EQUIVALENT GENERATING FUNCTIONS AND PERIODIC POINTS

KENNETH R. MEYER

Abstract. This note shows that to an area preserving map with a fixed point at the origin whose multipliers are n-th roots of unity there is associated a generating function which has an n-fold symmetry. The critical points of the function other than the one at the origin correspond to periodic points of the map of period n. A maximum or minimum of the generating function corresponds to an elliptic periodic point of the map and a saddle point of the generating function corresponds to hyperbolic periodic point of the map. Thus the study of periodic points is reduced to catastrophe theory with an n-fold symmetry.

1. Introduction.

Poincaré [12] associated a generating function to an area preserving map of a subset of the plane with the property that fixed points of the map correspond to critical points of the generating function. In particular a maximum or minimum of the generating function corresponds to an elliptic fixed point of the area preserving map and a saddle point of the generating function corresponds to hyperbolic fixed point of the map. Thus a rough idea of the dynamics of the area preserving map can be obtained by plotting the level curves of the generating function. Of course Poincaré's construction of the generating function is local and there are some mild eigenvalue conditions imposed. See Meyer [9] for details.

In my 1970 study of the generic bifurcation of periodic points of area preserving mappings I use Poincaré's generating function to analyze the bifurcations of fixed points of a map since the bifurcation of critical points was well studied in catastrophe theory. Catastrophe theory gave a nice clean solution to that part of the problem. For the bifurcation of periodic points of higher period I used normal form methods to study

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the bifurcations. As is well known the normalizing transformation does not converge in general, but it does provide enough information to eke out the nature of the bifurcations – an adequate but not aesthetic solution. In the intervening years normal form methods have been used by many authors on a variety of problems. Also the methods of catastrophe and singularity theory have been adapted to bifurcation problems [4, 6, 7]. The methods are distinct, but yield the same results in general. I discuss the interconnections between the methods below.

In this note I will show that to an area preserving map with a fixed point at the origin whose multipliers are n-th roots of unity there is associated a generating function which has an n-fold symmetry. The critical points of the function other than the one at the origin correspond to periodic points of the map of period n. As with Poincaré's generating function a maximum or minimum of the generating function corresponds to an elliptic periodic point of the map and a saddle point of the generating function corresponds to hyperbolic periodic point of the map. Thus the study of periodic points is again reduced to catastrophe theory but now the functions must have an n-fold symmetry.

This result is carried out in detail in Bridges and Furter [4] using a Lagrangian variational formulation pioneered by Aubry [3]. Those schooled in singularity theory may find their presentation easier, but I think a differential equationists will find the proof given here closer to their heritage. The construction of the generating function uses the classical technique called the Liapunov-Schmidt method or the alternative method by Hale – see Chapter IX of Hale [8] for details and historical remarks. Here I will modify the presentation of this method as given in Moser [11] to the present problem. Moser makes it very clear that his proof carries over to the n-degrees of freedom problem with the only addition expence being some more notation. Considering the 2-dimensional problem simplifies the presentation without loss of essential features.

In the next section, Section 2 , the relation between periodic systems and Poincaré maps for Hamiltonian systems is summarized and some standard results about normal forms for periodic Hamiltonian systems are given. This gives some motivation for the results presented later. Section 3 sets the notation and states the main result of the paper about the existence of the generating function. In Section 4 only a sketch is given of the proof because it is only a minor modification of the proof given in Moser [11] and his proof is straight forward. Section 5 indicates some extensions and applications of the main result.

2. Periodic Systems, Mappings, and Normal Forms.

Most of the results of this section are well known and can be found in Meyer and Hall [10]. The page numbers refer to that book. Bifurcations occur in systems which depend on various parameters, but for the time being the dependence on parameters of the various maps and equations will be suppressed.

Let $\mathcal{O}$ be an open neighborhood of the origin in $\mathbf{R}^2$ and $P : \mathcal{O} \to \mathbf{R}^2$ be a smooth area preserving mapping which has the origin as a fixed point. Smooth will mean $C^k, k \geq 1$, or real analytic and area preserving means that the determinant of the Jacobian matrix of P is identically equal to $+1$, i.e. $\det DP(\xi) \equiv +1$ for $\xi \in \mathcal{O}$. Let $B = DP(0)$ so $P(\xi) = B\xi + \cdots$. Assume that the eigenvalues of B are n-th roots of unity, $n > 2$, so $B^n = I$. (The case when $n = 2$ will be discussed in the last section.) The symplectic matrix B has a Hamiltonian logarithm A, i.e. $e^A = B$ and trace $A = 0$ – see (page 54). (Remark: The statement of the lemma on page 67 is incorrect and should be replaced by the statement of Theorem II.E.2.)

