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Stability of equilibria and fixed points of conservative systems

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Abstract. We consider the stability and instability of an equilibrium point of a Hamiltonian system of two degrees of freedom in certain resonance cases. We also consider the stability or instability of a fixed point of an area-preserving mapping in certain resonance cases. The stability criteria are established by Moser's invariant curve theorem and the instability is established by Chetaev's theorem.

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1. Introduction

We present two general lemmas which can be used to establish a wide variety of stability and instability results for Hamiltonian systems. These lemmas are established using the ideas presented in the classic book by Markeev [9]. The stability criteria are established by Moser's invariant curve theorem [12] and the instability is established by Chetaev's theorem [5]. In particular, we consider Hamiltonians which are in resonance where angle terms appear in the normalized Hamiltonian.

The main lemmas are presented in section 2. These lemmas are applied to the study of the stability or instability of a fixed point of an area-preserving mapping in section 3 and to the stability or instability of an equilibrium point of a Hamiltonian system of two degrees of freedom in certain resonance cases in section 4. We give some examples to show that symmetries can cause some difficulties in stability analysis.

2. Main stability lemma

In this section we will give two lemmas which are simple applications of Moser's invariant curve theorem and Chetaev's theorem. These lemmas will be used in subsequent sections to establish the stability or instability of systems in various resonance cases.

Lemma 2.1. *Let $K(r,\phi,t) = \Psi(\phi)r^n + \mathrm{O}(r^{n+\frac{1}{2}})$, where $n = m/2$ with $m \geqslant 3$, an integer. Suppose that K is an analytic function of $\sqrt{r}, \phi, t$, τ-periodic in ϕ and T-periodic in t. If $\Psi(\phi) \neq 0$, for all ϕ, then the origin $r = 0$ is a stable equilibrium for the Hamiltonian system*

$$\dot{r} = \frac{\partial K}{\partial \phi}, \qquad \dot{\phi} = -\frac{\partial K}{\partial r},$$

in the sense that given $\epsilon > 0$, there exists $\delta > 0$ such that if $r(0) < \delta$, then the solution is defined for all t and $r(t) < \epsilon$. If $\Psi(\phi)$ has a simple zero, i.e., if there exists ϕ^* such that $\Psi(\phi*) = 0$ and $\Psi'(\phi^*) \neq 0$, then the equilibrium $r = 0$ is unstable.

Proof. Suppose that $\Psi(\phi) \neq 0$, for all ϕ, say, $\Psi(\phi) > 0$. Consider the truncated Hamiltonian

$$k = \Psi(\phi)r^n$$

and define, for each $k > 0$, the variable $I = I(k)$ by

$$I = \frac{1}{2\pi}\int_0^\tau r(k,\phi)\,\mathrm{d}\phi,$$

where

$$r(k,\phi) = \frac{k^{1/n}}{\Psi(\phi)^{1/n}}.$$

Consider a generating function $S(I,\phi)$ defined by

$$S(I,\phi) = \int_0^\phi r(k,\theta)\,\mathrm{d}\theta.$$

By eliminating the factor $k^{1/n}$, we get

$$S(I,\phi) = \beta I G(\phi),$$

where

$$\beta = 2\pi \Bigg/ \int_0^\tau \frac{\mathrm{d}\theta}{\Psi(\theta)^{1/n}}, \qquad G(\phi) = \int_0^\phi \frac{\mathrm{d}\theta}{\Psi(\theta)^{1/n}}.$$

Now, S defines a symplectic transformation $(r,\phi) \to (I, W)$ by the relations

$$W = \frac{\partial S}{\partial I} = \beta G(\phi), \qquad r = \frac{\partial S}{\partial \phi} = \beta I G'(\phi),$$

and the original Hamiltonian

$$K(r,\phi,t) = \Psi(\phi)r^n + \mathrm{O}(r^{n+\frac{1}{2}})$$

is transformed into the new Hamiltonian (analytic in $\sqrt{I}, W, t$)

$$K(I, W, t) = \beta^n I^n + \mathrm{O}(I^{n+\frac{1}{2}}),$$

since $G'(\phi) = \Psi(\phi)^{-1/n}$, hence, $\Psi(\phi)G'(\phi)^n = 1$.

