THEOREM F. *Let M be a finite semigroup which is the union of groups. Then* $\#(M) - 2 \leqq \#(IG(M)) \leqq \#(M)$.

The techniques employed to prove Theorem C and its corollaries fall into two classes. The standard facts about finite semigroups as presented in reference 3, and in particular Rees Theorem, are used to provide the lower bounds to the complexity number. The machine techniques introduced in reference 1 are systematically used together with the so-called Schützenberger-Preston representations to construct the upper bounds for the complexity. The two estimates meet (within 1) in Theorem C for semigroups which are the union of groups.

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LIAPUNOV FUNCTIONS FOR THE PROBLEM OF LUR'E

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The search for Liapunov functions for the system of differential equations (1) known as the Lur'e system has been extensive in recent years. A significant change in the course of this research occurred after the appearance of the 1961 paper of Popov.[1] Popov obtained a criterion for the asymptotic stability of the Lur'e system without the use of Liapunov functions. Moreover, he proved that his criterion is satisfied if there exists a positive definite Liapunov function of the type quadratic form plus the integral of the nonlinearity, that is a function of the form (3), whose derivative along the trajectories of (1) is negative definite.

Several authors have proved partial converses to Popov's theorem on the existence of Liapunov functions. Two noteworthy papers are those of Yakubovich[2] and Kalman.[3] Yakubovich was able to prove a partial converse by strengthening the Popov criterion [see (2)] below from $P(i\omega) \geqq 0$ to $P(i\omega) > 0$, and Kalman was able to establish a partial converse by requiring that the system be completely controllable and completely observable (defined below). This note is to announce the result that the Popov criterion implies the existence of a positive definite Liapunov function of the type quadratic form plus integral of the nonlinearity that can be used to prove asymptotic stability in the large without the above restrictions. Also Lemma 1 which is used to prove this result can be used to establish further results such as Theorem 2.

Let E^n denote Euclidean n-space and I the $n \times n$ identity matrix. Let A be a real $n \times n$ matrix and b a real n-vector. The vector b will be considered as a column vector, and b' will be its transpose. The cyclic subspace generated by b relative to A will be denoted by $[A, b]$; that is, $[A, b] = \{x \in E^n: x = \alpha_0 b + \ldots$

$+ \alpha_{n-1}A^{n-1}b$, where the α_j's are real numbers$\}$. Let $[A, b]^0$ denote the orthogonal complement of $[A, b]$ in E^n. The pair (A, b) is said to be *completely controllable* if $[A, b] = E^n$ and (A, b') is said to be *completely observable* provided (A', b) is completely controllable.

The main result is the following lemma:

LEMMA 1. *Let A be a real $n \times n$ matrix all of whose characteristic roots have negative real parts; let τ be a real nonnegative number, and let b, k be two real n-vectors. If*

$$\tau + 2 \operatorname{Re} k'(i\omega I - A)^{-1}b \geqq 0$$

for all real ω, then there exists two real $n \times n$ symmetric matrices B and D, and a real n-vector q such that (a) $A'B + BA = -qq' - D$; (b) $Bb - k = \sqrt{\tau}q$; (c) *D is positive semidefinite and B is positive definite;* (d) $\{x \epsilon E^n\colon x'Dx = 0\} \cap [A', q]^0 = \{0\}$; (e) $q \notin [A, b]^0$; *and* (f) *if $i\omega$, ω real, is a root of $-q'(zI - A)^{-1}b + \sqrt{\tau}$, then it is a root of $b'(zI - A)^{-1}D(zI - A)^{-1}b$.*

The beginning of the proof of this lemma is similar to the proof given by Kalman in reference 3. A second lemma which is easy to establish gives a generalization of Popov's theorem to the critical cases. It is

