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Continuation of periodic solutions in three dimensions

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Abstract

We use the methods of symplectic scaling and reduction to show that the reduced spatial three-body problem with one small mass is to the first approximation the product of the spatial restricted three-body problem and a harmonic oscillator. This allows us to prove that a nondegenerate periodic solution of the spatial restricted three-body problem can be continued into the reduced three-body problem with one small mass.

The spatial three-body problem and the spatial restricted three-body problem admit two time-reversing symmetries. A solution which hits the fixed set of one of the symmetries at time 0 and the fixed set of the other at time T will be periodic of period $4T$ and its orbit will be symmetric with respect to both symmetries. Such solutions are called doubly symmetric. We prove that a nondegenerate doubly symmetric periodic solution of the spatial restricted three-body problem can be continued into the reduced three-body problem with one small mass.

Keywords: Periodic solutions; Restricted three-body problem; Three-body problem; Reduction; Symplectic scaling

1. Introduction

The restricted three-body problem is said to be a limit of the three-body problem as one of the masses tends to zero, and so a periodic solution of the restricted problem sometimes can be continued into the full three-body problem when one mass is small. Indeed, many times the existence of a periodic solution of a certain type is first established for the restricted problem and later the same type periodic solution is established in the full three-body problem. The spatial three-body problem has nine degrees of freedom, whereas the spatial restricted problem has only three degrees of freedom. The size difference seems large until one realizes that the three-body problem admits translations and rotations as symmetries and linear and angular momenta as integrals. By holding the integrals fixed and identifying symmetric configurations, the problem can be reduced to a four-degree-of-freedom problem which we will call the *spatial reduced three-body problem* or simply the *reduced problem.*

Using symplectic scaling, we will show that the spatial reduced problem with one small mass is to the first approximation the product of the spatial restricted problem and a harmonic oscillator. Thus it follows from standard results that a nondegenerate periodic solution of the spatial restricted problem whose period is not a multiple of 2π

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can be continued into the reduced three-body problem. Hadjidemetriou [4] established a similar result for the planar problem. Also see [6,9,11] for some generalizations and variations. As a corollary, the near collision nondegenerate periodic solutions of the spatial restricted problem established by Belbruno [2] can be continued into the reduced three-body problem.

We also show that a nondegenerate doubly symmetric solution of the restricted problem whose period is not a multiple of 2π can be continued into the reduced problem. This result shows that the doubly symmetric periodic solutions of the restricted problem found in [5] can be continued into the reduced problem. This class of periodic solutions of the three-body problem was first established in [15] and that paper was the motivation of this investigation.

2. Scaling and reduction

In this section we make a series of symplectic changes of variables in the three-body problem which show how to look at the spatial restricted problem as the limit of the spatial reduced problem with one small mass. To the first approximation the reduced problem with one small mass is separable, i.e. the Hamiltonian of the reduced problem to the first approximation is the sum of the Hamiltonian of the restricted problem and the Hamiltonian of a harmonic oscillator.

The three-body problem in three dimensional space is a nine-degree-of-freedom problem. By placing the center of mass at the origin and setting linear momentum equal to zero the problem reduces to a six-degree-of-freedom problem. This is easily done using Jacobi coordinates. The Hamiltonian of the three-body problem in rotating (about the z-axis) Jacobi coordinates $(u_0, u_1, u_2, v_0, v_1, v_2)$ is

$$\begin{aligned} H = {} & \frac{\|v_0\|^2}{2M_0} - u_0^{\mathrm{T}} J v_0 + \frac{\|v_1\|^2}{2M_1} - u_1^{\mathrm{T}} J v_1 - \frac{m_0 m_1}{\|u_1\|} \\ & + \frac{\|v_2\|^2}{2M_2} - u_2^{\mathrm{T}} J v_2 - \frac{m_1 m_2}{\|u_2 - \alpha_0 u_1\|} - \frac{m_2 m_0}{\|u_2 + \alpha_1 u_1\|}, \end{aligned} \tag{1}$$

where $u_i, v_i \in \mathbb{R}^3$,

$$\begin{aligned} & M_0 = m_0 + m_1 + m_2, \qquad M_1 = m_0 m_1/(m_0 + m_1), \\ & M_2 = m_2(m_0 + m_1)/(m_0 + m_1 + m_2), \\ & \alpha_0 = m_0/(m_0 + m_1), \qquad \alpha_1 = m_1/(m_0 + m_1), \end{aligned}$$

and

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In these coordinates u_0 is the center of mass, v_0 the total linear momentum, and total angular momentum is

$$A = u_0 \times v_0 + u_1 \times v_1 + u_2 \times v_2.$$

See [11] for details.

