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Effect of Delayed Neutrons on the Stability of a Nuclear Power Reactor

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An example is given to show that stability without delayed neutrons does not imply stability with delayed neutrons. A new criterion for stability in the large is derived. An example is given that shows that the new criterion predicts stability in some cases when the older criterion fails.

INTRODUCTION

This paper is concerned with the effect of delayed neutrons on the stability of a nuclear power reactor. This problem has been considered by several authors[1,2], but there seems to be some confusion in the literature.

In the important 1958 paper[1] of Popov, it was shown that, if stability can be proved by a particular type of Liapunov function without considering the effect of delayed neutrons, then the system is asymptotically stable when the effect of the delayed neutrons is taken into account.

It is important to note that the type of Liapunov function considered by Popov was a very particular type, and therefore, one cannot assume in general that delayed neutrons have a stabilizing effect. We give an example (in "Counter Example") of a system that is asymptotically stable without delayed neutrons and unstable when the effect of the delayed neutrons is considered.

We also prove (in a later section) a new criterion for stability in the large that takes into account the effect of delayed neutrons.

NOTATION AND STATEMENT OF THE PROBLEM

The two systems considered here are

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} - \boldsymbol{b}\eta$$

$$\frac{d\eta}{dt} = \kappa\eta$$

$$\kappa = \kappa_0 + \boldsymbol{c}'\boldsymbol{x} - \rho\eta \qquad (1)$$

and

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} - \boldsymbol{b}\eta$$

$$\frac{d\boldsymbol{y}}{dt} = B\boldsymbol{y} - \boldsymbol{d}\eta$$

$$\frac{d\eta}{dt} = \kappa\eta - \boldsymbol{e}'\frac{d\boldsymbol{y}}{dt} = (\kappa + \boldsymbol{e}'\boldsymbol{d})\eta - \boldsymbol{e}'B\boldsymbol{y}$$

$$\kappa = \kappa_0 + \boldsymbol{c}'\boldsymbol{x} - \rho\eta\,, \qquad (2)$$

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[1]V. M. POPOV, "Notes on the Inherent Stability of Nuclear Reactors," *Intern. Conf. Peaceful Uses Atomic Energy*, Geneva, 2nd Conf., p. 2458 (1958).

[2]E. P. GYFTOPOULOS and DEVOOGHT, "Effects of Delayed Neutrons on Nonlinear Reactor Stability," *Nucl. Sci. Eng.*, 8, (1960).

where

$\boldsymbol{x}$ is a real n-vector, whose components represent different temperatures in a reactor

η is a real positive scalar, representing neutron density

$\boldsymbol{y}$ is a real m-vector, whose components are positive and represent delayed-neutron emitter densities

A is an $n \times n$ matrix,

$$A = \text{diag}(-\theta_1, -\theta_2, \ldots, -\theta_n),$$

where θ_i (> 0) is the temperature time delay constant

B is an $m \times m$ matrix,

$$B = \text{diag}(-\lambda_1, \ldots, -\lambda_m),$$

where λ_j (> 0) is the decay constant of the delayed neutron emitter of group j

$\boldsymbol{b}$ is a real n-vector, whose components are negative and inversely proportional to the heat capacity

κ is the real scalar, representing reactivity divided by the neutron generation lifetime ℓ

$\boldsymbol{c}$ is a real n-vector, whose components are temperature coefficients of reactivity divided by ℓ

κ_0 is the power reactivity divided by ℓ

$\boldsymbol{e}$ is a real m-vector,

$$\boldsymbol{e}' = (1, 1, \ldots, 1)$$

$\boldsymbol{d}$ is a real m-vector, whose components are $-\beta_j/\ell$, where β_j is the effective yield of delayed neutrons of group j

$\boldsymbol{x}, \boldsymbol{y}, \eta, \kappa$ are functions of the real variable t, time.

The two systems, Eqs. (1) and (2), describe the kinetics of an n-temperature nuclear reactor without, and with, m-groups of delayed neutrons, respectively. An external control system is characterized by the scalar ρ.

