A NEW PROOF OF AND RESULTS RELATED TO THE POINCARE CENTER THEOREM

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1. Introduction.

In the third of a series of papers where Poincaré [5] laid the foundations of the geometric theory of differential equations, he studied the problem of centers by using a method which is now called normalization. He considered a two dimensional, autonomous, analytic system with a critical point at the origin which to the first approximation is a harmonic oscillator. In order to study the stability of the critical point he produced a series of transformations which brought the equations into a simple form to successively high order in the series expansion (Theorem 1.1 below). At each stage in the process two new terms are added to the equations in normal form, a frequency correction term and an amplitude correction term. If a amplitude correction term actually appears with a nonzero coefficient at some order then the amplitude either steadily decreases or steadily increases near the critical point so the critical point is either asymptotically stable or asymptotically unstable. In this case after a finite number of steps the stability or instability of the critical point is decided.

It remained the case where at every order the amplitude correction term was zero. Poincaré showed that in this case the infinite series of trans- formations actually converged so in the new limit coordinates the amplitude was constant and the origin was a center (Theorem 1.2 below). This result is usually referred to as the Poincaré center theorem. A careful proof using the method of majorants can be found in Siegel [6] or Siegel and Moser [7]. This was the first in a long series of works on the convergence of normal form transformations. The definitive work on the convergence of normal form transformations is Brjuno [1, 2].

In Moser [4] all the necessary ingredients for a simple contracting mapping proof of the original Poincaré center theorem are given. We observe in this note that Moser's proof yields Poincaré's center theorem and a slight generalization to the C^k category. The same proof gives a partial C^k version of the theorem. In the C^k case the equivalent of being formally a center is meaningless but a symmetry or an integral will suffice.

To be specific let U be a neighborhood of the origin in $\mathbb{R}^2$, $x \in U$, $\mathbf{f} : U \to \mathbb{R}^2$ be C^k or real analytic with $\mathbf{f}(0) = 0$,

$$D\mathbf{f}(0) = A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$f(x) = \mathbf{f}(x) - Ax$, and $\dot{x} = dx/\mathrm{dt}$. We consider the system of equations

$$\dot{x} = \mathbf{f}(x) = Ax + f(x). \tag{1}$$

1991 *Mathematics Subject Classification.* 34C25, 34D20.

This research supported by a grant from ACMP/DARPA administered by NBS..

The paper appeared in **Differential Equations and Applications**, Ed. A. R. Aftabizadeh, Ohio University Press, Athens, Ohio, 1988, 202-209.

The linearized system obtained from (1) by setting $f = 0$ is the equation of a harmonic oscillator written as a system. We seek a near-identity change of variables $x = u(y)$, $u(0) = 0$, $Du(0) = I$, which reduces (1) to

$$\dot{y} = \mathbf{g}(y) = Ay + g(y) \tag{2}$$

where g is in normal form, i.e.

$$g(e^{At}y) \equiv e^{At}g(y) \text{ or } Dg(y)Ay \equiv Ag(y). \tag{3}$$

If equation (2) is in normal form then the equations written in standard polar coordinates are independent of the polar angle and conversely. That is: if $y_1 = r\cos\theta$, $y_2 = r\sin\theta$ then (2) is in normal form if and only if the equation (2) in polar coordinates are of the form

$$\dot{r} = \alpha(r), \quad \dot{\theta} = -1 + \beta(r). \tag{4}$$

The linear system is obtained by setting $\alpha(r) \equiv \beta(r) \equiv 0$ in which case all solutions are periodic with constant amplitude since r is constant and constant frequency since $\dot{\theta}$ is constant. The function $\alpha(r)$ can be considered as a correction to the amplitude equation and the function $\beta(r)$ can be considered as a correction to the frequency equation. In the case when (4) is analytic

$$\alpha(r) = \sum_{k=1}^{\infty} a_k r^k.$$

If $\alpha(r) \not\equiv 0$ then for small r, $\alpha(r)$ is of one sign and a negative sign implies the solutions tend to the origin as t increases so the origin is asymptotically stable and similarly a positive sign implies the solutions tend away from the origin as t increases so the origin is unstable

Theorem 1.1. *If the f in system (1) is a power series (formal or convergent) then there is a formal power series $u(y) = y + \cdots$ such that the change of coordinates $x = u(y)$ reduces (1) to (2) where (2) is in normal form, i.e. g satisfies (3).*

Theorem 1.2. *If the f in system (1) is real analytic and the formal transformation give in Theorem 1.1 is such that $\alpha(r) \equiv 0$ in (4) then the formal transformation given in Theorem 1.1 is convergent.*

An integral $I(x)$ for (1) is *nondegenerate* if it is a perturbation of the energy integral for the harmonic oscillator, i.e. if $I(x) = x^T x/2 + \cdots$. If (1) is Hamiltonian then the Hamiltonian itself is a nondegenerate integral for (1). If (1) admits a nondegenerate integral then it is easy to see that the normal form has $\alpha(y) \equiv 0$ so the following follows easily from Theorem 1.2.