The set where $u_0 = v_0 = 0$ is invariant and setting these two coordinates to zero affects the first reduction. Setting $u_0 = v_0 = 0$ reduces the problem by three degrees of freedom.

We consider angular momentum to be nonzero. One way to reduce the problem by two more degrees is to hold the vector A fixed and eliminate the rotational symmetry about the A axis. Another way to reduce the problem is to note that A_z, the z-component of angular momentum, and $A = \|A\|$, the magnitude of angular momentum are integrals in involution. Two independent integrals in involution can be used to reduce a system by two degrees of freedom, see [16]. In either case we pass to the reduced space as defined in [8] or [7].

Assume that one of the particles has small mass by setting $m_2 = \varepsilon^2$ where ε is to be considered as a small parameter. Also set $m_0 = \mu$, $m_1 = 1 - \mu$ and $\nu = \mu(1-\mu)$ so that

$$M_1 = \nu = \mu(1-\mu), \qquad M_2 = \varepsilon^2/(1+\varepsilon^2) = \varepsilon^2 - \varepsilon^4 + \cdots,$$
$$\alpha_0 = \mu, \qquad \alpha_1 = 1 - \mu.$$

The Hamiltonian becomes

$$H = K + \tilde{H},$$

where

$$K = \frac{1}{2\nu}\|v_1\|^2 - u_1^{\mathrm{T}} J v_1 - \frac{\nu}{\|u_1\|}$$

and

$$\tilde{H} = \frac{(1+\varepsilon^2)}{2\varepsilon^2}\|v_2\|^2 - u_2^{\mathrm{T}} J v_2 - \frac{\varepsilon^2(1-\mu)}{\|u_2 - \mu u_1\|} - \frac{\varepsilon^2 \mu}{\|u_2 + (1-\mu)u_1\|}.$$

K is the Hamiltonian of the Kepler problem in rotating coordinates. We can simplify K by making the scaling $u_i \to u_i$, $v_i \to \nu v_i$, $K \to \nu^{-1}K$, $\tilde{H} \to \nu^{-1}\tilde{H}$, $\varepsilon^2\nu^{-1} \to \varepsilon^2$ so that

$$K = \frac{1}{2}\|v_1\|^2 - u_1^{\mathrm{T}} J v_1 - \frac{1}{\|u_1\|} \tag{2}$$

and

$$\tilde{H} = \frac{(1+\nu\varepsilon^2)}{2\varepsilon^2}\|v_2\|^2 - u_2^{\mathrm{T}} J v_2 - \frac{\varepsilon^2(1-\mu)}{\|u_2 - \mu u_1\|} - \frac{\varepsilon^2 \mu}{\|u_2 + (1-\mu)u_1\|}. \tag{3}$$

K has a critical point at $u_1 = a = (1,0,0)^{\mathrm{T}}$, $v_1 = b = (0,1,0)^{\mathrm{T}}$ – it corresponds to a circular orbit of the Kepler problem. Expand K in a Taylor series about this point, ignore the constant term and make the scaling

$$u_1 \to a + \varepsilon q, \qquad v_1 \to b + \varepsilon p, \qquad K \to \varepsilon^{-2}K$$

to get $K = K_0 + \mathrm{O}(\varepsilon)$ where

$$K_0 = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_2 p_1 - q_1 p_2 + \tfrac{1}{2}(-2q_1^2 + q_2^2 + q_3^2). \tag{4}$$

Now scale $\tilde{H}$ by the above and

$$u_2 = \xi, \quad v_2 = \varepsilon^2 \eta, \quad \tilde{H} \longrightarrow \varepsilon^{-2}\tilde{H}.$$

