Symmetries and Integrals in Mechanics†

KENNETH R. MEYER
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota

1 Introduction

The literature of Hamiltonian mechanics has several special theorems dealing with Hamiltonian systems that admit additional integrals in involution. Two examples are: (1) If a Hamiltonian system admits p additional independent integral in involution, then the algebraic multiplicity of $+1$ as a characteristic multiplier of a periodic solution is greater than or equal to $2(p + 1)$ [3]; and (2) the integration of a Hamiltonian system of n degrees of freedom that admits p independent integrals in involution can be reduced to the integration of a Hamiltonian system of $n - p$ degrees of freedom with p parameters and additional quadratures [5]. After restating (2) in modern terminology, we shall generalize these theorems by dropping the assumption that the integrals are in involution. Recall that in the important example where the integrals are the three components of angular momentum the integrals are not in involution.

The key to these generalizations is found in [4]. It is well known that Hamiltonians which are invariant under a group of symmetries define flows which have additional integrals. Thus when studying such systems

† This research was supported by ONR contract number N00014-67-A-0113-0019.

it is natural to study the flows restricted to the set where the integrals are constant. Smale observed that a subgroup of the full group of symmetries leaves these sets invariant, and thus it acts as a group of symmetries for the flow on the sets where the integrals are constant. It turns out that the case when the integrals are in involution corresponds to the case where the group is abelian and the subgroup is the full group. The generalizations of (1) and (2) above exploit the additional symmetry of the flow on the sets where the integrals are constant.

Section 2 contains the statements of the main results of this chapter along with several examples. Section 3 and 4 contain the proofs of the theorems stated in Section 2.

2 General background and main results

First, we shall review the relationship between symmetries and integrals to fix our notation and then state the main results. For further background and details see Abraham [1].

Henceforth all manifolds, functions, etc., will be considered C^∞. Let M be a $2n$-dimensional symplectic manifold with symplectic form Ω or $\{\ ,\ \}$. The form Ω defines an isomorphism between the tangent space T_mM and the cotangent space $T_m{}^*M$ for each $m \in M$; let

$$\flat: T_mM \to T_m{}^*M: v \to v^\flat, \qquad \text{and} \qquad \sharp: T_m{}^*M \to T_mM: v \to v^\sharp$$

denote this isomorphism and its inverse. For each function $H: M \to R$ there is a well-defined vector field $(dH)^\sharp$ on M; $(dH)^\sharp$ is called the Hamiltonian vector field generated by the Hamiltonian H.

Let G be a p-dimensional connected Lie group, $A = T_eG$ its algebra and $\Psi: G \times M \to M$ an action of G on M such that $\Psi(g, \cdot): M \to M$ is symplectic for all $g \in G$. Let $a \in A$ and e^{at} be the one-parameter subgroup of G generated by a. Then

$$\Psi_a: R \times M \to M: (t, m) \to \Psi(e^{at}, m)$$

is a local Hamiltonian flow on M and so is generated by a local Hamiltonian flow on M and so is generated by a local Hamiltonian vector field on M. Assume that for each $a \in A$ there is a globally defined function $F_a: M \to R$ such that $(dF_a)^\sharp$ generates the flow Ψ_a. Let $a_1, \ldots, a_p$ be a basis for A and $F_1 = F_{a_1}, \ldots, F_p = F_{a_p}$, a fixed set of functions such that $(dF_i)^\sharp$ generates the Hamiltonian flow $\Psi_i = \Psi_{a_i}$.

REMARK In Smale [4] the manifold M is the cotangent bundle of another manifold M_1 and action Ψ is the natural extension of an action Ψ_1 of G on M_1. In this special case, he found that there always exists globally defined functions which generate the flows ψ_i. Also there is a natural way to fix the constants of integrations by defining the F_i to be zero on zero covectors. Thus in this case there is a natural way to specify the F_i uniquely, and this adds to the elegance of his treatment.

Let $H: M \to R$ be a Hamiltonian such that $H(\Psi(g, m)) = H(m)$, for all $m \in M$ and $g \in G$, i.e., H is invariant under the action of G. The fundamental result relating symmetries and integrals is:

THEOREM 1 $F_1, \ldots, F_p$ are integrals for the flow φ generated by H. (See [1] for the proof.)

EXAMPLE 1 Let $M = R^3 \times R^3$, $G = R^3$, $A = R^3$, and $\Psi(g, (x, y)) = (x + g, y)$, where $g \in G = R^3$ and $(x, y) \in R^3 \times R^3$. Let a_1, a_2, a_3 be the usual basis vectors for $A = R^3$, then $\psi_i(t, (x, y)) = (x + ta_i, y)$ is the flow generated by the Hamiltonian vector fields

$$\dot{x} = \partial I_i/\partial y = a_i, \qquad \dot{y} = \partial I_i/\partial x = 0, \qquad \text{where} \quad I_i = y^{\mathrm{T}} a_i.$$

Thus if $H(x + g, y) = H(x, y)$, or equivalently H is independent of x, the flow generated by H has the three components of linear momentum I_1, I_2, I_3 as integrals.

