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Institute of Thermomechanics Czechoslovak Academy of Sciences

ENTRAINMENT DOMAINS

Meyer K.R., Schmidt D.S.

University of Cincinnati, U.S.A.

ABSTRACT: This paper developes effective algorithms for computing the domains in a parameter space where a differential equations which admits an invariant torus has a periodic solution with fixed rotation number. The algorithms are based on the method of Lie transforms and have been implemented on a computer. The algorithms are applied to two variations of van der Pol's equation.

INTRODUCTION

This is a summary of the results contained in [4]. The majority of the literature on invariant tori for ordinary differential equations does not concern itself with the flow on the tori itself, since the work of Poincare and Denjoy have shown that these flows can be quite varied and complicated. However, we choose a specific class of equations with the goal of developing an effective procedure for obtaining quanitative information.

The prototype has been the forced van der Pol equation. Therefore, we illustrate our procedure by studying this equation and a system of two weakly coupled van der Pol equations introduced by Linkens [3] in a study of the electrical activity of the human gastrointestinal tract. For the forced van der Pol equation there are several studies on harmonic entrainment but few when the forcing frequency and natural frequency are considerably different. Hayashi [2] and others have considered the cases when the ratio of the natural and forcing frequency is near 1 to 2 and 1 to 3 in detail. He found that for a small range of detuning a periodic solution with frequency near the forcing frequency exists. Since other frequency ratios require long computations they were not considered until now.

GENERAL PROBLEM

In order to fix some definitions, consider a system of equations of the form

$$\dot{x} = f(x,\lambda) \qquad (1.1)$$

where f is a smooth function from $B^n \times B^k$ into R^n, B^n and B^k are open balls in R^n and R^k respectively, and $\dot{x} = dx/dt$. Suppose that for each $\lambda \in B^k$ the system (1.1) has a unique, smooth, two-dimensional invariant torus $T_\lambda \subset B^n$ and that it varies smoothly with λ. Also let C_λ be a smooth closed curve on T_λ which is a global cross section for the flow on T_λ and that C_λ varies smoothly with λ. Thus to the flow flow on T we may associate a real number $\rho(\lambda)$-the rotation number. Also $\rho(\lambda)$ is rational if and only if the flow on T_λ has a periodic solution. Let Γ_λ be another smooth, closed curve on T_λ which varies smoothly with λ and is such that C_λ and Γ_λ form a base for the first homology group of T_λ. If $\rho(\lambda)$ is rational, say $\rho(\lambda) = p/q$ where $(p,q) = 1$, then the periodic solution on T_λ are homologous to $pC_\lambda + q\Gamma_\lambda$. Thus a rational rotation number has a simple geometric interpretation: if the rotation number is p/q where $(p,q) = 1$ then the periodic solutions of (1.1) on T_λ wind p times around C_λ and q times around Γ_λ before closing.

Following Bushard we define the p/q entrainment domain to be $A_{p/q} = \rho^{-1}(p/q)$. Since ρ is a continuous, $A_{p/q}$ is closed in B^k. Clearly distinct rational numbers give rise to disjoint entrainment domains.

Even though the rotation number is continuous in λ it will not be differentiable in general. This is due to the "locking-in phenomenon" of the "entrainment of frequency phenomenon". Restricting our attention to the flow on the two-dimensional torus a periodic solution has 2 characteristic multipliers 1 and $\mu, \mu > 0$. If $\mu \neq 1$ the periodic solution is called hyperbolic - a source if $\mu > 1$ and a sink if $0 < \mu < 1$. If a periodic solution is hyperbolic, an easy application of the implicit function theorem implies that small perturbations of the equations have a periodic solution with the same rotation number. Thus if for $\lambda = \lambda_0$ equation (1.1) has a hyperbolic periodic solution with rotation number p/q then λ_0 is an interior point of $A_{p/q}$. In general one expects that most periodic solutions are hyperbolic. For generic one parameter families of flows on a torus, it is a consequence of the work of Sotomayor that the entrainment domains are unions of nontrivial closed intervals and these intervals do not cluster. Thus generically the rotation number as a function of a single parameter has the essential qualitative features of the Cantor ternary function.