P is the period map of a 1-periodic Hamiltonian system of the form

$$\dot{x} = F(x,t) = Ax + f(x,t) = J\nabla H(x,t) = JSx + J\nabla K(x,t) \tag{1}$$

where

$$H(x,t) = \frac{1}{2}x^T Sx + K(x,t) \text{ and } A = \text{ JS}. \tag{2}$$

Here F, f, H and K are as smooth as P in $x \in \mathcal{O} \subset \mathbf{R}^2$ and are 1-periodic and C^∞ in $t \in \mathbf{R}$. Also $f(0,t) \equiv 0, Df(0,t) \equiv 0, \nabla K(x,0) \equiv 0, D^2K(x,0) \equiv 0$ where the derivatives are with respect to x only (page $116ff$).(The neighborhood $\mathcal{O}$ may have to be shrunk a bit.)

For the rest of this section only assume that F, f, H and K are formal power series in x with coefficients that are smooth, periodic functions in t. Furthermore assume that they are in normal form, i.e. assume that

$$H(e^{At}x,t) \equiv H(x,0) \tag{3}$$

and

$$f(e^{At}x,t) \equiv e^{At}f(x,0) \tag{4}$$

(page 193). Make the time-periodic, symplectic change of variables $x = e^{At}y$ so the equations of motion 1 become

$$\dot{y} = V(y) = J\nabla L(y) \tag{5}$$

where

$$L(y) = K(y, 0), V(y) = f(y, 0) \tag{6}$$

Equations (3) and (4) imply that

$$L(By) = L(y), V(By) = BV(y). \tag{7}$$

Thus if the system (1) with Hamiltonian (2) is in normal form then a linear, symplectic, periodic change of variable reduces the system to an autonomous system (5) with the n-fold symmetry expressed by (7).

Let $x(t, \xi)$ and $y(t, \xi)$ be the solutions of (1) and (5) respectively through ξ at $t = 0$. So $P(\xi) = x(1, \xi) = e^A y(1, \xi) = By(1, \xi)$. Thus the normal form for P is $By(1, \xi)$ where y is the time one map of an autonomous system with the n-fold symmetry (7).

We can think of L as a formal analog of Poincaré's generating function. If ξ_0 is a critical point of L then by (7) so are $B\xi_0, B^2\xi_0, \ldots, B^n\xi_0 = \xi_0$. Since ξ_0 is a critical point of L it is a fixed point of $y(1, \xi)$ and hence a periodic point of period n of P. So the critical points of L give rise to periodic points of P of period n.

Let the Hessian of L at the critical point ξ_0 be R so the linearization of (5) about the critical point is $\dot{y}$ = JRy = Qy. Therefore the linearization of P^n at ξ_0 is $e^{nQ}\xi$. If ξ_0 is a nondegenerate maximum or a minimum of L then the eigenvalues of Q are of the form $\pm\omega i, \omega > 0$, and the eigenvalues of e^{nQ} are $e^{\pm n\omega i}$. Thus the periodic point ξ_0 of P is elliptic (provided $n\omega$ is not a multiple of π). Similarly, a nondegenerate saddle point of L corresponds to a hyperbolic periodic point of P.

Thus critical points of L other than the origin correspond to periodic points of P of period n. Saddle points of L correspond to hyperbolic points of P and most maxima and minima of L correspond to elliptic periodic points of P.

To illustrate this result use complex coordinates in the plane, i.e. change coordinates from real coordinates $\xi = (\xi_1, \xi_2)$ to complex coordinates $(z, \bar{z})$ by $z = \xi_1 + i\xi_2$, $\bar{z} = \xi_1 - i\xi_2$. If the matrix B is in normal form then

$$B = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$

where $\omega = 2\pi/n$ and the transformation $\xi \to B\xi$ is $z \to e^{\omega i}z$ in complex coordinates. The only monomials invariant under this transformation are of the form

$$(z\bar{z})^{\alpha} z^{n\beta} \bar{z}^{n\gamma} \tag{8}$$

where α, β, γ are non-negative integers. Thus the formal generating function L is a formal series in these terms.