EXAMPLE 2 Let $M = R^3 \times R^3$, $G = SO_3$, $A = so_3$, and $\Psi(g, (x, y)) = (gx, gy)$, where $g \in SO_3$ and $(x, y) \in R^3 \times R^3$. Let

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

be the usual basis for so_3. Then the flows

$$\psi_i(t, (x, y)) = (\exp a_i t\, x, \exp a_i t\, y)$$

are generated by the Hamiltonian equations

$$\dot{x} = a_i x = \partial J_i/\partial y, \qquad \dot{y} = a_i y = -\partial J_i/\partial x, \qquad \text{where} \quad J_i = y^{\mathrm{T}} a_i x.$$

Thus if $H: R^3 \times R^3 \to R$ is a Hamiltonian such that $H(gx, gy) = H(x, y)$, for all $g \in G$, $(x, y) \in R^3 \times R^3$, then the flow generated by H admits the three components of angular momentum J_1, J_2, J_3 as integrals. Thus $J = x \times y$ is a vector-valued integral with components J_1, J_2, J_3.

Consider the map $F = (F_1, \ldots, F_p): M \to R^p$ and let $s \in R^p$ be a fixed regular value of F such that $N = F^{-1}(s) \neq \emptyset$. Thus N is a submanifold of M of dimension $2n - p$. Consider

$$G_s = \{g \in G : \Psi(g, n) \in N \quad \text{for all } n \in N\}.$$

Clearly G_s is a closed subgroup of G and hence is a Lie group. Let A_s be the subalgebra of A corresponding to G_s and $\dim G_s = \dim A_s = q$. Let $a_1, \ldots, a_p$ be so chosen that $a_1, \ldots, a_q$ is a basis for A_s.

EXAMPLE 1 N is the affine subspace of $R^3 \times R^3$ defined by $(x, y) \in N$ if $y = s$ for the fixed $s \in R^3$. In this case $G_s = G$.

EXAMPLE 2 Let $s \in R^3$ and $s \neq 0$. Then $N = \{(x, y) \in R^3 \times R^3 : x \times y = s\}$. One can see that N is topologically $S^1 \times R^2$, since if $s \neq 0$, x can be chosen as an arbitrary nonzero vector in the plane orthogonal to s and y can be chosen on a line in this orthogonal plane. Thus x is parametrized by $R^2 - (0) = S^1 \times R^1$ and y by R^1.

The subgroup G_s is the group of rotations with axis of rotation along s. This follows from the fact that $(gx) \times (gy) = g(x \times y)$ for all $g \in SO_3$. Thus G_s is isomorphic to SO_2 or S^1.

The group G_s acts as a transformation group on N, and so it is natural to consider the quotient space $B = N/\sim$, where is the equivalence relation defined by $n_1 \sim n_2$ if there is a $g \in G_s$ such that $gn_1 = n_2$. We will denote equivalence classes by $[n]$. In general, B may not be a manifold but in some interesting examples it is.

EXAMPLE 1 Here $N = \{(x, y) : y = s\}$, $G_s = R^3$, and $\Psi: R^3 \times N \to N$: $(g, (x, s)) \to (x + g, s)$. Thus B is a point!

EXAMPLE 2 Here $N = \{(x, y) \in R^3 \times R^3 : x \times y = s\}$, where $s \neq 0$. Choose coordinates in R^3 so that $s = (0, 0, s^3)$, $s^3 \neq 0$. Then N is a subset of $\{(x, y) \in R^3 \times R^3 : x^3 = y^3 = 0\}$, i.e., N lies in a 4-dimensional coordinate plane. By ignoring the last component, let us consider N as the subset of $R^2 \times R^2$ such that $x^1y^2 - x^2y^1 = s^3$. In $R^2 \times R^2$, introduce the canonical polar coordinates

$$\varrho^2 = (x^1)^2 + (x^2)^2, \qquad \theta = \tan^{-1} x^2/x^1,$$
$$R = (x^1y^1 + x^2y^2)/\varrho, \qquad \Theta = x^1y^2 - x^2y^1$$

(in classical notation $R = \dot{\varrho}$ and $\Theta = \varrho^2\dot{\theta}$). B is obtained by holding Θ

fixed and identifying points differing only in their Θ coordinate. Thus B is coordinatized by ϱ, $0 < \varrho < \infty$ and R, $-\infty < R < \infty$, and so B is topologically R^2. However, as seen above, B inherited a symplectic structure $d\varrho \wedge dR$ from M (cf. Theorem 3).