Notice that, since $\Psi(\phi)$ is τ-periodic,

$$\begin{aligned} W(r,\phi+\tau) = \beta G(\phi+\tau) &= \beta\int_0^{\phi+\tau} \frac{\mathrm{d}\theta}{\Psi(\theta)^{1/n}} \\ &= \beta\int_0^{\tau} \frac{\mathrm{d}\theta}{\Psi(\theta)^{1/n}} + \beta\int_\tau^{\phi+\tau} \frac{\mathrm{d}\theta}{\Psi(\theta)^{1/n}} = 2\pi + W(r,\phi), \end{aligned}$$

so W is a true angular variable, therefore $K(I, W, t)$ is 2π-periodic in W, and of course, T-periodic in t.

Consider the change of variables $(I, W) \to (J, \psi)$ defined by

$$I = \sigma\gamma J, \qquad W = \psi,$$

where $\sigma > 0$ is a small parameter, $1 \leqslant J \leqslant 2$ and γ is chosen so that $\beta^n\gamma^{n-1} = 1/n$. This is a symplectic change of variables with multiplier $\sigma\gamma$, so the new Hamiltonian is given by

$$\mathcal{K}(J,\psi,t) = \frac{1}{n}\sigma^{n-1}J^n + \mathrm{O}(\sigma^{n-\frac{1}{2}}),$$

and the corresponding Hamiltonian equations are

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \mathrm{O}(\sigma^{n-\frac{1}{2}}), \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t} = -\sigma^{n-1}J^{n-1} + \mathrm{O}(\sigma^{n-\frac{1}{2}}),$$

with the right-hand side analytic in J, ψ, t, 2π-periodic in ψ and T-periodic in t, with $1 \leqslant J \leqslant 2$.

Integrating between $t = 0$ and $t = T$, and denoting by J, ψ the initial values and by J_1, ψ_1 the final values, we obtain the map

$$\begin{aligned} J_1 &= J + \sigma^{n-\frac{1}{2}} F_1(J, \psi, \sigma) \\ \psi_1 &= \psi - \sigma^{n-1} T J^{n-1} + \sigma^{n-\frac{1}{2}} F_2(J, \psi, \sigma), \end{aligned}$$

defined and analytic in the region $1 \leqslant J \leqslant 2$, $\psi \in \mathbb{R}$, $|\sigma| < \sigma_0$, with F_1, F_2 periodic in ψ. This map is area preserving by virtue of the Hamiltonian character of the differential equations. Therefore, by Moser's invariant curve theorem [7,12] there exist, for small σ, invariant curves $J = J(\psi) = J(\psi + 2\pi)$, near to circles, that is, with $J'(\psi) \sim 0$. Since $I = \sigma\gamma J$, the corresponding curve $I = I(\psi)$ can be taken inside small neighbourhoods of the origin (by taking σ sufficiently small). In the three-dimensional space (I, ψ, t), identifying the sections $t = 0$ and $t = T$ we get a torus formed by solutions curves that begin on the closed curve $I = I(\psi)$. By uniqueness of solutions, any solution $(I(t), \psi(t))$ that starts at a point inside the region bounded by the curve $I = I(\psi)$ cannot cross the torus and therefore, $I(t)$ remains small. Therefore, $I(t)$ remains small for all t. Since the solutions stay inside a compact set it is defined for all time. This proves the stability, in the sense defined in the statement of the theorem.

Now we will give the proof of the instability statement. Assume that $\Psi(\psi^*) = 0$ and $\Psi'(\phi^*) > 0$. Choose $\delta > 0$ so small that

$$\Psi(\phi) \neq 0, \qquad \text{and} \qquad \Psi'(\phi) > 0 \qquad \text{for} \quad 0 < |\phi - \phi^*| \leqslant \delta.$$

Consider the function

$$V = r^n \sin\Phi,$$

where $\Phi = (\pi/2\delta)(\phi - \phi^* + \delta)$.

Define a region Ω as the set of points (r, ϕ, t) such that

$$\phi^* - \delta < \phi < \phi^* + \delta.$$

Then, $V > 0$ in Ω and $V = 0$ on $\partial\Omega$, the boundary of Ω. The derivative of V along the solutions of the system of equations

$$\begin{aligned} \dot{r} &= \frac{\partial K}{\partial \phi} = r^n \Psi'(\phi) + \mathrm{O}(r^{n+\frac{1}{2}}) \\ \dot{\phi} &= -\frac{\partial K}{\partial r} = -nr^{n-1}\Psi(\phi) + \mathrm{O}(r^{n-\frac{1}{2}}) \end{aligned}$$

is given by

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial r}\dot{r} + \frac{\partial V}{\partial \phi}\dot{\phi} + \frac{\partial V}{\partial t} \\ &= nr^{2n-1}\left[\Psi'(\phi)\sin\Phi - \frac{\pi}{2\delta}\Psi(\phi)\cos\Phi\right] + \mathrm{O}(r^{2n-\frac{1}{2}}). \end{aligned}$$