LEMMA 2. *Let A be a real $n \times n$ matrix all of whose characteristic roots are simple, distinct, and have zero real parts. If the residues of $k'(zI - A)^{-1}b$ are all positive, then there exists a positive definite matrix B such that*

$$A'B + BA = 0 \quad \text{and} \quad Bb - k = 0.$$

The two lemmas above can be used in the analysis of several different systems that occur in control theory. For example, consider

$$\dot{x} = Ax - b\phi(\sigma), \qquad \sigma = c'x, \tag{1}$$

where x, b, and c are real n-vectors; A is a real $n \times n$ matrix, and $\phi(\sigma)$ is a real continuous scalar function of the real scalar σ such that $\sigma\phi(\sigma) > 0$ for $\sigma \neq 0$. Both x and σ are functions of the real variable t and $\dot{x} = dx/dt$. Let (1) be such that through each $x_0 \epsilon E^n$ there passes a unique trajectory of (1). The system (1) is said to be completely controllable and completely observable provided that (A, b) and (A, c') are respectively completely controllable and completely observable. For this system, the Lemmas 1 and 2 can be used to prove:

THEOREM 1. *Let there exist two constants $\alpha \geqq 0, \beta \geqq 0, \alpha + \beta > 0$ such that*

$$\operatorname{Re} P(i\omega) \geqq 0 \text{ where}$$
$$P(i\omega) = (\alpha + i\omega\beta)c'(i\omega I - A)^{-1}b \tag{2}$$

for all real ω, and if $i\omega_0$, ω_0 real, is a characteristic root of A, then the pole of $P(z)$ at $i\omega_0$ is simple and has positive residues.[4] *Then there exists a Liapunov function V of the form*

$$V = x'Bx + \beta \int_0^\sigma \phi(\tau)d\tau, \tag{3}$$

where B is a symmetric matrix such that V is positive definite for all x and $\dot{V} =$ (grad $V)\cdot(Ax - b\phi(\sigma)) \leqq 0$ *for all x. If the linear system $\dot{x} = \{A - \mu bc'\}x$ is asymptotically stable for all $\mu > 0$, then no nonzero trajectory of (1) remains in the set where $\dot{V} = 0$.*

Moreover, if when $\alpha = 0$ *and* A *is singular,* $\int_0^\sigma \phi(\tau)d\tau$ *tends to* $+\infty$ *with* $|\sigma|$, *then* V *tends to* $+\infty$ *with* $\|x\|$. *Thus, under the above conditions, the solution* $x = 0$ *of* (*1*) *is asymptotically stable in the large.*

Lemmas 1 and 2 can equally well be used to consider the system (1) when $\phi(\sigma)$ is restricted to $0 < \sigma\phi(\sigma) < k\sigma^2$ and a sharper result can be obtained. Another example of the consequences of Lemma 1 is the following:

THEOREM 2. *Let* A *be any real* $n \times n$ *matrix and* b, c *as before. Let* λ *be any real number that is strictly greater than the real part of all the characteristic roots of* A. *If*

$$\mathrm{Re}\, c'(zI - A)^{-1}b \geqq 0$$

for all $z = i\omega + \lambda$, ω *real, then there exists a nonnegative function* K *defined on* $[0, \infty)$ *such that*

$$\|x(t)\| \leqq K(\|x_0\|)e^{\lambda t}$$

for all $t \geqq 0$ *where* $x(t)$ *is the solution of* (*1*) *such that* $x(0) = x_0$.

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[4] *Note added in proof: Assume also that if* $i\omega_0$, ω_0 *real, is a characteristic root of* A *of multiplicity* ν, *then* $\lim_{z \to i\omega_0} (z - i\omega_0)^\nu c'(zI - A)b \neq 0$.

STRUCTURE OF THE CHROMATIN IN SEA URCHIN SPERM*

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The nuclei of the sea urchin sperm, the subject of the present work, have been studied both chemically and by electron microscopy, and information is available about their DNA,[1,2] histones,[3] nucleoprotein,[4] and the ultrastructure of the sperm head.[5] X-ray diffraction data on the sperm heads of sea urchins have been reported.[6]

In the present study, some earlier observations of the effects of divalent cations and ionic strength on the behavior of the nucleoproteins[4,7] have been adapted to techniques of spreading and staining molecules for high-resolution electron microscopy. The chromatin has also been studied after several enzymatic and extraction procedures.

Materials and Methods.—Fresh sperm from the sea urchin *Strongylocentrotus purpuratus* was used in this work. Several techniques for the preparation of the electron microscopical specimens were used.