REMARK Example 1 is typical of the case when the integrals $F_1, \ldots, F_p$ are in involution, i.e., when $\{F_i, F_j\} = 0$, where $\{ , \}$ denotes the Poisson bracket. Since $\{F_i, F_j\} = 0$, the function F_i is an integral for the flow ψ_j generated by $(dF_j)^\#$. Thus $\psi_j(t, \cdot): N \to N$ for all t, and all $j = 1, \ldots, p$. Since $\psi_j(t, m) = \Psi(\exp a_i t, m)$ and G is connected, it follows that $\Psi(g, \cdot): N \to N$ for all $g \in G$. Thus $G = G_s$. Since $d\{F_i, F_j\}^\# = -[(dF_i)^\#, (dF_j)^\#]$, where $[,]$ is the Lie bracket, it follows that G must be abelian. If $\dim G = \frac{1}{2}\dim M = n$, then $\dim N = n$, and one would expect that $B = N/\sim$ would be zero dimensional.

The fact that $G = G_s$ when the integrals $F_1, \ldots, F_p$ are in involution seems to be the special property needed to prove the two classical results stated in Section 1.

Now let H be a Hamiltonian on M and $\varphi: R \times M \to M$ the flow generated by $(dH)^\#$. Let $n \in N \subset M$ be such that $\varphi(t, n)$ is a nonconstant periodic solution of least positive period T. Since $\varphi(T, n) = n$, $d\varphi(T, n)$: $T_nM \to T_nM$ (here the differential is with respect to the second argument only), and the eigenvalues of this linear transformation are called the characteristic multipliers of the periodic solution. The characteristic multiplier $+1$ is particularly troublesome in perturbation analysis, so one would like to know its multiplicity. The first result of this paper is:

THEOREM 2 If $dH(n)$ is independent of $dF_1(n), \ldots, dF_p(n)$, then the geometric multiplicity of $+1$ as a characteristic multiplier is greater than or equal to $p + 1$, and the algebraic multiplicity of $+1$ as a characteristic multiplier is greater than or equal to $p + q + 2$.

REMARKS Since this is basically a local result, it is unnecessary to assume that $s = F(n)$ is a regular value of F. If n is a regular point of F, i.e., $dF_1(n), \ldots, dF_p(n)$ are independent, then one can find a small neighborhood of the periodic solution such that s is a regular value of F restricted to that neighborhood. The first statement about the geometric multiplicity is well known [3], so the new result is the statement about the algebraic multiplicity. If $F_1, \ldots, F_p$ are in involution then by the remark above $G = G_s$, so $q = p$. Thus Theorem 2 is a generalization of the first classical result in Section 1.

Thus if one is studying a periodic solution of a Hamiltonian system with symmetries one should expect the characteristic multiplier $+1$ to have a high multiplicity. The classical approach is to reduce the dimension of the problem by using the integrals; i.e., one studies the flow on N. If $F_1, \ldots, F_p$ are in involution there is a classical local reduction which goes even further (cf. the second theorem in Section 1).

The global version of this classical result is:

For each point $n \in N$, let $W_n = \text{span}\{dF_1(n), \ldots, dF_p(n)\} \subset T_n{}^*M$. Then

$$T_nN = W_n{}^\circ = \{v \in T_nM : f(v) = 0 \text{ for all } f \in W_n\}.$$

Let $W_n{}^\# = \{v \in T_nM : v = u^\#, u \in W_n\}$. Since $\{dF_i(n), dF_j(n)\} = 0$ for $i = 1, \ldots, p$, and $j = 1, \ldots, q$, it follows that

$$\text{span}\{dF_1{}^\#(n), \ldots, dF_q{}^\#(n)\} \subset W_n{}^\circ \cap W_n{}^\#.$$

Assume that

$$\text{span}\{dF_1{}^\#(n), \ldots, dF_q{}^\#(n)\} = W_n{}^\circ \cap W_n{}^\# \qquad \text{for each} \quad n \in N.$$

THEOREM 3 Let N be a fiber bundle over B with fibers G_s. Then B is a symplectic manifold with symplectic form ω and if $\pi: N \to B$ is the projection $d\pi^\#: \omega \to \Omega \mid N$. If $b \in B$ is such that $\pi(n) = b$, then T_bB is isomorphic to $W_n{}^\circ / W_n{}^\circ \cap W_n{}^\#$.

Also one wants to study the flow on B which comes from the flow φ generated by $(dH)^\#$ on M. Let N and B be as above. Since $\varphi \mid R \times N$ is invariant under the action of G_s, i.e., $\Psi(g, \varphi(t, n)) = \varphi(t, \Psi(g, n))$, for all $t \in R$, $n \in N$, and $g \in G_s$, the flow projects to a flow ζ on B. Namely $\zeta(t, [n]) = [\varphi(t, n)]$, where $[n] \in N/\!\sim\, = B$. Also $H \mid N$ is invariant under the action of G_s, i.e., $H(n) = H(\Psi(g, n))$, for all $g \in G_s$ and $n \in N$. Thus $K([n]) = H(n)$, $[n] \in B$ is a well-defined function on B. By the assumption that N is locally trivial over B, it is clear that ζ and K are smooth. The next natural result is then:

THEOREM 4 The flow ζ on B is generated by $(dK)^\#$, where $\#$ is the isomorphism defined by ω.

