hamiltonian function, the same critical points are also found in a noncanonical coordinate system. We can thus use an additional near identity transformation to bring this function into Sylvester normal form. This transformation will be carried out with the help of the Lie transformation of Deprit, see [10] and [18].

Let $u = (\psi, x_n, \xi, y_n)$ be the coordinate vector of the function $K(u, \mu)$, whose quadratic terms are already in normal form. By rotating a given configuration by a fixed angle around the polar axis another configuration is found. In order to remove this rotational symmetry we place the first vortex on the x-axis, that is, we set $\varphi_0 = 0$. It means that ψ_0 is determined by the other variables and $K(u, \mu)$ will not depend on ψ_0. Thus, u has dimension $2n+1$.

Let ϵ be a formal parameter and scale all variables including μ by ϵ. We then write

$$K_\star(u, \mu, \epsilon) = \sum_{i=0} \frac{\epsilon^i}{i!} K_i^0(u, \mu)$$

where $K_\star(u, \mu, 1) = K(u, \mu)$ and K_i^0 are homogeneous polynomials of degree $i+2$ in the $2n+2$ variables of u and μ. The Lie transformation of Deprit, see also [19], constructs a near identity change of variables

$$u = u(v, \mu, \epsilon) = v + \cdots$$

where u is the solution of a system of differential equations given by

$$\frac{du}{d\epsilon} = W(u, \mu, \epsilon), \qquad u|_{\epsilon=0} = v.$$

The vector function W has the formal expansion

$$W(u, \mu, \epsilon) = \sum_{i=0} \frac{\epsilon^i}{i!} W_{i+1}(u, \mu).$$

In the new coordinates write

$$K^\star(v, \mu, \epsilon) = K_\star(u(v, \mu, \epsilon), \mu, \epsilon) = \sum_{j=0} \frac{\epsilon^j}{j!} K_0^j(v, \mu).$$

The functions $K_\star$ and $K^\star$ are related by the double indexed array K_i^j, which agrees with previously defined values when i or j are zero. The array can be computed iteratively via

$$K_i^j = K_{i-1}^{j+1} + \sum_{k=0}^{i} \binom{i}{k} \left[K_{i-k}^{j-1}, W_{k+1} \right],$$

where $[\ ,\ ]$ is the Lie derivative operator on functions given by

$$[K, W] = \frac{\partial K}{\partial u} W.$$

The transformed function is then obtained from $K^\star(v,\mu,1)$. By selecting W_i appropriately at each step of the algorithm a normal form for $K(u,\mu)$ can be achieved, where K consists of a sum of quadratic terms, plus a function G, which starts with at least third order terms and whose variables are determined by the nullspace of the Hessian. In catastrophe theory [24] this is called the splitting lemma. It shows that finding additional critical points of K near the origin depends on finding additional critical points of the function G. The Hessian is never degenerate with respect to the angular variables ψ_1 to ψ_{n-1}. It means that for any critical points of K the angular variables remain unchanged, i.e. $\psi_1 = \cdots = \psi_{n-1} = 0$. Stated differently it means that a new configuration which is found by this bifurcation analysis near κ_k is obtained from the original one by displacing the vortices along their lines of longitude.

For κ_0, and for $\kappa_{n/2}$ when n is even, the nullspace of the Hessian is one dimensional. Besides the dependency on μ the function G depends only on ξ_0 or on $\xi_{n/2}$ in the second case. These two cases can be treated completely for all values of n. We do this first before considering the more interesting cases of the two dimensional nullspaces in the next section.

When the critical value of the vorticity at the north pole is κ_0 and we set $\kappa = \kappa_0 + \mu$ we find that the function G has the form

$$G(\xi_0,\mu) = a\mu\xi_0^2 + b\xi_0^3 + \text{h.o.t.},$$

that is the desired bifurcation equation appears already among the third order terms in the normalized function. The coefficients a and b will depend on the parameter z and they can be found analytically. Since all terms in K_1^0 can be eliminated except those which depend on ξ_0 and μ we can set

$$\begin{aligned} z_j &= z + \xi_0/\sqrt{n} && \text{for } j = 0,\dots,n-1 \\ \varphi_j &= 2\pi j/n && \text{for } j = 0,\dots,n-1 \\ x_n &= y_n = 0 \end{aligned} \tag{6.2}$$

and find the third order terms in the functions (4.2). The function simplifies to

$$H = \frac{1}{2}\sum_{i<j} \ln\left(1-(z+\frac{\xi_0}{\sqrt{n}})^2\right)\left(1-\cos\frac{2\pi(j-i)}{n}\right) + \frac{\kappa_0+\mu}{2}\sum_{i=0}^{n-1}\ln\left(1-z-\frac{\xi_0}{\sqrt{n}}\right) - wn\left(z+\frac{\xi_0}{\sqrt{n}}\right).$$