If $\kappa_0 \neq 0$, systems (1) and (2) have two critical points. For Eqs. (1), the critical points are at

$$\boldsymbol{x}_1 = 0,$$
$$\eta_1 = 0$$

and at

$$\boldsymbol{x}_2 = A^{-1}\boldsymbol{b}(\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})^{-1}\kappa_0,$$
$$\eta_2 = (\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})^{-1}\kappa_0;$$

and for Eqs. (2), the critical points are at

$$\boldsymbol{x}_1 = 0,$$
$$\boldsymbol{y}_1 = 0,$$
$$\eta_1 = 0$$

and at

$$\boldsymbol{x}_2 = A^{-1}\boldsymbol{b}(\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})^{-1}\kappa_0,$$
$$\boldsymbol{y}_2 = B^{-1}\boldsymbol{d}(\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})^{-1}\kappa_0,$$
$$\eta_2 = (\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})^{-1}\kappa_0.$$

We shall assume throughout that $\kappa_0 > 0$ and $\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} > 0$, and so $\eta_2 > 0$. This does not limit the generality of our discussion, since asymptotic stability in the small of the critical point $\boldsymbol{x}_2$, $\boldsymbol{y}_2$, η_2 implies $\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} > 0$ and physically $\eta_2 > 0$.

The critical point with subscript 1 is the shut-down equilibrium state, and the critical point with subscript 2 in the operating equilibrium state.

In general, stability and asymptotic stability shall be in the sense of Liapunov.

We shall say that system (1) (or (2)) is absolutely stable, provided all solutions of Eqs. (1) (or Eqs. (2)), which start at $t = 0$ in the domain where $\eta > 0$ (or $\eta > 0$, $y_i > 0$, where y_i is the i'th component of $\boldsymbol{y}$), remain in this domain and tend to $\boldsymbol{x}_2$, η_2 (or to $\boldsymbol{x}_2$, $\boldsymbol{y}_2$, η_2) as $t \to \infty$ for all values of κ_0. Thus, absolute stability is "asymptotic stability in the large for all power levels."

We use the term absolute stability, since by the change of coordinates $\sigma = \ln(\eta/\eta_2)$ and $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_2$, the system (1) becomes

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} - \boldsymbol{b}\phi(\sigma)$$
$$\phi(\sigma) = \eta_2(e^{\sigma} - 1)$$
$$\frac{d\sigma}{dt} = \boldsymbol{c}'\boldsymbol{u} - \rho\phi(\sigma), \qquad (3)$$

which is a system of the Lurie type, and the concept of absolute stability of Eqs. (1), as defined above, is the same as that found in control theory for Eqs. (3).

Let $A_z = zI - A$ and $A_z^{-1} = (zI - A)^{-1}$, where I is the identity matrix. The following is the best result known to date on the stability of Eqs. (3) or (1):

The system (3) or (1) is absolutely stable provided there exist two constants $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$\tilde{\alpha} \geqq 0, \ \tilde{\beta} \geqq 0 \text{ and } \tilde{\alpha} + \tilde{\beta} > 0, \qquad (4a)$$

$$\text{Re}\,(\tilde{\alpha} + i\omega\tilde{\beta})\left\{\frac{\rho + \boldsymbol{c}'A_{i\omega}^{-1}\boldsymbol{b}}{i\omega}\right\} \geqq 0 \quad \text{for all real } \omega, \qquad (4b)$$

$$\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} > 0. \qquad (4c)$$

For a proof of this theorem, see papers by Kalman[3] or Meyer[4]. For those referring to these papers, it should be noted that the extra condition, when $\tilde{\alpha} = 0$, is not needed here due to the special nature of $\phi(\sigma)$.

When $\tilde{\alpha} = 0$ and $\tilde{\beta} = 1$, Eqs. (4) reduce to Welton's criterion[5], and so the above theorem contains Welton's criterion as a special case.

A theorem of Popov[1] can be applied when $\tilde{\alpha} = 0$ and $\tilde{\beta} = 1$, to show that the system (2) is absolutely stable for any number of delayed neutrons.

The system (2) is absolutely stable provided

$$\mathrm{Re}(\rho + \boldsymbol{c}' A_{i\omega}^{-1} \boldsymbol{b} \geqq 0, \quad \text{for all real } \omega \tag{5}$$

and

$$\rho - \boldsymbol{c}' A^{-1} \boldsymbol{b} > 0.$$

The questions to be considered here are: 1) Is it true in general that the stability of Eqs. (1) implies the stability of Eqs. (2)? 2) Can the criterion for stability, given above for Eqs. (1), be extended to Eqs. (2) in the cases where $\tilde{\alpha} \neq 0$?