Before proceeding with the proofs of these theorems, it would be instructive to look at a theorem of Poincaré on the existence of "periodic" solutions in the three-body problem. We claim no improvement on Poincaré's result, but the general theorems given above give a nature interpretation to his theorem [3].

EXAMPLE 3 (periodic orbits of the first kind) Consider the planar three-body problem. Let the particles have masses m_0, m_1, m_2; position vectors r_0, r_1, r_2, and momentums $p_0 = m_0\dot{r}_0$, $p_1 = m_1\dot{r}_1$, $p_2 = m_2\dot{r}_2$, respectively. The Hamiltonian of the problem is

$$H = \sum_{i=0}^{2} \| p_i \|^2/2m_i - \sum_{0\leq j<k\leq 2} \frac{m_j m_k}{\| r_j - r_k \|}, \tag{1}$$

where the gravitational constant is taken to be 1. It is clear that

$$H(r_0 + g, r_1 + g, r_2 + g, p_0, p_1, p_2) = H(r_0, r_1, r_2, p_0, p_1, p_2) \quad \text{for all} \quad g \in R^2,$$

so H is invariant under the action of R^2 on $(R^2)^3 \times (R^2)^3$ by translations in the first arguments. As in Example 1, this implies that the total linear momentum $p_0 + p_1 + p_2$ is a vector-valued integral. Let us use the standard device of changing coordinates so that one new position vector represents the position of the center of mass. Make the linear symplectic change of variables

$$\begin{aligned} u_0 &= m_0 r_0 + m_1 r_1 + m_2 r_2, & v_0 &= p_0 + p_1 + p_2, \\ u_1 &= -m_0 r_0 + (1 - m_1) r_1 - m_2 r_2, & v_1 &= -(m_1/m_0) p_0 + p_1, \\ u_2 &= -m_0 r_0 - m_1 r_1 + (1 - m_2) r_2, & v_2 &= -(m_2/m_0) p_0 + p_2, \end{aligned} \tag{2}$$

where we have assumed that $m_0 + m_1 + m_2 = 1$. After some computations we find that the Hamiltonian in the new coordinates is independent of u_0, so v_0 is an integral (total linear momentum). By changing the origin of our coordinate system by a Galilean transformation, we may assume that $v_0 = 0$. By taking $v_0 = 0$ and forgetting about u_0, we pass in the classical way to the quotient space $(R^2)^2 \times (R^2)^2$. The new Hamiltonian is

$$\begin{aligned} H = \frac{1}{2}\left\{\frac{\| v_1 \|^2}{m_1} + \frac{\| v_2 \|^2}{m_2} - \| v_1 + v_2 \|^2\right\} \\ - \left\{\frac{m_0{}^2 m_1}{\| (m_0 + m_1)u_1 + m_2 u_2 \|} + \frac{m_0{}^2 m_2}{\| m_1 u_1 + (m_0 + m_2)u_2 \|} \right. \\ \left. + \frac{m_1 m_2}{\| u_1 - u_2 \|}\right\} \end{aligned} \tag{3}$$

We wish to study the periodic solutions of this system when m_0 is near 1 and m_1 and m_2 are small.

We shall consider Eq. (3) as defining the basic system to be studied. One can see that the Hamiltonian H is invariant under the action $(g, (u_1, u_2, v_1, v_2) \to (gu_1, gu_2, gv_1, gv_2)$, where $g \in SO_2$, so as in Example 2 the equations defined by Eq. (3) admit the total angular momentum $J = u_1 \times v_1 + u_2 \times v_2$ as an integral. Since J is a scalar (a one-vector) it is clearly in involution, so SO_2 acts as a transformation group on

$$N_J = \{(u_1, u_2, v_1, v_2) \in (R^2)^2 \times (R^2)^2 : J = u_1 \times v_1 + u_2 \times v_2) \quad \text{for} \quad J \neq 0.$$

It will be instructive to compute the topological type of N and $B = N/\sim$.