The third order terms then follow from

$$\frac{n(n-1)}{4}\ln\left(1-(z+\frac{\xi_0}{\sqrt{n}})^2\right) + \frac{\kappa_0+\mu}{2}n\ln\left(1-z-\frac{\xi_0}{\sqrt{n}}\right)$$

to give

$$G(\xi_0,\mu) = \frac{(n-1)\xi_0^3}{6\sqrt{n}(1-z)(1+z)^3} - \frac{\mu\xi_0^2}{4(1-z)^2} + \text{h.o.t.}$$

Besides the extremum when $\xi_0 = 0$ another one is found from G for small μ when

$$\xi_0 = \frac{\sqrt{n}}{n-1}\frac{(1+z)^3}{(1+z)}\mu + O(\mu^2).$$

On the other hand (6.2) indicates that the latitude of all vortices of the ring is changed by the same amount. The solution found is the same ring of vortices at a different latitude. The bifurcation analysis has not given anything new in this case.

More interesting is the case when n is even, i.e. $n = 2m$, in which case we have

$$\kappa_{n/2} = \kappa_m = \frac{(m-1)^2-(2m-1)z^2}{(1+z)^2}.$$

The bifurcation equation will have the form

$$G(\xi_m, \mu) = a\mu\xi_m^2 + b\xi_m^4 + \text{h.o.t.}$$

The function has an extremum at

$$\xi_m = \sqrt{-\frac{a}{2b}\mu} + O(\mu). \tag{6.3}$$

It means that to first order

$$\begin{aligned} z_j &= z + (-1)^j \xi_m/\sqrt{n} && \text{for } j = 0, \ldots, n-1 \\ \varphi_j &= 2\pi j/n && \text{for } j = 0, \ldots, n-1 \\ x_n &= y_n = 0. \end{aligned}$$

The equations for the z_j's show that half of the vortices move north, the other half south. In other words the original ring of $2m$ vortices splits into two rings at different latitudes with m vortices each. The bifurcation into the two rings occurs either for $\mu > 0$ or $\mu < 0$ depending on the signs of a and b.

The coefficient a in the function $G(\xi_m, \mu)$ can only come from what was called H_2 and it is easily computed to be $a = -1/(4(1-z)^2)$. The coefficient b can be traced back to third order terms in H, which must have the factor ξ_m^2 and to the fourth order term with ξ_m^4 in H. Due to symmetries among the required terms only the coefficient of $\xi_0\xi_m^2$ is nonzero in H. Let us call it b_3 and the coefficient of ξ_m^4 will be called b_4, so that the terms of interest in H are $b_3\xi_0\xi_m^2 + b_4\xi_m^4$.

The normalization of these terms produces

$$b = -\frac{b_3^2}{2\tilde{\lambda}_0} + b_4 = \frac{b_3^2(1-z^2)^2}{m^2} + b_4,$$

where $\tilde{\lambda}_0$ stands for the eigenvalue λ_0 evaluated at $\kappa = \kappa_m$. The coefficients b_3 and b_4 can be computed with the help of Mathematica for a given value of m. Unfortunately the general formula appears to be complicated so that we restrict ourselves to give (6.3) for a few values of m.

Case $n = 4$:

$$\xi_2 = (1+z)\sqrt{\frac{-8(1-z^2)\mu}{7+9z^2}},$$

Case $n = 6$:

$$\xi_3 = (1+z)(1-z^2)\sqrt{\frac{-54\mu}{184+56z+15z^5-70z^3-50z^4}},$$

Case $n = 8$:

$$\xi_4 = 8(1+z)(1-z^2)\sqrt{\frac{-\mu}{559+180z-38z^2-140z^3-49z^4}}.$$

None of the denominators has a zero in the interval $-1 < z < 1$.