For example when $n = 3$ the first few terms are $az\bar{z} + bz^3 + \bar{b}\bar{z}^3 + \cdots$. The prototype bifurcation which occurs at a fixed point whose multipliers are cube roots of unity with one parameter μ can be modeled with $a = \mu$ and $b \neq 0$. In this case you can rotating coordinates so that b is real and so the model generating function is

$$L = \mu z\bar{z} + b(z^3 + \bar{z}^3) \tag{9}$$

in complex coordinates or

$$L = \mu r^2 + 2br^3 \cos 3\theta \tag{10}$$

The critical points other than the one at the origin occur in two sets namely $r = -\mu/3b, \theta = 0, 2\pi/3, 4\pi/3$ and $r = +\mu/3b, \theta = \pi/3, \pi, 5\pi/3$. Both sets are saddle points for L and correspond to a single periodic orbit of period three. This is the standard bifurcation as given in Meyer [9].

3. Main result.

It is well known that it is impossible in general to construct a change of variables which will linearize equation (1) – even a formal change of variables. It is also known that in general you cannot find a smooth transformation which reduces a system of the form (1) to normal form. The alternative method takes another tack. It asks what terms must be subtracted from (1) in order that a linearizing transformation exist.

In particular, we shall show that (1) can be modified to the form

$$\dot{x} = F(x,t) + w(x,t) = Ax + f(x,t) + w(x,t) \tag{11}$$

so that there is a change of variables

$$x = u(y,t) \tag{12}$$

which reduces (1) to the linear equation

$$\dot{y} = Ay. \tag{13}$$

Of course the u and w are not arbitrary. They are as smooth as f and 1-periodic in t. u is invertible with inverse $y = u^{-1}(x,t)$, and most importantly

$$w(x,t) = v(u^{-1}(x,t),t) \tag{14}$$

where $v(y, t)$ satisfies

$$v(e^{At}y, t) = e^{At}v(y, 0) \tag{15}$$

for all t and s.

The importance of these properties comes from the following observation. For each y_0 the function $x(t) = u(e^{At}y_0, t)$ is an periodic solution of (11) of period n, so if $w(x(t), t) \equiv v(e^{At}y_0, t) \equiv 0$ then it is a n-periodic solution of the original equation (1). But by (15) $v(e^{At}y_0, t) \equiv 0$ if and only if

$$V(y_0) = v(y_0, 0) = 0. \tag{16}$$

Thus the search for n-periodic solutions of (1) is reduced to solving the system of equations (16) which are called the determining equations.

It follows from (15) that

$$V(By) = BV(y) \tag{17}$$

so V has an n-fold symmetry. Also, since the equation (1) is Hamiltonian $V(y)$ is the gradient of a function $L(y)$ and by (17) L satisfies

$$L(By) = L(y). \tag{18}$$

This L serves as the analog of the Poincaré generating function for the study of the bifurcation of periodic points of period n. In contrast to the normal form approach outlined in the previous section this is not a formal procedure since we will establish the existence of smooth u and v satisfying the relations above. Let $U(x) = u(x, 0) = u(x, 1)$; U has a local inverse U^{-1}.

In summary: *Let $P : \mathcal{O} \longrightarrow \mathbf{R}^2 : x \longrightarrow Bx + \cdots$ ($\mathcal{O}$ a neighborhood of the origin in $\mathbf{R}^2$) be a smooth area preserving mapping which has the origin as a fixed point. Assume that the eigenvalues of B are n^{th} roots of unity, $n > 2$, so $B^n = I$. Then there exist a local diffeomorphism $y = U(x)$ and a smooth $L : \mathcal{O}' \to \mathbf{R}$, ($\mathcal{O}'$ a neighborhood of the origin in $\mathbf{R}^2$) such that $L(By) = L(y)$. If y_0 is a critical point of L other than at the origin then $x_0 = U^{-1}(y_0)$ is a periodic point of P of period n and conversely. Nondegenerate maxima or minima of L correspond to elliptic periodic points of P and nondegenerate saddle points of L correspond to hyperbolic periodic points of P. If P depends smoothly on parameters then so does U and L.*

4. Proof Outline.

Moser [11] gives a variational derivation for the various formulas given below. I shall simple verify the results. Since there are two variables $y \in \mathcal{O} \subset \mathbf{R}^2$ and $t \in \mathbf{R}$, let D denote the derivative with respect to y ($D = \partial/\partial y$) and an overdot the derivative with respect to t ($\dot{} = \partial/\partial t$). Let the matrices A and B be in normal form so $\| e^{At} \| = \| B \| = 1$.

The first step is to solve the linearized functional equation for the transformation sought. Here it is given by simple formulas (see (24 and (17). Let $\mathcal{D}$ be the differential operator defined by

$$\mathcal{D}v(y,t) = Dv(y,t)Ay - Av(y,t) + \dot{v}(y,t). \tag{19}$$

Now $\mathcal{D}v = 0$ implies $\frac{d}{ds} e^{-As} v(e^{As}y, t+s) = 0$ or $v(e^{As}y, t+s) = e^{As}v(y,t)$. Define a projection operator P by

$$Ph(y,t) = \int_0^1 e^{-As} h(e^{As}y, t+s)ds. \tag{20}$$

Then

$$Pv(y,t) = \int_0^1 e^{-As} v(e^{As}y, t+s)ds = \int_0^1 v(y,t)ds = v(y,t),$$

so $Pv = v$.