Let Q be the space of quaterians and consider $(R^2)^2 \times (R^2)^2$ as coordinatized by $Q \times Q$ as follows: To

$$(u_1^1, u_1^2, u_2^1, u_2^2, v_1^1, v_1^2, v_2^1, v_2^2) \in (R^2)^2 \times (R^2)^2,$$

associate the pair of quaterians $(x, y) \in Q \times Q$ as follows:

$$x = u_1^1 + u_1^2 i + u_2^1 j + u_2^2 k \quad \text{and} \quad y = v_1^2 + v_1^1 i - v_2^2 j + v_2^1 k.$$

One computes that $yx = J + \alpha i + \beta j + \gamma k$, where $J = u_1 \times v_1 + u_2 \times v_2$, and α, β, γ are combinations of the components of u_1, u_2, v_1, v_2. Thus for a given $J \neq 0$, the space $N \subset Q \times Q$ is given by

$$\{(x, y) \in Q \times Q : x \neq 0 \text{ and } y = (J + \alpha i + \beta j + \gamma k)x^{-1}\}.$$

Thus N is coordinatized by $x \in Q - \{0\} \simeq S^3 \times R^1$ and $(\alpha, \beta, \gamma) \in R^3$. Thus N is topologically just $S^3 \times R^4$.

The action of SO_2 on $(R^2) \times (R^2)$ is equivalent to the action of S^1 on $Q \times Q$ given by $(\theta, (x, y)) \to (r(\theta)x, yr(\theta)^{-1})$, where $\theta \in S^1$ and $r(\theta) = \cos\theta + i \sin\theta$. Thus we identify points

$$(x, \{J + \alpha i + \beta j + \gamma k\}x^{-1}) \quad \text{and} \quad (r(\theta)x, \{J + \alpha i + \beta j + \gamma k\}(r(\theta)x)^{-1})$$

on N. Note that the identified points both have the same coordinates in the last three places, namely α, β, and γ. Thus $B = (Q - \{0\}/\sim) \times R^3$, where $x_1 \sim x_2 (x_1, x_2 \in Q)$ if there is a $\theta \in S^1$ such that $x_1 = r(\theta)x_2$. For each sphere about $0 \in Q$, the action $(r(\theta), x) \to r(\theta)x$ is just the standard action giving rise to the Hopf fibration, so $(Q - \{0\}/\sim) = S^2 \times R^1$. Thus $B = S^2 \times R^4$.

Return to the perturbation problem: Since these equations admit an additional first integral, a periodic solution will have characteristic multipliers $+1$ of at least geometric multiplicity 2 and algebraic multiplicity 4. Thus a straightforward application of the implicit function theorem would fail unless the integral and the symmetry are taken into account. We shall accomplish this by doing the perturbation analysis on B.

Let us assume for simplicity that $m_1 = m_2 = \varepsilon$ and make the change of scale $v_1 \to \varepsilon v_1, v_2 \to \varepsilon v_2$. The new Hamiltonian becomes

$$\varepsilon^{-1}H = H_1 + O(\varepsilon), \tag{4}$$

where

$$H_1 = \tfrac{1}{2}\{\| v_1 \|^2 + \| v_2 \|^2\} - \| u_1 \|^{-1} - \| u_2 \|^{-1}. \tag{5}$$

Thus to the first approximation, the equations of motion are like two central force problems. We note that the change of scale leaves the position vectors u_1 and u_2 of order 1, and since v_1 and v_2 are momentum the velocities will also be order 1.

We wish to show, following Poincaré, that the circular solutions of this pair of central force problems can be continued into the full three body problem for $\varepsilon \neq 0$. In order to do this we shall use the polar coordinates of Example 2 to introduce local coordinates about these periodic solutions in B. Use the polar coordinates $\varrho_i, \theta_i, R_i = \dot\varrho_i, \Theta_i = \varrho_i\dot\theta_i$, $i = 1, 2$, so that

$$H_1 = \sum_{i=1}^{2} \left\{\frac{1}{2}\left(R_i^2 + \frac{\Theta_i^2}{\varrho_i^2}\right) - \frac{1}{\varrho_i}\right\}, \tag{6}$$

and $J = \Theta_1 + \Theta_2$. Change coordinates by $\varphi_1 = \tfrac{1}{2}(\theta_1 + \theta_2)$, $\varphi_2 = \tfrac{1}{2}(\theta_1 - \theta_2)$ $\Phi_1 = \Theta_1 + \Theta_2$, $\Phi_2 = \Theta_1 - \Theta_2$. Note that φ_1 and φ_2 are periodic of period π. Thus

$$H_1 = \frac{1}{2}\left\{R_1^2 + \frac{(J + \Phi_2)^2}{4\varrho_1^2}\right\} - \frac{1}{\varrho_1} + \frac{1}{2}\left\{R_2^2 + \frac{(J - \Phi_2)^2}{4\varrho_2^2}\right\} - \frac{1}{\varrho_2}. \tag{7}$$