For $0 < \phi - \phi^* < \delta$, we have $\pi/2 < (\pi/2\delta)(\phi - \phi^* + \delta) < \pi$ so that $\cos\Phi < 0$. Also for $-\delta < \phi - \phi^* < 0$, we have $\cos\Phi > 0$ and therefore,

$$\Psi(\phi)\cos\Phi < 0 \qquad \text{on} \quad 0 < |\phi - \phi^*| < \delta.$$

Since $\Psi'(\phi) > 0$ and $\sin\Phi > 0$ on $|\phi - \phi^*| < \delta$, we have $\Psi'(\phi)\sin\Phi > 0$ in this interval. Since for the function inside the brackets the two summands do not vanish simultaneously on the compact interval $|\phi - \phi^*| \leqslant \delta$, it follows that it has a positive minimum, and therefore, for small r, we conclude that $\mathrm{d}V/\mathrm{d}t > 0$ on Ω, if r is sufficiently small. It then follows from Chetaev's theorem [5] that the equilibrium is unstable. □

Lemma 2.2. *Let $K(r, \phi, t, \epsilon) = \epsilon^m \Psi(\phi) r^n + \mathrm{O}(\epsilon^{m+1})$, where m and $2n$ are positive integers. Suppose that K is an analytic function of r, ϕ, t, τ-periodic in ϕ, T-periodic in t for all $\frac{1}{2} \leqslant r \leqslant 3$ and all $0 \leqslant \epsilon \leqslant \epsilon_0$. If $\Psi(\phi) \neq 0$, for all ϕ, then if ϵ_0 is sufficiently small any solution of*

$$\dot{r} = \frac{\partial K}{\partial \phi}, \qquad \dot{\phi} = -\frac{\partial K}{\partial r},$$

which starts with $|r(0)| \leqslant 1$ for $0 \leqslant \epsilon \leqslant \epsilon_0$ satisfies $|r(t)| \leqslant 2$ for all t.

Proof. As in the proof of the previous lemma we show that there are invariant curves for the section map which separate $r = 1$ from $r = 2$. □

3. Hamiltonian periodic systems and area-preserving maps

As the first application we consider stability and instability of an equilibrium point of a periodic analytic Hamiltonian system of one degree of freedom $H = H(q, p, t)$. We also discuss the equivalent problem of the stability and instability of a fixed point of an analytic area-preserving mapping. The stability of a periodic solution of an autonomous Hamiltonian system of two degrees of freedom can be reduced to either of these cases.

First consider the periodic case. The classical Liapunov theory shows that the origin is unstable unless the multipliers have unit modulus [7], and so we consider a system whose multipliers are λ, λ^{-1} with $|\lambda| = 1$. We consider the case where the monodromy matrix is diagonalizable.

We are interested in the resonance case so we will assume the system is 2π-periodic and the multiplier is a root of unity, i.e., $\lambda^{\pm 1} = \mathrm{e}^{\pm a 2\pi \mathrm{i}/b}$ where a, b are relatively prime positive integers. By the discussion in [7], chapter VII we can assume the equilibrium is at the origin, and that a series of periodic symplectic changes of variables have been made so that the Hamiltonian is of the form

$$H(I, \theta, t) = \frac{a}{b} I + \beta_2 I^2 + \cdots + \beta_l I^l + \Psi(at + b\theta) I^m + H^\dagger(I, \theta, t) \tag{1}$$

where

- $m = l + \frac{1}{2}$ or $m = l + 1$ with $l \geqslant 1$,
- $\beta_2, \ldots, \beta_l$ are constants,
- $\Psi(\cdot)$ is 2π-periodic and has a finite Fourier series in a single angle,
- $H^\dagger(I, \theta, t)$ is analytic in $\sqrt{I}$, θ, and t and 2π-periodic in θ and t,
- $H^\dagger(I, \theta, t)$ is at least of order $I^{m+1/2}$.

Here we have used action-angle variables I, θ where $q = \sqrt{2I}\cos\theta$ and $p = \sqrt{2I}\sin\theta$. Usually one assumes that the Hamiltonian is analytic in the original variables q, p and hence the Poisson series must have the d'Alembert character [7].