The totality is a symplectic scaling with multiplier ε^{-2} and so the Hamiltonian of the three-body problem becomes $H_{\mathrm{R}} + K_0 + \mathrm{O}(\varepsilon)$ where K_0 is given in (4) and H_{R} is the Hamiltonian of the restricted problem, i.e.

$$H_{\mathrm{R}} = \frac{1}{2}\|\eta\|^2 - \xi^{\mathrm{T}} J\eta - \frac{(1-\mu)}{\|\xi - (\mu,0,0)\|} - \frac{\mu}{\|\xi + (1-\mu,0,0)\|}. \tag{5}$$

To obtain the expansions above recall $u_1 = (1,0,0) + \mathrm{O}(\varepsilon)$.

We have already reduced the problem by using the transitional invariance and the conservation of linear momentum, so now we will complete the reduction by using the rotational invariance and the conservation of angular momentum.

Recall that angular momentum in the original coordinates is $A = u_1 \times v_1 + u_2 \times v_2$ and in the scaled coordinates it becomes

$$A = (a + \varepsilon q) \times (b + \varepsilon p) + \varepsilon^2 \xi \times \eta \tag{6}$$

and so holding angular momentum fixed by setting $A = a \times b$ imposes the constraint

$$0 = a \times p + q \times b + \mathrm{O}(\varepsilon) = (-q_3, -p_3, p_2 + q_1) + \mathrm{O}(\varepsilon). \tag{7}$$

Now let us do the reduction when $\varepsilon = 0$ so that the Hamiltonian is $H = H_{\mathrm{R}} + K_0$ and holding angular momentum fixed is equivalent to $q_3 = p_3 = p_2 + q_1 = 0$. Notice that the angular momentum constraint is only on the q, p variables. Make the symplectic change of variables

$$r_1 = q_1 + p_2, R_1 = p_1, \qquad r_2 = q_2 + p_1, R_2 = p_2, \qquad r_3 = q_3, R_3 = p_3, \tag{8}$$

so that

$$K_0 = \tfrac{1}{2}(r_2^2 + R_2^2) + \tfrac{1}{2}(r_3^2 + R_3^2) + r_1 R_2 - r_1^2. \tag{9}$$

Notice that holding angular momentum fixed in these coordinates is equivalent to $r_1 = r_3 = R_3 = 0$, that R_1 is an ignorable coordinate, and r_1 is an integral. Thus passing to the reduced space reduces K_0 to

$$K_0 = \tfrac{1}{2}(r_2^2 + R_2^2). \tag{10}$$

Thus when $\varepsilon = 0$ the Hamiltonian of the reduced three-body problem becomes

$$H = H_{\mathrm{R}} + \tfrac{1}{2}(r^2 + R^2), \tag{11}$$

which is the sum of the Hamiltonian of the restricted three-body problem and a harmonic oscillator. Here in (11) and henceforth we have dropped the subscript 2. The equations and integrals all depend smoothly on ε and so for small ε the Hamiltonian becomes

$$H = H_{\mathrm{R}} + \tfrac{1}{2}(r^2 + R^2) + \mathrm{O}(\varepsilon). \tag{12}$$

We can also introduce action–angle variables (I, ι) by

$$r = \sqrt{2I}\cos\iota, \qquad R = \sqrt{2I}\sin\iota,$$

to give

$$H = H_{\mathrm{R}} + I + \mathrm{O}(\varepsilon). \tag{13}$$

The spatial three-body problem on the reduced space with one small mass is approximately the product of the spatial restricted problem and a harmonic oscillator.

3. Continuation of nondegenerate periodic solutions

A periodic solution of a conservative Hamiltonian system always has the characteristic multiplier $+1$ with algebraic multiplicity at least 2. If the periodic solution has the characteristic multiplier $+1$ with algebraic multiplicity exactly equal to 2 then the periodic solution is called *nondegenerate* or *elementary*. A nondegenerate periodic solution lies in a smooth cylinder of periodic solutions which are parametrized by the Hamiltonian. Moreover, if the Hamiltonian depends smoothly on parameters then the periodic solution persists for small variations of the parameters. See [11, pp. 133–136; 155–156] for complete details.