Note that we have replaced Φ_1 by J, which we will consider as a constant. The action of SO_2 in these coordinates is just a translation of φ_1, so we may drop φ_1. Thus we have the local symplectic coordinates $\varrho_1, \varrho_2, \varphi_2, R_1, R_2, \Phi_2$ on B. The equations of motion for $\varepsilon = 0$, on B are then

$$\dot\varrho_i = R_i, \qquad \dot R_1 = \frac{(J + \Phi_2)^2}{4\varrho_1^3} - \frac{1}{\varrho_1^2}, \qquad \dot R_2 = \frac{(J - \Phi_2)^2}{4\varrho_2^3} - \frac{1}{\varrho_2^2},$$

$$\dot\varphi_2 = \frac{1}{4}\left\{\frac{(J + \Phi_2)}{\varrho_1^2} - \frac{(J - \Phi_1)}{\varrho_2^2}\right\}, \qquad \dot\Phi_2 = 0. \tag{8}$$

Since these are the equations of first approximation, we need only compute the characteristic multipliers of the circular orbits on B, and if $+1$ has algebraic multiplicity 2, then we can be sure by the usual continuation theorems that these periodic solutions will persist when $\varepsilon \neq 0$ and ε small.

By the last equation, Φ_2 is a constant. Thus the equations for ϱ_i, R_i are independent and have a critical point at $\varrho_1{}^\circ = \frac{1}{4}(J + \Phi_2)^2$, $\varrho_2{}^\circ = \frac{1}{4}(J - \Phi_2)^2$, $R_1 = R_2 = 0$. These, of course, correspond to the circular orbits on B. The linearized equations about these critical points are $\ddot{\varrho}_i + \omega_i{}^2\varrho_i = 0$, where $\omega_1 = 8(J + \Phi_2)^{-3}$ and $\omega_2 = 8(J - \Phi_2)^{-3}$. The time T for φ_2 to increase by π is

$$T = \frac{\pi}{4}\left\{\frac{(J + \Phi_2)^3(J - \Phi_2)^3}{(J - \Phi_2)^3 - (J + \Phi_2)^3}\right\}.$$

Thus the characteristic multipliers of these periodic solution on B are $+1$, $+1$, $\exp \omega_1 Ti$, $\exp -\omega_1 Ti$, $\exp \omega_2 Ti$, $\exp -\omega_2 Ti$. Thus we must require that $\omega_j T \neq k2\pi$, $k \in \mathbb{Z}$. If $\tau_j, j = 1, 2$, represents the period of the two circular solutions in the central force problem, then one calculates that

$$\omega_1 T = 2\pi\tau_2(\tau_2 - \tau_1)^{-1} \qquad \text{and} \qquad \omega_2 T = 2\pi\tau_1(\tau_1 - \tau_2)^{-1}.$$

Thus the multipliers $+1$ will have algebraic multiplicity 2 if

$$\tau_1/\tau_2 \neq 1 - 1/k \qquad \text{for} \quad k \in Z. \tag{9}$$

Thus if (9) is satisfied then for ε small, there will exist a period solution of the full three-body problem in the reduced space B, which is near this circular orbit. However these periodic solutions may represent quasi-periodic solutions in the full phase space $(R^2)^2 \times (R^2)^2$.

3 Characteristic multipliers

Use the notation of the previous section: Let the flow φ defined by the Hamiltonian H on M be such that $\varphi(t, m_0)$ is T periodic, where T is the last positive period. Then the eigenvalues of

$$X = \frac{\partial}{\partial m}\varphi(T, m_0) : T_{m_0}M \to T_{m_0}M$$

are called the characteristic multipliers of the periodic solution $\varphi(t, m_0)$.

Let

$$\alpha_0 = \frac{\partial \varphi}{\partial t}(0, m_0) = (dH(m_0))^\#, \qquad \alpha_i = \frac{\partial \psi_i}{\partial t}(0, m_0) = (dF_i)^\#,$$
$$i = 1, \ldots, p,$$

and assume that $\alpha_0, \alpha_1, \ldots, \alpha_p$ are linearly independent. Since the Poisson bracket of H with F_i is zero, the flows φ and ψ_i commute, so $\varphi(t, \psi(s, m)) = \psi(s, \varphi(t, m))$. Differentiating this last expression with respect to s and setting $t = T$, $s = 0$, $m = m_0$ yields $X\alpha_i = \alpha_i$, $i = 1, 2, \ldots, p$. Also $\varphi(t, \varphi(s, m)) = \varphi(t, \varphi(s, m))$ yields in the same way $X\alpha_0 = \alpha_0$. Thus $\alpha_0, \alpha_1, \ldots, \alpha_p$ are linearly independent eigenvectors of X corresponding to the eigenvalue $+1$, and so the geometric multiplicity of $+1$ as a characteristic multiplier is greater than or equal to $p + 1$. This proves the first and well-known part of Theorem 1.