If one of the β_i is nonzero then Moser's invariant curve theorem [7, 12] implies that the equilibrium point is stable. So we will consider the degenerate case when

$$\beta_2 = \cdots = \beta_l = 0.$$

In this case we have the following.

Theorem 3.1. *If $\Psi(\psi)$ is never zero then the equilibrium is stable. If Ψ has a simple zero, that is, if there exists ψ^* such that $\Psi(\psi^*) = 0$ and $\Psi'(\psi^*) \neq 0$, then the equilibrium solution is unstable.*

Proof. Make the time-dependent symplectic change of variables

$$r = I, \qquad \phi = \frac{a}{b}t + \theta,$$

which is generated by the function $S(r, \theta) = r(\theta + at/b)$. The Hamiltonian becomes

$$H = \Psi(b\phi)r^m + \cdots,$$

and so the theorem follows from lemma 2.1. □

The discussion in chapter VII of [7] shows the study of the fixed point of an area-preserving map can be reduced to the study of an equilibrium point of a periodic Hamiltonian system as given above. Consider an analytic area-preserving mapping of a neighbourhood of the origin in $\mathbb{R}^2$ with fixed point at the origin. Again assume that the multipliers of this fixed point are roots of unity, specifically $\lambda^{\pm 1} = \mathrm{e}^{\pm a 2\pi \mathrm{i}/\mathrm{b}}$ where a, b are relatively prime positive integers. By a series of symplectic changes of coordinates we may assume that the mapping, $(I, \theta) \rightarrow (I', \theta')$, is of the form

$$\begin{aligned} I' &= I + b\Psi'(b\theta)I^m + \cdots, \\ \theta' &= \theta + \frac{2\pi a}{b} + \beta_1 I + \cdots + \beta_l I^l - m\Psi(b\theta)I^{m-1} + \cdots, \end{aligned} \tag{2}$$

where

- $m = l + \frac{1}{2}$ or $m = l + 1$ with $l \geqslant 1$,
- $\beta_2, \ldots, \beta_l$ are constants,
- $\Psi(\cdot)$ is 2π-periodic and has a finite Fourier series in a single angle,
- the ellipses are terms of higher order in I and periodic in θ.

Again we have used action-angle variables I, θ where $q = \sqrt{2I}\cos\theta$ and $p = \sqrt{2I}\sin\theta$.

If one of the β_i is nonzero then Moser's invariant curve theorem [7, 12] implies that the fixed point is stable. So we will consider the case when

$$\beta_2 = \cdots = \beta_l = 0.$$

In this case we have the following.

Corollary 3.1. *If $\Psi(\psi)$ is never zero then the fixed point is stable. If Ψ has a simple zero, that is, if there exists ψ^* such that $\Psi(\psi^*) = 0$ and $\Psi'(\psi^*) \neq 0$, then the fixed point is unstable.*

Proof. By the discussion in chapter VII of [7] the period map of a system of the form (1) is of the form (2) and conversely a mapping of the form (2) can be suspended in a periodic system of the form (1). Thus the two results are the same. □

3.0.1. Example: cube root of unity. In the case of an analytic mapping when the multipliers of the fixed point are cube roots of unity then the mapping can be put into the form

$$I' = I - 2\gamma_1 I^{3/2}\sin 3\theta + \cdots,$$
$$\theta' = \theta + \frac{2\pi a}{3} + \gamma_1 I^{1/2}\cos 3\theta + \cdots,$$

where $a = 1$ or 2 and γ_1 is a constant. Generically γ_1 is nonzero and so the fixed point is unstable. However, if the mapping has a special symmetry or depends on parameters it may happen that $\gamma_1 = 0$. In that case the system can be normalized further to get

$$I' = I + 3\Psi'(3\theta)I^3 + \cdots,$$
$$\theta' = \theta + \frac{2\pi a}{3} + \beta_1 I + 3\Psi(3\theta)I^2 + \cdots,$$

where

$$\Psi(3\theta) = \beta_2 + \gamma_2\sin 6\theta + \gamma_3\cos 6\theta + \gamma_4\sin 3\theta + \gamma_5\cos 3\theta.$$

Now if β_1 is nonzero the fixed point is stable otherwise the stability depends on whether Ψ is nonzero or has a simple zero etc. Of course there is the ever elusive case when Ψ has a degenerate zero.