7. Bifurcations under symmetry

The Hamiltonian function H of (4.2) possesses symmetries which are not obvious in the variables which are used in connection with the circulant matrices. With $\alpha = 2\pi/n$ the transformation is listed here in some details:

$$z_j = z + \sqrt{\frac{2}{n}}(\zeta_0/\sqrt{2} + \zeta_1 \cos j\alpha + \zeta_2 \cos 2j\alpha + \cdots + \zeta_{n-2} \sin 2j\alpha + \zeta_{n-1} \sin j\alpha)$$

$$\varphi_j = j\alpha + \sqrt{\frac{2}{n}}(\phi_0/\sqrt{2} + \phi_1 \cos j\alpha + \phi_2 \cos 2j\alpha + \cdots + \phi_{n-2} \sin 2j\alpha + \phi_{n-1} \sin j\alpha)$$

for $j = 0, \ldots, n-1$. When replacing $\zeta_j \to -\zeta_{n-j}$ for $j = \lfloor n/2 \rfloor + 1, \ldots, n-1$ and leaving the other ζ_j's unchanged we achieve $z_j \to z_{n-j}$. Similarly replacing $\phi_j \to -\phi_j$ for $j = 0, \ldots, \lfloor n/2 \rfloor$ we achieve $\varphi_j \to \varphi_{n-j} \bmod 2\pi$. If in addition we set $x_n \to x_n$ and $y_n \to -y_n$ then the Hamiltonian function (4.2) remains invariant. This symmetry will show up in the normal form when the nullspace is two dimensional, that is in

$$G = G(x_n, y_n, \mu) \qquad \text{and in} \qquad G = G(\xi_j, \xi_{n-j}, \mu) \qquad \text{for } j = 2, 3, \ldots.$$

Both cases can be considered together in the function $G = G(x, y, \mu)$ which has the symmetry $G(x, y, \mu) = G(x, -y, \mu)$. The discussion follows the one given in [20]. Let $x = r\cos\varphi$, $y = r\sin\varphi$. We can expect G to have the form

$$G = p_1 r^2 + p_2 r^4 + \cdots + q_1 r^d \cos d\varphi + \cdots$$

with the p's and q's functions of μ and $d > 2$ a divisor of n. Generically we can expect these functions to be different from zero. From the discussion of the Hessian we can expect to find $p_1(\mu) = a\mu r^2 + \cdots$ with a a nonzero constant. We also expect that $q_1(0) = c$ is nonzero and when $d = 4$ then with $p_2(0) = b$ also $b \neq c$ will hold.

Three cases can occur:

Case $d = 3$:

$$G = a\mu r^2 + cr^3 \cos 3\varphi + \cdots.$$

From $\partial G/\partial r = \partial G/\partial\varphi = 0$ we find $\varphi = 0, \pi/3, 2\pi/3, \ldots$ and $r = \pm 2a\mu/(3c) + \cdots$. Since in this case μ can be positive and negative we can restrict ourselves to the solution with $\varphi = 0$, that is to

$$x = -\frac{2a}{3c}\mu + \cdots \qquad\qquad y = 0 + \cdots.$$

The other solutions can be obtained via rotations by a multiple of $\pi/3$.

Case $d \geqslant 5$:

$$G = a\mu r^2 + br^4 + \cdots + cr^d \cos d\varphi + \cdots.$$

Besides the extremum for $r = 0$ others occur when $\varphi = k\pi/d$ with $k = 0, 1, \ldots, 2d-1$ and $r = \sqrt{-a\mu/(2b)} + \cdots$. Real solutions exist only for $\mu > 0$ when $ab < 0$ or for $\mu < 0$ when $ab > 0$. Different solutions are found with $k = 0$ or $k = 1$. The other solutions can be obtained from those via a rotation.

Case $d = 4$:

$$G = a\mu r^2 + r^4(b + c\cos 4\varphi) + \cdots.$$

New extrema can occur when $\varphi = 0$ or $\varphi = \pi/4$. As before $\varphi = k\pi/4$ with $k = 2, \ldots, 7$ gives only solutions, which can be found from the first two by a rotation. The derivative of G with respect to r gives $2a\mu r + 4r^3(b \pm c) + \cdots = 0$. It now depends on the signs of $b + c$ and $b - c$. If $b + c$ and $b - c$ have the same sign then new solutions are only found on one side of $\mu = 0$. In the other case solutions exist for $\mu > 0$ and for $\mu < 0$, but these solutions can not be considered to be a continuation of each other, as was the case for $d = 3$.

For the function $G(x_n, y_n, \mu)$ we found always that $d = n$, but for the other cases it was less predictable what d might be. For larger values of n the complexity of the computations prevented us from keeping z as a formal parameter, as one has to compute terms of order n in order to obtain the required resonance terms. For $n = 3$ and $n = 4$ at most fourth order terms are needed and the two cases could be handled with the help of Mathematica. These two cases are also typical examples for the cases when $d = 3$ or $d = 4$.