The answer to question 1 is, in general, no; but it is yes in some interesting special cases. It has not been possible to completely answer question 2, but a new criterion is given which is analogous to the case when $\tilde{\alpha} \neq 0$ and $\tilde{\beta} = 0$.

COUNTER EXAMPLE

We shall give in this section an example of a reactor system that is asymptotically stable in the small without delayed neutrons and unstable when the effect of the delayed neutrons is taken into account. This example contradicts the general theorem that the delayed neutrons always have a stabilizing effect. Let us consider the reactor kinetics Eqs. (2) and introduce the new coordinates $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_2$, $\boldsymbol{v} = \boldsymbol{y} - \boldsymbol{y}_2$, and $\xi = \eta - \eta_2$. The system (2) then takes the form

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} - \boldsymbol{b}\xi$$

$$\frac{d\boldsymbol{v}}{dt} = B\boldsymbol{v} - \boldsymbol{d}\xi$$

$$\frac{d\xi}{dt} = \kappa(\xi + \eta_2) + \boldsymbol{e}'\boldsymbol{d}\xi - \boldsymbol{e}' B\boldsymbol{v}$$

$$\kappa = \boldsymbol{c}'\boldsymbol{u} - \rho\xi . \tag{6}$$

The Linearized System

By a general theorem of Lyapunov, the system (6) is asymptotically stable in the small, provided the matrix of the linear part has only characteristic roots with negative real parts. By the well-known criterion of Nyquist, the matrix of the linear part of Eqs. (6) will have negative real parts for all power levels, i.e., for all $\eta_2 > 0$, provided

$$\rho - \boldsymbol{c}' A^{-1} \boldsymbol{b} > 0 \tag{7a}$$

$$F(i\omega) = \frac{\rho + \boldsymbol{c}' A_{i\omega}^{-1} \boldsymbol{b}}{i\omega(1 - \boldsymbol{e}' B_{i\omega}^{-1} \boldsymbol{d})} \neq \text{a negative real number for all real } \omega . \tag{7b}$$

Thus, the conditions of Eqs. (7a) and (7b) are sufficient for the asymptotic stability in the small of the operating point $(\boldsymbol{x}_2, \boldsymbol{y}_2, \eta_2)$ for all power levels.

From continuity considerations, the matrix of the linear part of Eqs. (6) has a characteristic root with positive real part for some $\eta_2 > 0$, if the curve $F(i\omega)$ intersects the negative real axis for some nonzero $i\omega_0$, and moreover, if in some neighborhood of the intersection the curve is not on one side of the negative axis, i.e., if $F(i\omega)$ definitely crosses the negative axis at some point. If the matrix of the linear part of Eqs. (6) has a positive characteristic root for some power level, then the system (6) will be unstable for that power level.

The above considerations remain valid if $\boldsymbol{e} = 0$, $\boldsymbol{d} = 0$, $B = 0$; that is, if we are considering Eqs. (1).

If we denote by $F_1(z)$ and $F_2(z)$ the functions in Eq. (7b) without delayed neutrons and with delayed neutrons, respectively, then

$$F_1(z) = z^{-1}\left\{\rho + \sum_{i=1}^{n} \frac{\zeta_i}{z + \theta_i}\right\}$$

$$F_2(z) = z^{-1}\left\{\rho + \sum_{i=1}^{n} \frac{\zeta_i}{z + \theta_i}\right\}\left\{1 + \frac{1}{\ell}\sum_{i=1}^{m} \frac{\beta_i}{z + \lambda_i}\right\}^{-1}, \tag{8}$$

where $\zeta_i = \boldsymbol{c}_i \boldsymbol{b}_i$.

In the following example, every attempt has been made to make the constants of the correct magnitude. To construct the example, it is necessary to take a model with four different and large temperature-time constants:

Special Case $\rho = 0$, $n = 4$, $m = 1$.