Theorem. A nondegenerate periodic solution of the spatial restricted three-body problem whose period is not a multiple of 2π can be continued into the reduced three-body problem.

More precisely:

Theorem. Let $\xi = \bar{\xi}(t)$, $\eta = \bar{\eta}(t)$ be a periodic solution with period T of the restricted problem with Hamiltonian (5) with multipliers $+1, +1, \beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}$. Assume that $T \neq n2\pi$ for all $n \in \mathbb{Z}$ and $\beta_1 \neq +1$ and $\beta_2 \neq +1$. Then the reduced three-body problem with Hamiltonian (12) has a periodic solution of the form $\xi = \bar{\xi}(t) + \mathrm{O}(\varepsilon)$, $\eta = \bar{\eta}(t) + \mathrm{O}(\varepsilon)$, $r = \mathrm{O}(\varepsilon)$, $R = \mathrm{O}(\varepsilon)$ whose period is $T + \mathrm{O}(\varepsilon)$. Moreover, its multipliers are $+1, +1, \beta_1 + \mathrm{O}(\varepsilon), \beta_1^{-1} + \mathrm{O}(\varepsilon), \beta_2 + \mathrm{O}(\varepsilon), \beta_2^{-1} + \mathrm{O}(\varepsilon), \mathrm{e}^{iT} + \mathrm{O}(\varepsilon), \mathrm{e}^{-iT} + \mathrm{O}(\varepsilon)$.

Proof. When $\varepsilon = 0$ the reduced problem with Hamiltonian (12) has the periodic solution $\xi = \bar{\xi}(t)$, $\eta = \bar{\eta}(t)$, $r = 0$, $R = 0$ with period T. The multipliers of this periodic solution are $+1, +1, \beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \mathrm{e}^{\mathrm{i}T}, \mathrm{e}^{-\mathrm{i}T}$. By the assumption $T \neq n2\pi$ it follows that $\mathrm{e}^{\pm\mathrm{i}T} \neq +1$ and so this periodic solution is nondegenerate. The classical continuation theorem can be applied to show that this solution can be continued smoothly into the problem with ε small and nonzero. See [3, p. 142] or [11, pp. 154–56]. □

The planar version of this theorem is due to Hadjidemetriou [4]. One of the most interesting families of nondegenerate periodic solution of the spatial restricted problem can be found in Belbruno [2]. He regularized double collisions when $\mu = 0$ and showed that some spatial collision orbits are nondegenerate periodic solutions in the regularized coordinates. Thus, they can be continued into the spatial restricted problem as nondegenerate periodic solutions for $\mu \neq 0$. Now these same orbits can be continued into the reduced three-body problem.

4. Continuation of doubly symmetric periodic solutions

Very few nondegenerate periodic solutions of the spatial restricted problem have been established rigorously, but there are interesting families of periodic solutions which have been established using symmetry arguments. Jefferys [5] used two time-reversing symmetries of the spatial restricted problem to establish the existence of periodic solutions which are symmetric with respect to two planes in phase space – hence the name doubly symmetric periodic solutions.

Write the Hamiltonian of the restricted problem in components by letting $\xi = (x, y, z)$, $\eta = (X, Y, Z)$ so that

$$H_{\mathrm{R}} = \frac{1}{2}(X^2 + Y^2 + Z^2) + (yX - xY) - \frac{1-\mu}{\{(x-\mu)^2 + y^2 + z^2\}^{1/2}} - \frac{\mu}{\{(x-1+\mu)^2 + y^2 + z^2\}^{1/2}}. \tag{14}$$

This Hamiltonian is invariant under the two anti-symplectic reflections:

$$\begin{aligned} \mathcal{R}_1 &: (x, y, z, X, Y, Z) \longrightarrow (x, -y, -z, -X, Y, Z), \\ \mathcal{R}_2 &: (x, y, z, X, Y, Z) \longrightarrow (x, -y, z, -X, Y, -Z). \end{aligned} \tag{15}$$