For $n = 3$ we computed

$$r = -\frac{3(1+z)^4(1-3z)^3\sqrt{1-z^2}}{8z^2(1+3z)^2(9+20z+3z^2)}\mu + \cdots.$$

Of interest are the zeros and poles of this function in $-1 < z < 1$. The function is zero for $z = 1/3$, that is where g_3 has a pole. It indicates that no bifurcation is possible at this value of z. The poles of r at $z = 0$ and $z = -1/3$ can be an indication for the fact that the stability boundaries g_2 and g_3 intersect each other at these values. On the other hand it is surprising that the quadratic term in the denominator of r has a zero at $z = (-10 + \sqrt{73})/3$. This value inside $-1 < z < 1$ does not appear to have any relationship to features of the regions of stability.

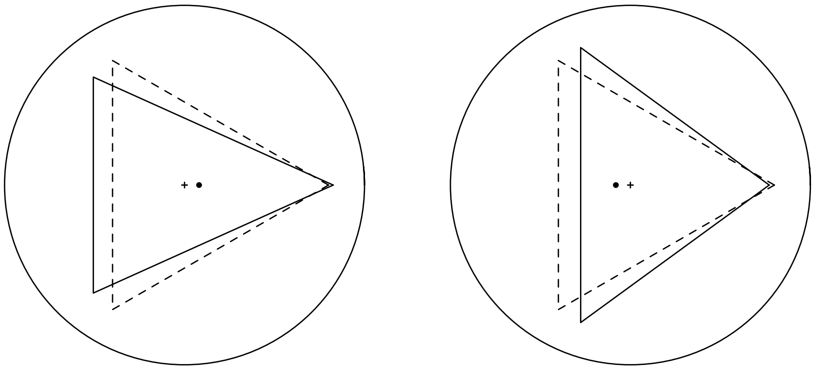

Fig. 8. Dashed lines indicate original configuration at a latitude $z = 0.6$. Solid lines give new configuration, drawn for a large value of $\mu = 5.0$ in order to show that the vortex at the north pole has moved, as indicated by the small solid disk. The configuration on the right is drawn for $\mu = -5.0$

For $n = 4$ we also have one 2-dimensional null space and the resonance terms occur with $d = 4$. Thus, we have from the bifurcation equations two values for r^2. For the case $\varphi = 0$ we computed

$$r^2 = -\frac{4(1-2z)^2(1-z)(1+z)^7(1-12z^2-32z^3-12z^4)}{3z(1+4z)(4+12z-48z^2-219z^3+\cdots+1548288z^{19})}\mu + \cdots \tag{7.1}$$

and for $\varphi = \pi/4$ we found

$$r^2 = -\frac{8(1-2z)^4(1-z)(1+z)^7(1-4z-2z^2)}{3z(1+4z)(8-8z-76z^2+\cdots+448z^{11})}\mu + \cdots \tag{7.2}$$

Of interest are the real roots in $-1 < z < 1$ of the numerators and denominators for r. Both numerators are zero in this interval for $z = 1/2$ and $z = (-2 + \sqrt{6})/2$. The denominator of (7.1) is zero when z is one of the values in the list (-0.8343, -0.5300, -0.3003, -0.25, 0). The zeros of the denominator in (7.2) are given by one of the values (-0.3022, -0.25, 0, 0.264). All zeros indicate that the bifurcation that existed on one side of the critical value of the vorticity now happens at the other side.

For $n > 4$ we were not able to carry out the calculations with z kept as a formal parameter. The expansion of the Hamiltonian function H into a sum of homogeneous polynomials proved to be too time consuming, as terms through order n have to be included. Instead we selected a value for z from the beginning, so that the coefficients of theses polynomials would be numerical values and could be stored as floating point numbers in the computer. The computations were carried out with our own algebraic processor Polypack. Not surprisingly the results look very similar to those obtained for the planar problem in [20].