Corollary 1.1. *If the f in system (1) is real analytic and the system admits a nondegenerate integral then the formal transformation given in Theorem 1.1 is convergent.*

Let R be the 2×2 matrix $R = \mathrm{diag}\,(1, -1)$. The system (1) is called *reversible* if $\mathbf{f}(Rx) = -R\mathbf{f}(x)$ or equivalently $f(Ry) = -Rf(y)$. A transformation $x = u(y)$ is called *R-preserving* if $u(Ry) = Ru(y)$. It is easy to check that an R-preserving transformation takes a reversible system to a reversible system. It is easy to see that a reversible system in normal form has $\alpha(y) \equiv 0$.

Corollary 1.2. *If the f in system (1) is real analytic and reversible then the formal transformation given in Theorem 1.1 is convergent. Moreover, the transformation $x = u(y)$ is R-preserving.*

In Theorem 1.2 or Corollaries 1.1 and 1.2 the normal form is further restricted. In this case the equations are of the form

$$\dot{y} = Ay + a(|y|^2)Ay \tag{5}$$

where $a(\zeta)$ is a scalar, real analytic function of a single variable. Note that the change of the dependent variable give by $d\tau = (1 + a(|y|^2))dt$ transforms (5) to the linear equation $dy/d\tau = Ay$.

The formal transformations which reduce the system (1) to (2) are not unique and one cannot expect all such transformations to converge. The convergent series given in Theorem 1.2 will satisfy additional conditions of a technical nature given later. We will define reversible and nondegenerate later; either of these conditions insure that $\alpha(r) \equiv 0$ and so the corollaries follows from Theorem 1.2. Theorem 1.2 is Poincaré's center theorem. We shall give a new proof of Theorem 1.2 based on the lemmas in Moser [4]. The concept of a formal transformation does not make sense in the C^k case so we have no analog of Theorem 1.2 in the C^k case, but we do prove a C^k version of Corollaries 1.1 and 1.2.

2. The Lemmas.

Here we shall outline the essential lemmas we shall need to prove the theorems and corollaries discussed in the introduction. In some cases an outline of the proofs will be given and the reader is referred to Moser [4] for a more complete proof and the motivation. Let $\delta > 0$ be given, $N_\delta = \{y \in \mathbb{C}^2 :| y |< \delta\}$,and define the Banach space $\mathcal{A}_\delta = \mathcal{A} = \{u : u : N_\delta \to \mathbb{C}^2, u(0) = 0, u \quad \text{bounded and real analytic } \}$ with the supremum norm. Let $M_\delta = \{y \in \mathbb{R}^2 :| x |\le \delta\}$ and $\mathcal{C}^k = \mathcal{C} = \{u : M_\delta \to \mathbb{R}^2, u(0) = 0, u \quad \text{has continuous derivatives to the } k^{th} \text{ order } \}$, $k \ge 1$, with the usual supremum norm on the derivatives. Let $\mathcal{B}_\delta = \mathcal{B}$ be either of the Banach spaces $\mathcal{A}$ or $\mathcal{C}$.

Lemma 2.1. *For each $g \in \mathcal{B}$ there exists a unique pair u,$v \in \mathcal{B}$ such that*

$$Du(y)Ay - Au(y) - v(y) = g(y) \tag{6}$$

$$Dv(y)Ay - Av(y) = 0 \tag{7}$$

Moreover, $\|u\| \le \|g\|$ and $\|v\| \le \|g\|$.

Outline. Moser shows that the solutions are given by the formulas

$$u(y) = \int_0^1 se^{-As}g(e^{As}y)ds$$

and

$$v(y) = -\int_0^1 e^{-As}g(e^{As}y)ds$$

and he verifies the uniqueness of the solutions.

Remark 1. Since

$$\frac{d}{dt}\left\{e^{-At}v(e^{At}y)\right\} = e^{-At}Dv(e^{At}y)Ae^{At}y - Ae^{-At}v(e^{At}y)$$

formula (6) implies

$$v(e^{At}y) = e^{At}v(y). \tag{8}$$

This implies $v(y) = \{a(y)I + b(y)A\}y$ where a and b are scalar real analytic or C^k and satisfy $a(e^{At}y) = a(y)$ and $b(e^{At}y) = b(y)$. To see this simply write down the equations for and b in terms of the two components of v and solve.