These symmetries are time-reversing symmetries of the problem, so if $(x(t), y(t), z(t), X(t), Y(t), Z(t))$ is a solution, then so are $(x(-t), -y(-t), \pm z(-t), -X(-t), Y(-t), \mp Z(-t))$. The fixed set of these two symmetries are Lagrangian subplanes, i.e.

$$\mathcal{L}_1 = \{(x, 0, 0, 0, Y, Z)\}, \qquad \mathcal{L}_2 = \{(x, 0, z, 0, Y, 0)\},$$

are fixed by the symmetries $\mathcal{R}_1, \mathcal{R}_2$. If a solution starts in one of these Lagrangian planes at time $t = 0$ and hits the other at a later time $t = T$ then the solution is $4T$-periodic and the orbit of this solution is carried into itself by both symmetries – see [12]. We shall call such a periodic solution *doubly symmetric*. If the orbit meets these Lagrangian planes transversally then we shall call it a *nondegenerate doubly symmetric periodic solution*. Geometrically, an orbit hits $\mathcal{L}_1$ if it hits the x-axis perpendicularly and it hits $\mathcal{L}_2$ if it hits the x, z-plane perpendicularly.

To be more specific, let

$$(\tilde{x}(t, \alpha, \beta, \gamma), \tilde{y}(t, \alpha, \beta, \gamma), \tilde{z}(t, \alpha, \beta, \gamma), \tilde{X}(t, \alpha, \beta, \gamma), \tilde{Y}(t, \alpha, \beta, \gamma), \tilde{Z}(t, \alpha, \beta, \gamma)) \tag{16}$$

be a solution which starts at $(\alpha, 0, 0, 0, \beta, \gamma) \in \mathcal{L}_1$ when $t = 0$, i.e.

$$\begin{aligned} &\tilde{x}(0, \alpha, \beta, \gamma) = \alpha, \quad &&\tilde{y}(0, \alpha, \beta, \gamma) = 0, \quad &&\tilde{z}(0, \alpha, \beta, \gamma) = 0, \\ &\tilde{X}(0, \alpha, \beta, \gamma) = 0, \quad &&\tilde{Y}(0, \alpha, \beta, \gamma) = \beta, \quad &&\tilde{Z}(0, \alpha, \beta, \gamma) = \gamma. \end{aligned} \tag{17}$$

The solution with $\alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0$ will be doubly symmetric periodic with period $4T$ if it hits the $\mathcal{L}_2$ plane after a time T, i.e.

$$\tilde{y}(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad \tilde{X}(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad \tilde{Z}(T, \alpha_0, \beta_0, \gamma_0) = 0. \tag{18}$$

This solution will be *nondegenerate* if the Jacobian

$$\frac{\partial(\tilde{y}, \tilde{X}, \tilde{Z})}{\partial(t, \alpha, \beta, \gamma)}(T, \alpha_0, \beta_0, \gamma_0) \tag{19}$$

has rank 3.

It follows from the implicit function theorem that nondegenerate doubly symmetric periodic solutions lie in a one parameter family. Also, a nondegenerate doubly symmetric periodic solution can be continued in the parameter μ. In general, a nondegenerate doubly symmetric periodic solution may not be nondegenerate in the sense of the previous section, i.e. a nondegenerate doubly symmetric periodic solution may have all its multipliers equal to 1. Also, a one-parameter family of nondegenerate doubly symmetric periodic solutions may not be locally isolated since there may be nearby non-symmetric periodic solutions.

In a similar manner the Hamiltonian of the three-body problem is invariant under reflections. Let $u_i = (x_i, y_i, z_i)$, $v_i = (X_i, Y_i, Z_i)$ for $i = 0, 1, 2$. Then the Hamiltonian H in (1) is invariant under the two reflections

$$\begin{aligned} &\mathcal{R}_1 : (x_i, y_i, z_i, X_i, Y_i, Z_i) \longrightarrow (x_i, -y_i, -z_i, -X_i, Y_i, Z_i) \quad \text{for } i = 0, 1, 2, \\ &\mathcal{R}_2 : (x_i, y_i, z_i, X_i, Y_i, Z_i) \longrightarrow (x_i, -y_i, z_i, -X_i, Y_i, -Z_i) \quad \text{for } i = 0, 1, 2. \end{aligned} \tag{20}$$