Lemma 4.1. *Let $g(y,t)$ smooth for $y \in \mathcal{O}$, $t \in \mathbb{R}$ and be 1-periodic in t, then there exists functions $u(y,t)$ and $v(y,t)$, smooth for $y \in \mathcal{O}$, $t \in \mathbb{R}$ and be 1-periodic in t such that*

$$\mathcal{D}v = 0 \tag{21}$$

$$\mathcal{D}u - v = g \tag{22}$$

$$Pu = 0 \tag{23}$$

Proof. Let

$$v(y,t) = -\int_0^1 e^{-As} g(e^{As}y, t+s)ds, \tag{24}$$

$$u(y,t) = \int_0^1 (\frac{1}{2} + s) e^{-As} g(e^{As}y, t+s)ds \tag{25}$$

To prove (21):

$$\begin{aligned}
Dv(y,t)Ay &= -\int_0^1 e^{-As} Dg(e^{As}y, t+s)e^{As}Ayds \\
&= -\int_0^1 e^{-As}\left(\tfrac{dg}{ds}(e^{As}y, t+s) - \dot{g}(e^{As}y, t+s)\right) ds \\
&= -\int_0^1 Ae^{-As}g(e^{As}y, t+s)ds - \dot{v}(y,t) \\
&= Av(y,t) - \dot{v}(y,t).
\end{aligned}$$

The second to last step is an integration by parts – the nonintegral terms drop out by the periodicity of e^{As}.

To prove (22):

$$\begin{aligned}
Du(y,t)Ay &= \int_0^1 (\tfrac{1}{2}+s)e^{-As} Dg(e^{As}y, t+s)e^{As}Ayds \\
&= \int_0^1 (\tfrac{1}{2}+s)e^{-As}\left(\tfrac{dg}{ds}(e^{As}y, t+s) - \dot{g}(e^{As}y, t+s)\right) ds \\
&= (\tfrac{1}{2}+s)e^{-As}g(e^{As}y, t+s)\ |_0^1 \\
&\qquad - \int_0^1 \left(-A(\tfrac{1}{2}+s)+1\right) e^{-As}g(e^{As}y, t+s)ds - \dot{u}(y,t) \\
&= g(y,t) + Au(y,t) + v(y,t) - \dot{u}(y,t).
\end{aligned}$$

To prove (23):

$$\begin{aligned}
Pu(y,t) &= \int_0^1 e^{-As}u(e^{As}y, t+s)ds \\
&= \int_0^1 e^{-As}\int_0^1 \left(\tfrac{1}{2}+\tau\right) e^{-A\tau}g(e^{As}e^{A\tau}y, t+s+\tau)d\tau ds \\
&= -\tfrac{1}{2}Pv(y,t) + \int_0^1 \tau e^{-A\tau}\int_0^1 e^{-As}g(e^{As}e^{A\tau}y, t+s+\tau)dsdt \\
&= -\tfrac{1}{2}v(y,t) + \int_0^1 \tau e^{-A\tau}v(e^{A\tau}y, t+\tau)d\tau \\
&= -\tfrac{1}{2}v(y,t) + \int_0^1 \tau v(y,t)d\tau = 0.\square
\end{aligned}$$

Thus having solved the linearized functional equation the next step is to solve the nonlinear equation by the contracting mapping principle.

Lemma 4.2. *Let $F(x,t) = Ax + f(x,t)$ be smooth for $x \in \mathcal{O}$, $t \in \mathbb{R}$, be 1-periodic in t, and $f(0,t) = 0, Df(0,t) \in 0$. Then there is an open neighborhood $\mathcal{Q} \subset \mathcal{O}$ and there are functions $u(y,t)$ and $v(y,t)$, smooth for $y \in \mathcal{Q}$, $t \in \mathbb{R}$ and be 1-periodic in t such that*

(26) $$\mathcal{D}v = 0$$

$$\mathcal{D}u - v = f \circ u \tag{27}$$

$$(Pu)(y,t) = y \tag{28}$$

$Du(y,t) = I$. *(Equation (27) means* $(\mathcal{D}u)(y,t) - v(y,t) = f(u(y,t),t)$*.).*