Bushard [1] has shown that there are positive numbers ε_0, δ_0 and continuous functions

$$a,b : I = [-\varepsilon_0,\varepsilon_0] \to J = [p/q - \delta_0,\ p/q + \delta_0]$$

such that $a(0) = b(0) = p/q$ and

$$A_{p/q} = (I \times J) = \{(\varepsilon,\omega) : 0 \leq \varepsilon \leq \varepsilon_0 \text{ and } a(\varepsilon) \leq \omega \leq b(\varepsilon)\}.$$

That is, close to $(0,p/q)$ the entrainment domain, $A_{p/q}$, is a sector bounded above and below by continuous curves which pass through $(0,p/q)$. The two curves a and b will be called the local boundary curves for $A_{p/q}$ and the set $A_{p/q} \cap (I \times J)$ will be the local boundary sector of $A_{p/q}$.

In order to calculate these domains we develop an algorithm based on the method of Lie transforms which transform a system of equations of the form

$$\dot{x} = Ax + g(x,\varepsilon) \tag{2.1}$$

where A is diagonalizable, $g(x,0) = 0$ and g has a formal expansion in ε with coefficients which are polynomials in x into a system

$$\dot{y} = Ay + h(y,\varepsilon) \tag{2.2}$$

where h is of the same form as g with the additional property that $h(e^{At}y,\varepsilon) \equiv h(y,\varepsilon)$.

With A. Deprit we have written a PL/I program which performs the algebraic manipulations to effect the transformation described above. The abundance of examples given in [4] illustrate the effectiveness of the programs and the method.

EXAMPLES

We consider two examples, the forced van der Pol equation

$$\ddot{u} + \varepsilon(u^2 - 1)\dot{u} + \omega_1^2 u = A\cos\omega_2 t \tag{3.1}$$

and a pair of weakly coupled van der Pol equations

$$\begin{aligned} &\ddot{u}_1 + \varepsilon\{(u_1 + \varepsilon\lambda u_2)^2 - 1\}\dot{u}_1 + \omega_1^2(u_1 + \varepsilon\lambda u_2) = 0 \\ &\ddot{u}_2 + \varepsilon\{(u_2 + \varepsilon\lambda u_1)^2 - 1\}\dot{u}_2 + \omega_2^2(u_2 + \varepsilon\lambda u_1) = 0 \end{aligned} \tag{3.2}$$

Both of these equations are in the form (2.1) and by a classical theorem admit an invariant torus for small ε provided $\omega_1 \neq \omega_2$. We shall illustrate the general theorems contained in [4] by stating them for the forced van der Pol equation (3.1) only. Consider the first quadrant in the parameter plane (ε,Ω) where $\Omega = \omega_1^2/\omega_2^2$.

Consider the case when $p + q$ is odd and let $m = 3p + q - 1$. We develop an effective finite algorithm for computing a constant C.

Theorem 2: For the forced van der Pol equation (3.1) the local boundary curves have order of contact at least equal to $m-1$. If $C \neq 0$ then the local boundary curves are analytic in ε and have an order of contact equal to $m - 1$. For (ε,Ω) interior to the local sector there are 2 stable and 2 unstable periodic solutions with rotation number p/q on the invariant torus. For (ε,Ω) on the boundary curves there are two semi-stable periodic solutions with rotation number p/q on the invariant torus.

The functions a and b have expansions of the form

$$\Omega = \omega_1^2/\omega_2^2 = p/q^2(p + \Delta_2\varepsilon^2 + \cdots + \Delta_{m-2}\varepsilon^{m-2}$$
$$+ (\Delta_m \pm C)\varepsilon^m + O(\varepsilon^{m+1}).$$

We have developed a PL/I program which computes the constants $\Delta_2,\ldots,\Delta_m$ and C and used this program on (3.1) when

$p/q = 1/3, 1/5, 1/7, 1/2, 3/1, 1/9, 1/11, 1/4, 3/5, 2/1, 1/13, 3/7, 5/1, 1/15,$
$1/6, 2/3, 5/3.$

Except for $p/q = 1/7$ the constant $C \neq 0$ and this case was considered separately by hand.

References

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