In coordinates $(r_1, r_2, r_3, R_1, R_2, R_3)$ used above in (8) these reflections are

$$\begin{aligned} &\mathcal{R}_1 : (r_1, r_2, r_3, R_1, R_2, R_3) \to (r_1, -r_2, -r_3, -R_1, R_2, R_3), \\ &\mathcal{R}_2 : (r_1, r_2, r_3, R_1, R_2, R_3) \to (r_1, -r_2, r_3, -R_1, R_2, -R_3). \end{aligned} \tag{21}$$

It is easy to see that K_0 in (9) is invariant under these reflections.

Our local coordinates on the reduced space are $(\xi, r, \eta, R) = (x, y, z, r, X, Y, Z, R)$ (recall $r = r_2$, $R = R_2$) and so the reflections on the reduced space are

$$\begin{aligned} \mathcal{R}_1 &: (x, y, z, r, X, Y, Z, R) \longrightarrow (x, -y, -z, r, -X, Y, Z, -R), \\ \mathcal{R}_2 &: (x, y, z, r, X, Y, Z, R) \longrightarrow (x, -y, z, r, -X, Y, -Z, -R). \end{aligned} \tag{22}$$

These symmetries are time-reversing symmetries of the reduced problem; therefore, if $(x(t), y(t), z(t), r(t)X(t), Y(t), Z(t), R(t))$ is a solution, then so are $(x(-t), -y(-t), \pm z(-t), r(-t), -X(-t), Y(-t), \mp Z(-t), -R(-t))$. The fixed set of these two symmetries are Lagrangian subplanes:

$$\mathcal{L}_1 = \{(x, 0, 0, r, 0, Y, Z, 0)\}, \qquad \mathcal{L}_2 = \{(x, 0, z, r, 0, Y, 0, 0)\}.$$

Again, if a solution of the reduced problem starts in one of these Lagrangian planes at time $t = 0$ and hits the other at a later time $t = T$ then the solution is $4T$-periodic and the orbit of this solution is carried into itself by both symmetries – *doubly symmetric periodic solutions.*

Let

$$\begin{aligned} &(\tilde{x}(t, \alpha, \beta, \gamma, \delta), \tilde{y}(t, \alpha, \beta, \gamma, \delta), \tilde{z}(t, \alpha, \beta, \gamma, \delta), \tilde{r}(t, \alpha, \beta, \gamma, \delta), \\ &\tilde{X}(t, \alpha, \beta, \gamma, \delta), \tilde{Y}(t, \alpha, \beta, \gamma, \delta), \tilde{Z}(t, \alpha, \beta, \gamma, \delta), \tilde{R}(t, \alpha, \beta, \gamma, \delta)) \end{aligned} \tag{23}$$

be a solution which starts at $(\alpha, 0, 0, \delta, 0, \beta, \gamma, 0) \in \mathcal{L}_1$ when $t = 0$, i.e.

$$\begin{aligned} &\tilde{x}(0, \alpha, \beta, \gamma, \delta) = \alpha, \quad \tilde{y}(0, \alpha, \beta, \gamma, \delta) = 0, \quad \tilde{z}(0, \alpha, \beta, \gamma, \delta) = 0, \quad \tilde{r}(0, \alpha, \beta, \gamma, \delta) = \delta, \\ &\tilde{X}(0, \alpha, \beta, \gamma, \delta) = 0, \quad \tilde{Y}(0, \alpha, \beta, \gamma, \delta) = \beta, \quad \tilde{Z}(0, \alpha, \beta, \gamma, \delta) = \gamma, \quad \tilde{R}(0, \alpha, \beta, \gamma, \delta) = 0. \end{aligned} \tag{24}$$

The solution with $\alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0, \delta = \delta_0$ will be doubly symmetric periodic with period $4T$ if it hits the $\mathcal{L}_2$ plane after a time T, i.e.

$$\tilde{y}(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad \tilde{X}(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad \tilde{Z}(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad \tilde{R}(T, \alpha_0, \beta_0, \gamma_0) = 0. \tag{25}$$

This solution will be *nondegenerate* if the Jacobian

$$\frac{\partial(\tilde{y}, \tilde{X}, \tilde{Z}, \tilde{R})}{\partial(t, \alpha, \beta, \gamma, \delta)}(T, \alpha_0, \beta_0, \gamma_0, \delta_0) \tag{26}$$

has rank 4.