In this case we have

$$F_1(z) = \frac{az^3 + bz^2 + cz + f}{z(z + \theta_1)(z + \theta_2)(z + \theta_3)(z + \theta_4)}$$

$$F_2(z) = F_1(z)\left\{\frac{z + \lambda}{z + \lambda + \frac{\beta}{\ell}}\right\},$$

where

λ is the average decay constant of delayed neutrons

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[3]R. E. KALMAN, "Lyapunov Functions for the Problem of Lur'e in Automatic Control," *Proc. Nat. Acad. Sci.*, U. S., (1963).

[4]K. R. MEYER, "Lyapunov Functions for the Problem of Lur'e," *Proc. Nat. Acad. Sci.*, U. S., (1965).

[5]H. B. SMETS, *Problems in Nuclear Power Reactor Stability*, Presses Universitaires de Bruxelles, (1962).

$\beta = \Sigma\beta_i$ is the effective yield of delayed neutrons

$$a = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$$

$$b = \zeta_1(\theta_2 + \theta_3 + \theta_4) + \zeta_2(\theta_3 + \theta_4 + \theta_1) + \zeta_3(\theta_4 + \theta_1 + \theta_2) + \zeta_4(\theta_1 + \theta_2 + \theta_3)$$

$$c = \zeta_1(\theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4) + \zeta_2(\theta_1\theta_3 + \theta_1\theta_4 + \theta_3\theta_4) + \zeta_3(\theta_1\theta_2 + \theta_1\theta_4 + \theta_2\theta_4) + \zeta_4(\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3)$$

$$f = \zeta_1\theta_2\theta_3\theta_4 + \zeta_2\theta_3\theta_4\theta_1 + \zeta_3\theta_4\theta_1\theta_2 + \zeta_4\theta_1\theta_2\theta_3 .$$

We choose $\theta_1 = 100$, $\theta_2 = 200$, $\theta_3 = 300$, $\theta_4 = 400$, and the roots of the polynomial $az^3 + bz^2 + cz + f$ as $z_{01} = -0.1$, $z_{02} = -(0.1 + i0.1)$, and $z_{03} = -(0.1 - i0.1)$. Let $a = 1$, and so $az^3 + bz^2 + cz + f = z^3 + 0.3z^2 + 0.04z + 0.002 = 0$. Thus, we have linear algebraic equations for ζ_i, whose exact solution is

$$\zeta_1 = -\frac{1}{6} + \frac{1}{2} \times 10^{-3} - \frac{2}{3} \times 10^{-6} + \frac{1}{3} \times 10^{-9}$$

$$\zeta_2 = 4 + 6 \times 10^{-3} + 4 \times 10^{-6} - 1 \times 10^{-9}$$

$$\zeta_3 = -13\frac{1}{2} + 13\frac{1}{2} \times 10^{-3} - 6 \times 10^{-6} + 1 \times 10^{-9}$$

$$\zeta_4 = 10\frac{2}{3} - 8 \times 10^{-3} + 2\frac{2}{3} \times 10^{-6} - \frac{1}{3} \times 10^{-9}$$

$$\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} = \frac{\zeta_1}{\theta_1} + \frac{\zeta_2}{\theta_2} + \frac{\zeta_3}{\theta_3} + \frac{\zeta_4}{\theta_4} = \frac{1}{12} \times 10^{-11} > 0.$$

Now we shall consider the argument of $F_1(z)$ for $z = i\omega$. The function $F_1(z)$ has 3 zeroes and 5 poles; the zeroes are $z_{01} = -0.1$, $z_{02} = -(0.1 + i0.1)$, $z_{03} = -(0.1 - i0.1)$, and the poles are $z_{p1} = -100$, $z_{p2} = -200$, $z_{p3} = -300$, $z_{p4} = -400$, and $z_{p5} = 0$. The pole-zero diagram is shown in Fig. 1.