The computations in the planar case were considerably simplified by the use of complex variables. The coordinates for the vortices on the sphere are given with real variables. We did not see a natural way to simplify them with the help of complex variables. It means that for a given n we have to manipulate twice as many variables as compared to the planar case and thus we could not go

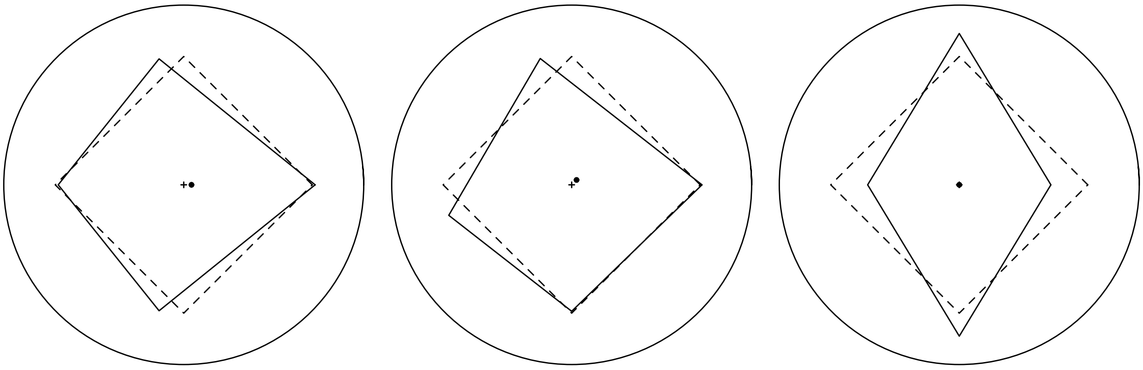

Fig. 9. Dashed lines indicate original configuration at a latitude $z = 0.7$. The kite and trapezoid on the left are generated by the critical value κ_1. The rhombus comes from κ_2.

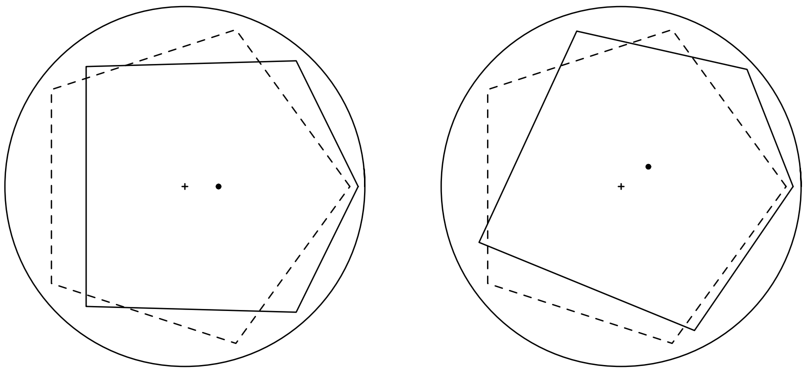

Fig. 10. Dashed lines indicate original configuration for $n = 5$ at $z = 0.4$. Bifurcations at κ_1

as far in n as in [20]. Since the results look very similar to those in [20] it suffices to give here just one sample. Figure 10 and 11 depict the bifurcations which are possible from the pentagon. Numerical computations do not reveal where the solutions of the bifurcation equations may have zeros or singularities unless one selects z to be exactly at one of these values. Nevertheless, the absolute values of the coefficients give some indications.

8. Bifurcation from two vortices at a fixed latitude with a vortex at the north pole

Although the equilibrium positions of three vortices on a sphere has been analyzed fully in [21], this section is added for completeness sake and to show where the change of stability of an equilibrium solution gives rise to another stationary solution. The Hamiltonian corresponding to the Hessian (5.1) is

$$H_2 = \frac{1}{2}\left(\sigma_1\phi_1^2 + \sigma_2 x_2^2 + \lambda_0\zeta_0^2 + \lambda_1\zeta_1^2 + \lambda_2 y_2^2\right) - \alpha\phi_1 y_2 + \beta\zeta_1 x_2 \tag{8.1}$$

where σ_1, λ_0, and λ_1 follow from the formulas given early with $n = 2$, but now

$$\alpha = \frac{\kappa\sqrt{1-z^2}}{\sqrt{2}(1-z)} \qquad\qquad \beta = -\frac{\kappa}{\sqrt{2(1-z^2)}(1-z)}.$$

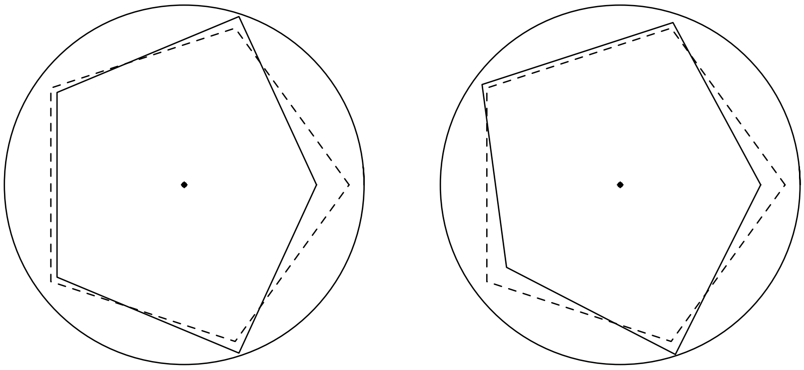

Fig. 11. Dashed lines indicate original configuration for $n = 5$ at $z = 0.4$. Bifurcations at κ_2