It follows from the implicit function theorem that nondegenerate doubly symmetric periodic solutions of the reduced problem lie in a one parameter family etc.

Theorem. A nondegenerate doubly symmetric periodic solution of the spatial restricted three-body problem whose period is not a multiple of 2π can be continued into the reduced three-body problem.

More precisely:

Theorem. Let $\xi = \bar{\xi}(t)$, $\eta = \bar{\eta}(t)$ be a nondegenerate doubly symmetric periodic solution with period $4T$ of the restricted problem with Hamiltonian (5). Assume that $4T \neq n2\pi$ for all $n \in \mathbb{Z}$. Then the reduced three-body problem, the system with Hamiltonian (12), has a nondegenerate doubly symmetric periodic solution of the form $\xi = \bar{\xi}(t) + O(\varepsilon)$, $\eta = \bar{\eta}(t) + O(\varepsilon)$, $r = O(\varepsilon)$, $R = O(\varepsilon)$ whose period is $4T + O(\varepsilon)$.

Proof. When $\varepsilon = 0$ the reduced problem with Hamiltonian (12) has the doubly symmetric periodic solution $\xi = \bar{\xi}(t)$, $\eta = \bar{\eta}(t)$, $r = 0$, $R = 0$ with period $4T$. Let the general solution of the restricted problem which starts in $\mathcal{L}_1$ be (16) satisfying (17). Let the given doubly symmetric periodic solution $\bar{\xi}(t)$, $\bar{\eta}(t)$ correspond to

$$\alpha = \alpha_0, \quad \beta = \beta_0, \quad \gamma = \gamma_0,$$

so that (19) has rank 3.

When $\varepsilon = 0$ the general solution of the restricted problem which starts in $\mathcal{L}_1$ is

$$\begin{aligned} &(\tilde{x}(t,\alpha,\beta,\gamma), \tilde{y}(t,\alpha,\beta,\gamma), \tilde{z}(t,\alpha,\beta,\gamma), \quad r(t,\delta) = \delta\cos t, \\ &(\tilde{X}(t,\alpha,\beta,\gamma), \tilde{Y}(t,\alpha,\beta,\gamma), \tilde{Z}(t,\alpha,\beta,\gamma), \quad R(t,\delta) = -\delta\sin t. \end{aligned} \tag{27}$$

This solution with $\alpha_0, \beta_0, \gamma_0, \delta = 0$ hits $\mathcal{L}_2$ when $t = T$ and so satisfies Eqs. (25). Clearly the Jacobian

$$\frac{\partial(\tilde{y}, \tilde{X}, \tilde{Z}, \tilde{R})}{\partial(t,\alpha,\beta,\gamma,\delta)}(T,\alpha_0,\beta_0,\gamma_0,0) = \begin{pmatrix} \dfrac{\partial(\tilde{y}, \tilde{X}, \tilde{Z})}{\partial(t,\alpha,\beta,\gamma)}(T,\alpha_0,\beta_0,\gamma_0) & 0 \\ 0 & -\sin T \end{pmatrix} \tag{28}$$

has rank 4 when T is not a multiple of π, and so this doubly symmetric periodic solution is nondegenerate when $\varepsilon = 0$. By the implicit function theorem Eqs. (25) can be solved for ε not zero but small. □

5. Application from Jefferys to Soler

Jefferys [5] considers the spatial restricted problem (5) as a perturbation of the Kepler problem in rotating coordinates by treating the mass ratio parameter μ as a small parameter. He shows that there are doubly symmetric circular solutions of the Kepler problem with arbitrary inclination which are nondegenerate and so can be continued into the restricted problem for small nonzero μ.