The argument of $F_1(i\omega)$ is $\alpha_1 + \alpha_2 + \alpha_3 - (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \pm 90°$ and, therefore, is never $+180°$ or $-180°$. Thus, this special system without delayed neutrons is asymptotically stable in the small for all power levels. For example, when $\omega = 10$

$$\alpha_1, \alpha_2, \alpha_3 > 89°.$$

$$\gamma_1 < 6°,\ \gamma_2 < 3°,\ \gamma_3 < 2°,\ \gamma_4 < 2°,\ \gamma_5 = 90°.$$

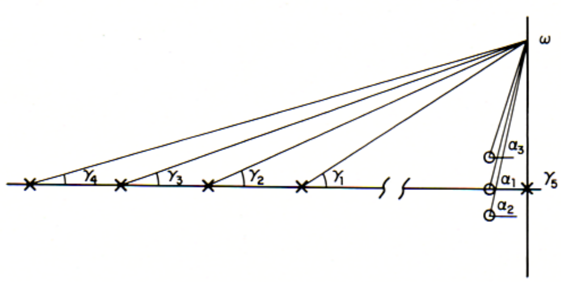

Fig. 1. A pole-zero diagram.

Thus,

$$\arg F_1(i10) > 164°.$$

Now let us consider the effect of one average group of delayed neutrons, which is characterized by constants $\lambda = 0.1\ \text{sec}^{-1}$ and $\beta = 0.0075$. Let us also suppose that $\ell = 10^{-4}$. Thus, $F_2(z)$ has one pole and one zero more than $F_1(z)$. The pole is at $z_p^d = -75$, and the zero is at $z_0^d = -0.1$. The argument of $F_2(i\omega)$ is greater than the argument of $F_1(i\omega)$ by an amount, $\arg(i\omega + 0.1) - \arg(i\omega + 75)$. Thus, ω equals 10, the argument of $F_2 > 164° + 82° = 246°$. For small ω, the argument of F_2 is approximately $-90°$, and as ω increases, the argument increases (not necessarily monotonically) and passes through 180°. Thus, Eq. (6) is unstable for some power levels. Hence, this example shows that, even if the linearized system without delayed neutrons is asymptotically stable for all power levels, the system is unstable for some power levels--an intermediate range--if the effect of the delayed neutrons is taken into account.

A NEW CRITERION FOR STABILITY

Let Γ denote the open set in the $(\boldsymbol{x}, \boldsymbol{y}, \eta)$ space such that $\boldsymbol{x}$ is arbitrary, $\eta > 0$, and $y_j > 0$ for $j = 1, \ldots, m$, where y_j is the j'th component of $\boldsymbol{y}$. The following lemma is self evident from the form of Eqs. (2):

Lemma 1. Let $\boldsymbol{x}(t, \boldsymbol{x}_0)$, $\boldsymbol{y}(t, \boldsymbol{y}_0)$, and $\eta(t, \eta_0)$ be the solution of Eqs. (2) that satisfies $\boldsymbol{x}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$, $\boldsymbol{y}(0, \boldsymbol{y}_0) = \boldsymbol{y}_0$, and $\eta(0, \eta_0) = \eta_0$. If $(\boldsymbol{x}_0, \boldsymbol{y}_0, \eta_0) \in \Gamma$, then $\boldsymbol{x}(t, \boldsymbol{x}_0)$, $\boldsymbol{y}(t, \boldsymbol{y}_0)$, $\eta(t, \eta_0)) \in \Gamma$, for all $t \geq 0$. Moreover, if $\kappa_0 > 0$, then $\eta(t, \eta_0)$ does not tend to zero as $t \to +\infty$.

The new criterion for stability is then:

The system (2) is absolutely stable provided the following conditions hold:

$$\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} > 0,\quad \kappa_0 > 0 \tag{9a}$$

$$\operatorname{Re}\left\{\frac{\rho + \boldsymbol{c}'A_{i\omega}^{-1}\boldsymbol{b}}{i\omega(1 - \boldsymbol{e}'B_{i\omega}^{-1}\boldsymbol{d})}\right\} \geq 0, \quad \text{for all real } \omega. \tag{9b}$$

Proof: As stated in the section, "Counter Example", we can change the origin of our coordinate system to the operating point. Thus, Eqs. (2) can be transferred to the form

$$\frac{d\boldsymbol{w}_1}{dt} = R\boldsymbol{w}_1 - \boldsymbol{f}\eta\kappa \quad \kappa = \boldsymbol{g}'\boldsymbol{w}_1, \tag{10}$$

where

$$R = \begin{pmatrix} A & 0 & -\boldsymbol{b} \\ 0 & B & -\boldsymbol{d} \\ 0 & -\boldsymbol{e}'B & \boldsymbol{e}'\boldsymbol{d} \end{pmatrix}$$

$$\boldsymbol{f} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\boldsymbol{g}' = (\boldsymbol{c}', 0, -\rho)$$

$$\boldsymbol{w}_1' = (\boldsymbol{u}', \boldsymbol{v}', \zeta) = (\boldsymbol{x}' - \boldsymbol{x}_2', \boldsymbol{y}' - \boldsymbol{y}_2', \eta - \eta_2).$$