Proof Outline. To solve (27) let $u(y,y) = y + u'(y,t)$ so equation (27) becomes

$$(Du')(y,t) - v(y,t) = f(y + u'(y,t), t) \tag{29}$$

and equation (28) becomes $Pu' = 0$. The linearization of these equations is precisely the equations given in Lemma 4.1. The proof from here on is a standard contracting mapping principle argument similar to the argument given in Moser [11]. Since the details are given in detail there I will omit them. □

This lemma reduces the problem of finding n-periodic solutions to solving a system of equations which with an n-fold symmetry. To see this let $x = u(y,t)$, and $y = u^{-1}(x,t)$ be the change of variables given by this lemma. If $y(t)$ satisfies

$$\dot{y} = Ay \tag{30}$$

then $x(t) = u(y(t),t)$ satisfies

$$\dot{x} = Ax + f(x,t) + v(u^{-1}(x,t),t). \tag{31}$$

So if

$$V(y_0) = v(y_0, 0) = 0 \tag{32}$$

then $v(e^{At}, t) = 0$ and $x(t) = u(e^{At}y_0, t)$ satisfies

$$\dot{x} = Ax + f(x,t). \tag{33}$$

Thus solving (32) gives rise to an n-periodic solution of (33). Since $\mathcal{D}v = 0$ it follows as before that

$$BV(y) = V(By).$$

Up to this point I have not used the fact that the equations are Hamiltonian.

Lemma 4.3. *Let the conditions of Lemma 4.2 hold. If the equations (1) are Hamiltonian with Hamiltonian (2) then there is a smooth function* $L : \mathcal{Q} \to \mathbb{R}$ *such that* $V(y) = v(y,0) = \nabla L(y)$ *and* $L(By) = L(y)$*.*

Proof. Here we assume that $f(x,t) = J\nabla K(x,t)$. Let

$$S(y,t) = \int_0^1 \left(\frac{1}{2}\frac{du}{ds}(e^{As}y, t+s)^T J u(e^{As}y, s+t) - K(u(e^{As}y, t+s), t+s) \right) ds.$$

Then

$$
\begin{aligned}
DS(y,t) &= \int_0^1 \left(\tfrac{du}{ds}(e^s y, t+s) - J\nabla K(u(e^{As}y, t+s)\right)^T Du(e^{As}y, s+t)e^{As}ds \\
&= \int_0^1 \left((\mathcal{D}u)(e^{As}y, t+s) - f(u(e^{As}y, t+s)\right) TJDue^{As}y, s+t)e^{As}ds \\
&= \int_0^1 \left(v(e^{As}y, t+s)\right)^T JDu(e^{As}y, s+t)e^{As}ds \\
&= v(y,t)^T J \int_0^1 e^{As} Du(e^{As}y, s+t)e^{As}ds \\
&= v(y,t)^T J.
\end{aligned}
$$

Therefore $v(y,t) = J\nabla S(y,t)$ or $V(y) = J\nabla L(y)$ where $L(y) = S(y,0)$. In the sequence of expressions for DS given above the first inequality requires an integration by parts, the second uses some definitions, the third uses (22), the forth uses (21) and the remark preceding Lemma 4.1, and the last comes from differentiating (23).

5. Final Remarks.

It remains now to discuss the unfoldings of the critical points of functions which have a n-fold symmetry. Fortunately, this has been done in detail in Bridges and Furter [4]. For example, the codimension 3 functions with a 3-fold symmetry with there unfoldings are

$$\frac{1}{3}z\bar{z} + \frac{1}{4}\alpha(z^3 + \bar{z}^3)$$

$$\frac{1}{4}(z\bar{z})^2 + \frac{1}{24}(z^3 + \bar{z}^3)^2 + \frac{1}{2}\alpha z\bar{z} + \frac{1}{6}\beta(z^3 + \bar{z}^3) + \frac{i}{4}\gamma(\bar{z}^3 - z^3)$$

where α, β, γ are the unfolding parameters. These functions are easy to analyse, but the analysis is tedious when there are many unfolding parameters. [4] contains many tables which give the complete catalog of functions with a n-fold symmetry with their unfoldings up to codimension 3.

This reduction to a problem in equivariant catastrophy theory is very pleasing, but if you just want to know the bifurcations in a particular situation it seems to me that a straight forward application of normal form methods will yield the answer quicker. I say this for two reasons. First of all, if you are interested in periodic solutions of a Hamiltonian system, you do not need to know the canonical form of the generating function. You really don't care about the function itself, only its critical points – all its other properties have no meaning. Secondly, normal form algorithms are fully developed and implimented a various computer platforms.

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Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221-0025

E-mail address: `ken.meyer@uc.edu`