Now $H(\psi_i(t, m_0)) = H(m_0)$. Differentiating this expression with respect to t and setting $t = 0$ gives

$$\frac{\partial}{\partial m} H(m_0)\left(\frac{\partial \psi_i}{\partial t}(0, m_0)\right) = \alpha_0^\flat(\alpha_i) = \{\alpha_0, \alpha_j\} = 0.$$

Also $\{\alpha_0, \alpha_0\} = 0$. Also

$$F_i(\psi_j(t, m_0)) = F_i(m_0), \qquad \text{for} \quad i = 1, \ldots, p, \quad \text{and} \quad j = 1, \ldots, q.$$

Differentiating this expression with respect to t and setting $t = 0$ gives

$$\frac{\partial F_i}{\partial m}(m_0)\left(\frac{\partial \psi_i}{\partial t}(0, m_0)\right) = \alpha_i^\flat(\alpha_j) = \{\alpha_i, \alpha_j\} = 0.$$

Thus in summary, we have $\alpha_0, \alpha_1, \ldots, \alpha_p$ are linearly independent eigenvectors corresponding to the eigenvalue $+1$ such that $\{\alpha_i, \alpha_j\} = 0$ for $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, q$. The second part of Theorem 1 follows from these facts and the lemma given below. The proof of this lemma follows the eloquent discussion of characteristic multipliers found in [2].

Let V be a symplectic linear space and $X\colon V \to V$ a symplectic linear transformation. Let $\eta_l = \ker(X - I)^l$. It is well known that for some least integer k one has $\eta_k = \eta_{k+1} = \eta_{k+2} = \cdots$, so let $\eta = \eta_k$. The geometric multiplicity of $+1$ as an eigenvalue is the dimension of η_1 and the algebraic multiplicity of $+1$ as an eigenvalues is the dimension of η. Let ζ be a subspace of η_1 and ζ a subspace of ζ such that $\{\xi, \zeta\} = 0$. Then we have

Lemma 1 $\dim \eta \geq \dim \zeta + \dim \xi$.

Remark The second part of Theorem 1 follows directly from this lemma taking

$$\zeta = \operatorname{span}\{\alpha_0, \alpha_1, \ldots, \alpha_p\} \qquad \text{and} \qquad \xi = \operatorname{span}\{\alpha_0, \alpha_1, \ldots, \alpha_q\}.$$

Proof Let V^* be the dual of V, X^* the dual of X, $\mathscr{R} = \operatorname{range}(X - I)^k$, $\mathscr{R}^* = \operatorname{range}(X^* - I)^k$, $\eta_1{}^* = \ker(X^* - I)$, and $\eta^* = \ker(X^* - I)^k$. For any $v \in V$, let $v^\flat = \{v, \cdot\} \in V^*$. For any $W \subset V$, let $W^\circ = \{f \in V^* : f(W) = 0\}$, and for any $W \subset V^*$, let $W^\circ = \{v \in V : f(v) = 0$ for all $f \in W\}$. Recalling the basic facts of the Jordan decomposition theorem, $V = \eta \oplus \mathscr{R}$, $V^* = \eta^* \oplus \mathscr{R}^*$, $\mathscr{R}^\circ = \eta^*$, and $\eta^\circ = \mathscr{R}^*$:

(a) $\mathscr{R}^* \subset \zeta^\circ$. Since $\eta \supset \eta_1 \supset \zeta$, we have $\zeta^\circ \supset \eta^\circ = \mathscr{R}^*$.
(b) $\xi^\flat \subset \zeta^\circ$. Let $v \in \xi$ and $u \in \zeta$. Then $0 = \{v, u\} = v^\flat(u)$, so $v^\flat \in \zeta^\circ$.
(c) $\xi^\flat \cap \mathscr{R}^* = \{0\}$. Let $v \in \zeta \subset \eta_1$, so $Xv = v$ or $X^{-1}v = v$. Then if $y \in V$, we have

$$v^\flat(y) = \{v, y\} = \{X^{-1}v, y\} = \{v, Xy\} = (X^* v^\flat)(y),$$

so $X^* v^\flat = v^\flat$. Thus $\zeta^\flat \subset \eta_1{}^*$. However $\xi^\flat \subset \zeta_c{}^\flat \eta_1{}^* \subset \eta^*$, and $\eta^* \cap \mathscr{R}^* = \{0\}$.