Clearly, the matrix R has characteristic roots $-\theta_1, \ldots, -\theta_n, -\lambda_1, \ldots, -\lambda_m$, and one characteristic root zero. By a linear nonsingular change of coordinates, the system (10) becomes

$$\begin{pmatrix} \frac{d\boldsymbol{w}}{dt} \\ \frac{d\nu}{dt} \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \nu \end{pmatrix} - \begin{pmatrix} \boldsymbol{r} \\ \delta \end{pmatrix} \eta\kappa$$

$$\kappa = \boldsymbol{k}'\boldsymbol{w} + \epsilon\boldsymbol{v} ,$$

where $\boldsymbol{w}$, $\boldsymbol{k}$, $\boldsymbol{r}$ are $(n+m)$-vectors and ν, ϵ, δ, and κ are scalars. Since $\epsilon\delta = (1 + \boldsymbol{e}'B^{-1}\boldsymbol{d})^{-1} (\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b})$, it follows that $\epsilon\delta > 0$. (See the appendix.) By Eq. (9b), there exists a positive definite $(n+m) \times (n+m)$-matrix Q, a positive semi-definite $(n+m) \times (n+m)$-matrix D, and a real $(n+m)$-vector $\boldsymbol{q}$, such that

$$S'Q + QS = -D - \boldsymbol{q}\boldsymbol{q}' \quad \text{and} \quad Q\boldsymbol{r} - \frac{1}{2}\boldsymbol{k} = 0.$$

(See the appendix for details.) Consider the following Liapunov function:

$$V = \boldsymbol{w}'Q\boldsymbol{w} + \frac{\epsilon}{2\delta}\nu^2 . \tag{11}$$

It is clear that V is positive definite in the whole $(\boldsymbol{w}, \nu)$ space and that $V \to \infty$ as $||\boldsymbol{w}|| + |\nu| \to \infty$. The derivative of Eq. (11) along the trajectories is

$$\frac{dV}{dt} = \boldsymbol{w}'(S'Q + QS)\boldsymbol{w} - 2(Q\boldsymbol{r} - \frac{1}{2}\boldsymbol{k})'\boldsymbol{w}\kappa\eta - \eta\kappa^2$$

$$= -(\boldsymbol{q}'\boldsymbol{w})^2 - \boldsymbol{w}'D\boldsymbol{w} - \eta\kappa^2 .$$

By lemma 1, any solution that starts in Γ for $t = 0$ remains in Γ for all $t \geqq 0$. Thus, dV/dt is negative if $(\boldsymbol{w}, \nu) \in \Gamma$, since $\eta > 0$. Thus, the critical point $\boldsymbol{w} = 0$, $\nu = 0$, or $\boldsymbol{x} = \boldsymbol{x}_2$, $\boldsymbol{y} = \boldsymbol{y}_2$, $\eta = \eta_2$ is stable, and all solutions that start in Γ are bounded.

Since $\kappa_0 > 0$, a solution that starts in Γ cannot tend to the critical point $\boldsymbol{x} = \boldsymbol{x}_1$, $\boldsymbol{y} = \boldsymbol{y}_1$, $\eta = \eta_1$ by lemma 1. Thus, by a theorem of LaSalle[6], the origin $\boldsymbol{w} = 0$, $\nu = 0$ is asymptotically stable provided the only solution in Γ that remains in the set, where $dV/dt = 0$ is the trivial solution $\boldsymbol{w} = 0$, $\nu = 0$.