Soler [15] considers the three-body problem with two small masses and one large mass. For him the three-body problem is a perturbation of two Kepler problems. He selects a circular orbit from each of the two Kepler problems with arbitrary relative inclination such that the pair is a nondegenerate doubly symmetric periodic solution of the two Kepler problems. Thus, he proved that there are truly three-dimensional periodic solutions of the three-body problem. Jefferys' paper is cryptic, but Soler's paper is a model of clarity.

Since both Jefferys and Soler use a variation of Delaunay–Poincaré variables, we need a coordinate-free definition of nondegenerate doubly symmetric periodic solution of the restricted problem. The phase space of the spatial restricted problem is an open subset of $\mathbb{R}^6$ considered as a symplectic manifold. Note that an anti-symplectic involution like $\mathcal{R}_1$ or $\mathcal{R}_2$ always has a Lagrangian manifold as a fixed set – see [10]. Thus, $\mathcal{L}_1$ and $\mathcal{L}_2$ are three-dimensional Lagrangian submanifolds and (α, β, γ) used in the definition of nondegenerate doubly symmetric periodic solution are just coordinates in the Lagrangian manifold $\mathcal{L}_1$. The image $\mathcal{I}$ of $\mathcal{L}_1$ under the flow of the restricted problem is locally a four-dimensional submanifold. It may intersect itself.

If $\mathcal{I}$ intersects the other Lagrangian manifold $\mathcal{L}_2$ then the point of intersection lies on a doubly periodic solution. Say the point of intersection comes from the point with coordinates $(\alpha, 0, 0, 0, \beta, \gamma)$ on $\mathcal{L}_1$ as given in (16)–(18). Then (19) is just the condition that these two manifolds meet transversally. This gives an intrinsic definition of nondegenerate doubly symmetric periodic solution.

Jefferys uses his variation of Delaunay–Poincaré variables to define the local coordinates on Lagrangian subplanes $\mathcal{L}_1$ and $\mathcal{L}_2$. He also expresses the transversality condition (19) in these coordinates. He then selects a doubly symmetric circular orbit of the Kepler problem with arbitrary inclination and shows that it is nondegenerate. Thus, it can be continued into the spatial restricted three-body problem as a nondegenerate doubly symmetric periodic solution for small μ.

Thus, the theorem of the last section implies that Jefferys' orbits can be continued into the reduced three-body problem for ε small and nonzero. These are of course Soler's solutions.

References

[1] R. Abraham and J. Marsden, Foundations of Mechanics (Benjamin Cummings, London, 1978).
[2] E.A. Belbruno, Celestial Mech. 25 (1981) 397–415.
[3] G.D. Birkhoff, Dynamical Systems (Amer. Math. Soc., Providence, RI, 1927).
[4] J.D. Hadjidemetriou, Celestial Mech. 12 (5) (1975) 155–174.
[5] W.H. Jefferys, Astron. J. 70 (1965) 393–394.
[6] J. Llibre and D. Saari, Trans. Amer. Math. Soc. 347 (8) (1995) 3017–3030.
[7] J. Marsden and A. Weinstein, Rep. Math. Phy. 5 (1974) 121–130.
[8] K.R. Meyer, in: Dynamical Systems, ed. M. Peixoto (Academic Press, New York, 1973) pp. 259–272.
[9] K.R. Meyer, J. Diff. Eqs. 39 (1) (1981) 2–38.
[10] K.R. Meyer, J. Diff. Eqs. 41 (2) (1981) 228–238.
[11] K.R. Meyer and G.R. Hall, Introduction to Hamiltonian Dynamical Systems and the N-body Problem (Springer, New York, 1992).
[12] M. Michalodimitrakis, Celest. Mech. 19 (1979) 263–277.
[13] H. Poincaré, Les méthodes nouvelles de la mécanique céleste (Gauther–Villars, Paris, 1899).
[14] C.L. Siegel and J.K. Moser, Lectures on Celestial Mechanics (Springer, New York, 1971).
[15] J.J. Soler, in: Advanced Series in Nonlinear Dynamics, eds. E.A. Lacomba and J. Llibre, Vol. 8 (World Scientific, Singapore, 1996).
[16] E.T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge University Press, Cambridge, 1996).