Remark 2. It is easy to see that if g is reversible, i.e. $g(Ry) = -Rg(y)$ then so is v. Also u is R-preserving, i.e. $u(Ry) = Ru(y)$.

Let $N'_\delta = \{(y,\lambda) \in \mathbb{C}^2 \times \mathbb{C}^1 : \|y\|, |\lambda - 1| < \delta\}$, and define the Banach space $\mathcal{A}'_\delta = \mathcal{A}' = \{u : N'_\delta \to \mathbb{C}^2,\ u$ bounded and real analytic $\}$ with the supremum norm. Let $M'_\delta = \{(y,\lambda) \in \mathbb{R}^2 : \|y\|, \|\lambda - 1\| \le \delta\}$ and $\mathcal{C}^k = \mathcal{C} = \{u : M \to \mathbb{R}^2, u$ has continuous derivatives to the k^{th} order $\}$ with the usual supremum norm on the derivatives. Let $\mathcal{B}'_\delta = \mathcal{B}'$ be either of the Banach spaces $\mathcal{A}'$ or $\mathcal{C}'$.

Lemma 2.2. *Let $f \in \mathcal{B}_\eta$, $f(0) = 0$, $Df(0) = 0$. There exists a $\delta < \eta$, $u \in \mathcal{B}'_{delta}$, and $v \in \mathcal{B}'_\delta$ such that*

$$D_1u(y,\lambda)Ay = \lambda\{Au(y,\lambda) + f(u(y,\lambda))\} + v(y,\lambda) \tag{9}$$

$$D_1v(y,\lambda)Ay - Av(y,\lambda) = 0 \tag{10}$$

Moreover, $D_1u(0,\lambda) = I$, the 2×2 identity matrix, and $D_1v(0,\lambda) = (1-\lambda)A$.

Outline. Define $F(u)$ by

$$\begin{aligned} F(u) &= \lambda\{Au(y,\lambda) + f(u(y,\lambda))\} - Au(y,\lambda) \\ &= (\lambda - 1)Au(y,\lambda) + f(u(y,\lambda)) \end{aligned}$$

and $L(u,v)$ by

$$L(u,v)(y,\lambda) = Du(y,\lambda)Ay - Au(y,\lambda) - v(y,\lambda).$$

So we must solve $L(u,v) = F(u)$. In the analytic case by Lemma 2.1 the operator L has a bounded inverse and by taking δ sufficiently small F has a small Lipschitz constant so the contracting mapping principle can be applied to show there is a fixed point and the fixed point is analytic. In the C^k case first show the existence of a continuous solution of this operator equation and the operator equation for the formal derivative of u and v by the contracting mapping principle. Then use estimates to show that the formal derivative is the actual derivative. The argument is standard but see Moser [4] for more details. An easier proof can be based on the differentiability of the composition map as given in Franks [3].

Remark 1: If f is reversible and u is R-preserving then $f \circ u$ is reversible and also the limit of reversible functions is reversible. Therefore since the solution of (4) and (5) are obtained by limits of iterations if f is reversible then so is v and u is R-preserving.

Remark 2. Moser [5] claims that the estimates can be made uniform in k the order of differentiability and so if f is C^∞ then so are u and v.

Lemma 2.3. *Continue the assumptions and conclusions of Lemma 2.2. There exist a real analytic or C^k function $\lambda(y)$ for $\|y\| < \alpha$, $\alpha < \delta$, such that if we define $u(y) = u(y,\lambda(y))$ and $v(y) = v(y,\lambda(y))$ then*

$$Du(y)Ay = \lambda(y)\{Au(y) + f(u(y))\} + v(y) \tag{11}$$

$$
Dv(y)Ay - Av(y) = 0 \tag{12}
$$

$$
Du(0) = I, \quad Dv(0) = 0, \quad \lambda(0) = 1 \tag{13}
$$

$$
D\lambda(y)Ay = 0, \quad \lambda(e^{At}y) = \lambda(y), \quad (v(u), Ay) = 0. \tag{14}
$$

Proof. Let $u(y, \lambda)$ and $v(y, \lambda)$ be as given in Lemma 2.2. As in the above remark (10) implies $v(e^{At}y, \lambda) = e^{At}v(y, \lambda)$ or v is of the form $v(y) = \{a(y, \lambda)I + b(y, \lambda)A\}y$ where a and b are scalar real analytic or C^k. By the moreover because of Lemma 2.2, $a(0, \lambda) = 0$ and $b(0, \lambda) = 1 - \lambda$. Consider the function

$$
h(y, \lambda) = (v(y, \lambda), Ay) = y^T\{a(y, \lambda)I + b(y, \lambda)A^T\}Ay = b(y, \lambda)y^T A^T Ay.
$$