Let $(\boldsymbol{w}_0, \nu_0)$ be a point in Γ so that the solution $\boldsymbol{w}(t)$, $\nu(t)$ of Eq. (10) satisfies $\boldsymbol{w}(0) = \boldsymbol{w}_0$, $\nu(0) = \nu_0$ and remains in the set where $dV/dt = 0$ for all $t \geqq 0$. It must be shown that $\boldsymbol{w}_0 = 0$, $\nu_0 = 0$. If $dV/dt = 0$ and $\eta \neq 0$, then $\kappa = 0$, and thus, $\boldsymbol{w}(t)$, $\nu(t)$, is a solution of $d\boldsymbol{w}/dt = S\boldsymbol{w}$, $dV/dt = 0$. Thus, $\boldsymbol{w}(t)$ equals $(\exp St)\boldsymbol{w}_0$, and ν equals γ_0 for $t \geqq 0$.

But, since $dV/dt = 0$ along this solution, it follows that $\boldsymbol{q}'(\exp St)\boldsymbol{w}_0 \equiv 0$ and $\boldsymbol{w}_0'D\boldsymbol{w}_0 = 0$. But this implies that $\boldsymbol{w}_0 = 0$. (See the appendix.) Since κ equals 0, the scalar ν_0 equals 0, also. Thus $\boldsymbol{w} = 0$, $\nu = 0$ is the only solution in Γ that remains in the set where $dV/dt = 0$, and so the critical point $\boldsymbol{w} = 0$, $\nu = 0$ or $\boldsymbol{x}_2$, $\boldsymbol{y}_2$, η_2 is absolutely stable.

[6]J. P. LaSALLE, "The Extent of Asymptotic Stability," *Proc. Nat. Acad. Sci.*, U. S., (1954).

DISCUSSION OF THE NEW CRITERION

One can give a simple interpretation of the new criterion as well as the criterion given by Eq. (5). This simple interpretation will indicate the conditions under which the two criteria predict stability. Let us assume throughout that $\rho - \boldsymbol{c}'A^{-1}\boldsymbol{b} > 0$, a necessary requirement in both cases. Observe that we need only check Eq. (5) or (9) for positive ω. Thus Eq. (5) is equivalent to

$$-90° \leqq \arg F(i\omega) \leqq 90°$$

for all positive ω, where $F(z) = \rho + \boldsymbol{c}'A_z^{-1}\boldsymbol{b}$, and Eq. (9) is equivalent to

$$\arg f(i\omega) \leqq \arg F(i\omega) \leqq 180° + \arg f(i\omega)$$

for all positive ω, where F is as before and $f(z) = 1 - \boldsymbol{c}'B_z^{-1}\boldsymbol{d}$.

Thus, if $\arg F(i\omega)$ is greater than 90° for some positive ω, then the criterion of Eq. (5) fails. Indeed, this was the case in the example given in the section, "Counter Example." However, sometimes the new criterion will predict stability.

Let us consider a simple example that will illustrate the point without any complicated algebra. Consider the case when $F(z)$ has two poles $-\theta_1$, $-\theta_2$ and two zeroes $-z_1$, $-z_2$. That is, consider the two-temperature model with $\rho \neq 0$. Let us also assume that $\theta_1 > \theta_2 >> z_1 > z_2$. In this case, $0 < \arg F(i\omega) = \alpha_1 + \alpha_2 - \gamma_1 - \gamma_2 < 180°$ for positive ω, and for some positive ω, $\arg F(i\omega) > 90°$. (See Fig. 2.) Thus, the criterion of Eq. (5) fails.

Let $f(z)$ have one zero $-\lambda$ and one pole $-p$; that is, consider one average group of delayed neutrons. Assume that $\theta_1 > p > \theta_2$ and $z_1 > \lambda > z_2$. Then, clearly, $\arg f(i\omega) \leqq \arg F(i\omega) \leqq 180° + \arg f(i\omega)$ for all positive ω, and so the new criterion predicts stability.