and

$$\sigma_2 = -\frac{\kappa(2 + 3z + \kappa(1 + z))}{2(1 - z^2)} \qquad \lambda_2 = \frac{\kappa(z + 2z^2 - \kappa(1 + z))}{2(1 - z^2)}$$

are the values computed in Section 5 when $n = 2$. Again we complete squares in H_2 and introduce noncanonical variables this time by

$$\phi_1 = \psi_1 + \frac{\alpha}{\sigma_1} y_2, \qquad \zeta_1 = \xi_1 - \frac{\beta}{\lambda_1} x_2,$$

with the other variables left unchanged. The quadratic terms of (8.1) are transformed into

$$K = \frac{1}{2}\left(\sigma_1\psi_1^2 + \lambda_0\psi_0^2 + \lambda_1\psi_1^2 + (\sigma_2 - \frac{\beta^2}{\lambda_1})x_2^2 + (\lambda_2 - \frac{\alpha^2}{\sigma_1})y_2^2\right).$$

When a coefficient of a quadratic term becomes zero the Hessian becomes degenerate and bifurcations are possible. Obviously $\lambda_1 = 0$ has to be excluded, due to our choice of coordinates. Since values of z and κ where $\lambda_1 = 0$ do not lead to a change of stability, this is not a restriction. Also $\kappa = 0$ reduces the rank of the Hessian by two, but this value has to be excluded due to the form of the differential equations as was mentioned in Section 6.

The rank of the Hessian is reduced by two when

$$\kappa = -\frac{z}{1 + z},$$

but the bifurcation equations have only the trivial solution as was the case for $n \geqslant 3$. Another place where the Hessian becomes degenerate occurs when $\lambda_0 = 0$. There the two vortices move to the same latitude nearby, so that again it is not an interesting case. There is one more value

$$\kappa = -\frac{2z + 3z^2}{(1 + z)^2}$$

where the Hessian is degenerate and reduces its rank by one. The value corresponds to what was denoted by κ_1 in Section 6. In this case we find by normalizing the function through fourth order, that for $\mu > 0$ or $\mu < 0$ the vortex at the north pole can move to

$$x_2 = \pm\frac{(1 + z)^3}{|1 + 2z|}\sqrt{\frac{2(1 - z^2)(1 - 2z - 2z^2)\mu}{z(4 + 4z - 8z^2 - 3z^3 + 4z^4 - 2z^5 + 4z^6)}}, \qquad y_2 = 0.$$

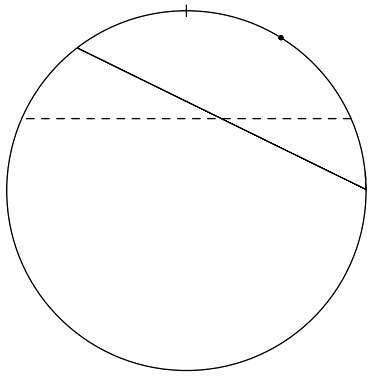

Fig. 12. Bifurcations when $n = 2$ as seen in the $x - z$ plane. In original configuration the vortices are at the end of the dashed line, and another one of strength κ_1 at the north pole. In new configuration vortices are at the end of the solid line, plus another one indicated by a small disk.

Since the x-axis always points to vortex 0, the whole configuration stays in the $x-z$ plane as illustrated in figure 12. With this the vortices remain on a great circle, which is one of the possibilities mentioned in [21].

Since the position of the vortex at the north pole was given in Cartesian coordinates both signs of the square root can be used. On the other hand the sign of μ is determined by the value of the terms under the square root as it depends on z in the interval $-1 < z < 1$. The sign of μ has to change at $z = (-1 + \sqrt{3})/2$, which is a zero of the numerator, and it has to change at $z = 0$, which is the only real zero of the denominator. Also the value $z = -1/2$ causes difficulties. By looking at the stability region in figure 7 it is not surprising that $z = -1/2$ and $z = 0$ are exceptional values. On the other hand $z = (-1 + \sqrt{3})/2$ does not correspond to any special feature of the stability region.

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