By the implicit function theorem we can solve $h(y, \lambda) = 0$ for λ as a function of y, let $\lambda(y)$ be that solution. Since

$$
\begin{aligned}
h(e^{At}y, \lambda) &= (v(e^{At}y, \lambda), Ae^{At}y) \\
&= (e^{At}v(y, \lambda), e^{At}Ay) \\
&= (v(y, \lambda), Ay) = h(y, \lambda)
\end{aligned}
$$

both $\lambda(y)$ and $\lambda(e^{At}y)$ are solutions of $h(y, \lambda) = 0$ so by the uniqueness of the solutions given by the implicit function theorem $\lambda(y) = \lambda(e^{\mathrm{At}}y)$. This gives all the formulas in (14). (9) and (14) imply (11) and (10) and the first formula in (14) imply (12). □

Remark: The two remarks following Lemma 2.2 hold here also. In the reversible case $\lambda(Ry) = \lambda(y)$.

3. Proof of The Center Theorems.

Let $f \in \mathcal{B}$ be given and u, v, λ as given in Lemma 2.3. Then the change of coordinate $x = u(y)$ transforms

$$
\dot{x} = \lambda(u^{-1}(x))\{Ax + F(x)\} + v(u^{-1}(x))
$$

to the linear equation

$$
\dot{y} = Ay.
$$

Use λ to change the time parameter by letting $\mu(x) = 1/\lambda(x)$, $d\tau = \lambda(x)dt$, $' = d/d\tau$ so $x = u(y)$ transforms

$$
x' = Ax + F(x) + \mu(u^{-1}(x))v(u^{-1}(x))
$$

to

$$
y' = \mu(y)Ay.
$$

Thus $x = u(y)$ transforms

$$
x' = Ax + f(x) \tag{15}
$$

to

$$
y' = \mu(y)Ay + Du^{-1}(y)v(y). \tag{16}
$$

If $Du^{-1}(y)v(y) \equiv 0$ then equation (16) is in normal form by the remarks in the introduction and (14). Thus we must show under the various assumptions on (15) that $v(y) \equiv 0$.

Proof of The Classical Center Theorem, Theorem 1.2. Assume that f is real analytic and hence u, v, λ are also. Assume v is not identically zero. Since $v(e^{At}y) = e^{At}v(y)$ and $v(y) = a(y)y$ the series expansion for v starts with a term $v(y) = \gamma\|y\|^{2n}y + \cdots$ with $\gamma \neq 0$. Since $Du(0) = I$ equation (16) looks like $y' = \mu(y)Ay + \gamma\|y\|^{2n}y + \cdots$. But this extra term is a term which establishes asymptotic stability or instability at a finite order since the derivative of the Liapunov function $V = y^T y/2$ is

$$\dot{V} = \gamma\|y\|^{2n+2} + \cdots$$

which is sign definite since $\gamma \neq 0$. Or if you like a different proof, change to polar coordinates r, θ and find that the r equation would be

$$\dot{r} = \gamma r^{2n+1} + \cdots .$$

Since the equation is formally a center $\gamma = 0$. Thus if (15) is formally equivalent to a center then $v(y) \equiv 0$ or (15) is transformed into normal form by $x = u(y)$. □

Proof of Corollary 1.1. As stated before the concept of being formally a center is meaningless in the C^k case but the Corollary 1.1 does make sense in the C^k case. Therefore we shall give a proof of this corollary which does not depend on the formal argument so holds in the C^k case also. Assume that (15) has an integral of the form $I(x) = x^T x/2 + \cdots$. Since $Du(0) = I$, (16) has a integral of the form

$$J(y) = y^T y/2 + \cdots$$

but

$$J'(y) = 2y^T\{\mu Ay + Du^{-1}(y)v(y)\} = 2a(y)y^T y + \cdots = 0$$

implies $a(y) \equiv 0$ for small y. □

Proof of Corollary 1.2. Let R be the 2×2 matrix, $R = \operatorname{diag}(1, -1)$. By the remarks in the previous section the function v is reversible and u is R-preserving. The function λ and hence μ satisfy $\lambda(Ry) = \lambda(y)$ and $\mu(Ry) = \mu(y)$. Since a R-preserving transformation takes a reversible system to a reversible system equation (16) is reversible. v is reversible and $v(y) = a(y)y$ so $v(Ry) = a(Ry)Ry = Ra(y)y = -Rv(y) = -Ra(y)y$ and $a(y) = -a(y)$. Thus $v(y) \equiv 0$. □

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