CONCLUDING REMARKS

In the counter example, the system without delayed neutrons, Eqs. (1), was only asymptotically stable in the small. If the system (1) satisfies the

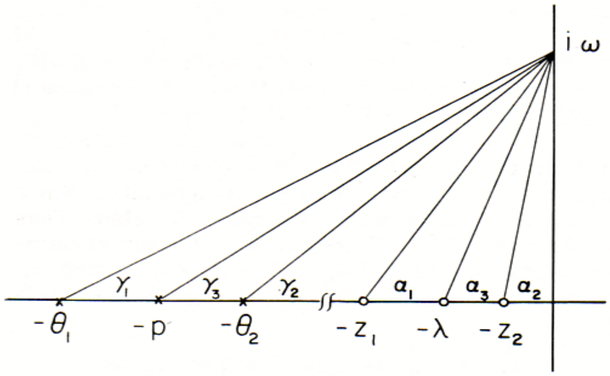

Fig. 2. The pole-zero diagram for the example.

conditions of Eqs. (4)—the best-known criterion for absolute stability—then the system with delayed neutrons, Eqs. (2), is asymptotically stable in the small for all power levels. This can be seen from the following: The conditions, Eqs. (4), imply that the argument of $F_1(i\omega)$ for positive ω is less than 90°. The increase in the argument of $F_2(i\omega)$ over $F_1(i\omega)$ for positive ω is positive and less than 90°. Thus, if the conditions of Eqs. (4) are satisfied, the argument of $F_2(i\omega)$ is less than 180° for all positive ω and, similarly, for negative ω.

It would be interesting to know if absolute stability of Eqs. (1) implies the absolute stability of Eqs. (2), or even if the conditions of Eqs. (4) imply the absolute stability of Eqs. (2). We have not been able to establish these results, nor have we been able to construct a counter example.

APPENDIX

Recently the second author has extended some work of Kalman and Yacubovich and established the following lemma[4].

Lemma 2: Assume that the characteristic roots of the $p \times p$ real matrix S have negative real parts. Let $\boldsymbol{r}$ and $\boldsymbol{k}$ be any real p-vectors. Then there exists a real positive definite symmetric $p \times p$-matrix Q, a real positive semi-definite symmetric $p \times p$-matrix D, and a real p-vector $\boldsymbol{q}$, such that

$$S'Q + QS = -D - \boldsymbol{q}\boldsymbol{q}' \quad \text{and}$$

$$Q\boldsymbol{r} - \boldsymbol{k} = 0,$$

if

$$\operatorname{Re} \boldsymbol{k}' S_{i\omega}^{-1} \boldsymbol{r} \geqq 0, \quad \text{for all real } \omega.$$

Furthermore, if $\boldsymbol{q}'(\exp St)\boldsymbol{w}_0 \equiv 0$ and $\boldsymbol{w}_0 D \boldsymbol{w}_0 = 0$, $\boldsymbol{w}_0$ equals 0.

In the section, "A New Criterion for Stability," it was assumed that

$$\operatorname{Re} \frac{\rho + \boldsymbol{c}' A_{i\omega}^{-1} \boldsymbol{b}}{i\omega(1 - \boldsymbol{c}' B_{i\omega}^{-1} \boldsymbol{d})} \geqq 0, \quad \text{for all real } \omega.$$

It is easy to see that

$$\boldsymbol{g}' R_z^{-1} \boldsymbol{f} = \frac{\rho + \boldsymbol{c}' A_z^{-1} \boldsymbol{b}}{z(1 - \boldsymbol{e}' B_z^{-1} \boldsymbol{d})} .$$

Since the function $\boldsymbol{g}' R_z^{-1} \boldsymbol{d}$ (the transfer function) is invariant under a linear nonsingular change of coordinates,

$$\boldsymbol{g}' R_z^{-1} \boldsymbol{f} = (\boldsymbol{k}', \epsilon) \begin{pmatrix} S_z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \delta \end{pmatrix} = \boldsymbol{k}' S_z^{-1} \boldsymbol{r} + \frac{\epsilon \delta}{z}$$

Thus,

$$\epsilon \delta = (1 + \boldsymbol{c}' B^{-1} \boldsymbol{d})^{-1} (\rho - \boldsymbol{c}' A^{-1} \boldsymbol{b}),$$

and

$$\operatorname{Re} \boldsymbol{k}' S_{i\omega}^{-1} \boldsymbol{r} \geqq 0, \quad \text{for all real } \omega.$$

The condition (9b) of the new criterion thus implies that $\operatorname{Re} \boldsymbol{k}' S_{i\omega}^{-1} \boldsymbol{r} \geqq 0$, for all real ω, and so, with lemma 2, it implies the existence of Q, D, and $\boldsymbol{q}$ that satisfy the above identities.