Now by (a)–(c), we have

$$\begin{aligned}
\zeta^\circ &\supset \mathscr{R}^* \oplus \xi^\flat, \\
\dim \zeta^\circ &\geq \dim \mathscr{R}^* + \dim \xi^\flat, \\
\dim V - \dim \zeta &\geq \dim V - \dim \eta^* + \dim \xi^\flat, \\
\dim \eta^* &\geq \dim \zeta + \dim \xi^\flat, \\
\dim \eta &\geq \dim \zeta + \dim \xi.
\end{aligned}$$

4 The reduced space

Again use the notation of Section 2. Since $s \in R^p$ is a regular value of $\mathbb{F} = (F_1, \ldots, F_p)\colon M \to R$, $N = \mathbb{F}^{-1}(s)$ is a submanifold of M and $G_s = \{g \in G : \Psi(g, n) \in N$ for all $n \in N\}$ acts as a transformation group of N. In this section, we shall explicitly use the assumption that N is a fiber bundle over B with fiber G_s. Let $\pi\colon N \to B$ be the projection map. The first step is to show that B is a symplectic manifold. The symplectic form Ω of M restricts to a (possibly degenerate) two-form Ω_N on N. Since Ω_N is invariant under the action of G_s and by the local product structure of N, it is clear that there is a smooth two form ω on B such that $d\pi^*(\omega) = \Omega_N$. It remains to show that ω is symplectic.

Let V be a symplectic linear space with symplectic inner product $\{\,,\,\}$, V^* the dual of V, and $W \subset V^*$ a linear subspace. Let

$$W^\circ = \{v \in V : f(v) = 0 \text{ for all } f \in W\} \quad \text{and} \quad W^\# = \{v^\# : v \in W\}.$$

LEMMA 2 For $[x], [y] \in W^\circ/(W^\# \cap W^\circ)$ the bilinear form given by $\{[x], [y]\} = \{x, y\}$ is a well-defined symplectic inner product on $W^\circ/(W^\# \cap W^\circ)$.

Proof If $v \in W^\#$ and $u \in W^\circ$, then $\{v, u\} = 0$ by definition. Thus if $x, y \in W^\circ$ and $\xi, \eta \in W^\# \cap W^\circ$ one has

$$\{[x + \xi], [y + \eta]\} = \{x + \xi, y + \eta\} = \{x, y\} = \{[x], [y]\},$$

so the form is well defined.

Now assume that $\{[x], [y]\} = 0$ for all $[y] \in W^\circ/(W^\circ \cap W^\#)$. Then $\{x, y\} = 0$, for all $y \in W^\circ$, or $\{x, \cdot\} \in W$. Thus $x \in W^\#$ or $[x] = 0$. Thus $\{\,,\,\}$ is nondegenerate on $W^\circ/(W^\circ \cap W^\#)$.

REMARK Theorem 3 follows from this lemma. If $b \in B$ and $n \in N$ are such that $\pi(n) = b$, let $V = T_nN$ and $W = \text{span}\{dF_1(n), \ldots, dF_p(n)\} \subset V^* = T_n{}^*N$. Clearly $W^\circ = T_nN$. $d\pi\colon T_nN \to T_bB$, and the kernel is the set of all $v \in T_nN$ such that

$$V = \frac{d}{dt}\Psi(e^{\alpha t}, n), \qquad \text{where} \quad \alpha \in A_s.$$

That is, the kernel is $\text{span}\{dF_1(n)^\#, \ldots, dF_q(n)^\#\}$, which is assumed to be $W^\circ \cap W^\#$.

Recall that the flow $\varphi \mid N$ is invariant under the action of G_s and so defines a flow ζ on B. Also $H \mid N$ is invariant under the action of G_s and so defines a function K on B. By Theorem 3, we know that B has a well-defined symplectic structure, so $(dK)^\#$ is a vector field on B. To prove Theorem 4, we must show that $(dK)^\#$ generates ζ or

$$\frac{d}{dt}\zeta(0, b_0) = (dK(b_0))^\#.$$

Let $v = dH(m_0)\colon V \to R$; then $v \mid W^\circ = \bar{v} = d(H/N)(m_0)\colon W^\circ \to R$. By construction

$$u = dK(b_0)\colon W^\circ/W^\circ \cap W^\# \to R\colon [x] \to \bar{v}(x) = v(x).$$

Let

$$w = v^{\#} = \frac{d}{dt}\,\varphi(0, m_0).$$

By construction, $z = (d/dt)\zeta(0, b_0) = [W]$. Now

$$U([x]) = v(x) = \{w, v\} = \{[w], [x]\} = z^{\flat}([x]),$$

so $u = z^{\flat}$ or

$$dK(b_0) = \left(\frac{d\zeta}{dt}\,(0, b_0)\right)^{\flat}.$$

Thus $dK^{\#}$ does generate ζ and Theorem 4 is established.

References

[1] R. Abraham, "Foundations of Mechanics," Benjamin, New York, 1967.
[2] A. Deprit and H. Henrard, A manifold of periodic solutions, *Advances Astron. Astrophys.* **6** (1968), 12.
[3] H. Poincaré, "Les methodes nouvelles de la mecanique celeste," Vol. 1, Gauthier-Villars, Paris, 1892.
[4] S. Smale, Topology and Mechanics I, *Invent. Math.* **10**, 1970.
[5] E. Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," 4th ed., Cambridge Univ. Press, London and New York, 1959.