Example: fourth root of unity. In the case of an analytic mapping when the multipliers of the fixed point are fourth roots of unity then the mapping can be put into the form

$$I' = I - 2\gamma_1 I^2\sin(4\theta) + \cdots,$$
$$\theta' = \theta + a\pi + 4\{\beta_1 + \gamma_1\cos 4\theta\}I + \cdots,$$

where $a = 1$ or 3 and β_1 and γ_1 are constants. When $|\beta_1| < |\gamma_1|$ the fixed point is unstable whereas if $|\beta_1| > |\gamma_1|$ the fixed point is stable. This example and others were considered in [10].

Example: odd forces, odd-harmonic forcing. Much of the classical literature on bifurcation of periodic solutions deals with the forced nonlinear oscillator of the form

$$\ddot{u} + f(u) = g(t)$$

where the force f is assumed to be odd, $f(-u) = -f(u)$, and the external forcing g is assumed to be 2π periodic and odd-harmonic, $g(t+\pi) = -g(t)$. Duffing's equation $\ddot{u} + u + u^3 = \cos t$ and the forced pendulum equation $\ddot{u} + \sin u = \sin t$ are prime examples. Written as a periodic Hamiltonian system with Hamiltonian $H(u, v, t)$ these systems admit the symmetry

$$H(-u, v, t+\pi) \equiv H(u, v, t),$$

or if written in action-angle coordinates I, ϕ the Hamiltonian admits the symmetry

$$H(I, \phi, t) \equiv H(I, \phi + \pi, t + \pi).$$

This symmetry condition places restrictions on the angle dependent terms which occur in the Hamiltonian. Consider for example the case when the multipliers are cube roots of unity. The normalized Hamiltonian is of the form

$$H = \frac{a}{3}I + \gamma_1 I^{3/2}\cos(3\theta + at) + \beta_2 I^2 + \cdots,$$

where $a = 1, 2$. Generically without symmetry $\gamma_1 \neq 0$ and so the periodic solution is unstable. However, the above symmetry implies that the cosine term must be zero when $a = 2$ and so the first term in the normalized Hamiltonian is $\beta_2 I^2$ with $\beta_2 \neq 0$ in general. So generically in the presence of symmetry the periodic solution is unstable if $a = 1$ but stable if $a = 2$. Not all cube roots of unity are the same!

4. Hamiltonian systems with two degrees of freedom

As the second application we consider stability and instability of an equilibrium point of an analytic autonomous Hamiltonian system with two degrees of freedom $H = H(q_1, q_2, p_1, p_2)$. The classical Liapunov theory shows that the origin is unstable unless the eigenvalues of the linearized system are pure imaginary, and so we consider a system whose linear part has eigenvalues $\pm\omega_1\mathrm{i}$, $\pm\omega_2\mathrm{i}$. In this case if the frequencies are of the same sign, the Hamiltonian is sign definite, Dirichlet's theorem [6] asserts that the equilibrium is stable. Therefore, we shall consider the case when the frequencies ω_1, ω_2 have opposite sign, i.e. the Hamiltonian has an indefinite quadratic part. Furthermore, we assume that the frequencies satisfy the resonance relation

$$a\omega_1 - b\omega_2 = 0 \tag{3}$$

where a and b are relatively prime positive integers or $a = b = 1$. If $a = b = 1$, we assume also that the matrix of the linearized system is diagonalizable.

We write the Hamiltonian in action-angle variables $(I, \phi) = (I_1, I_2, \phi_1, \phi_2)$ defined by

$$q_j = \sqrt{2I_j}\cos\phi_j, \qquad p_j = \sqrt{2I_j}\sin\phi_j, \qquad (j = 1, 2)$$

and assume that the Hamiltonian H is the normal form through terms of order m where $m = 2l - 1$ or $m = 2l$, i.e.,

$$H(I, \phi_1, \phi_2) = H_2(I) + \cdots + H_{2l-2}(I) + H_m(I, a\phi_1 + b\phi_2) + \cdots \tag{4}$$

where:

- $H_2 = \omega_1 I_1 - \omega_2 I_2$,
- H_{2j} is a homogeneous polynomial of degree j in I_1, I_2,
- $H_m(I, a\phi_1 + b\phi_2)$ is a homogeneous polynomial of degree m in $\sqrt{I_1}, \sqrt{I_2}$ with coefficients which are finite Fourier series in the single angle $a\theta_1 + b\theta_2$,
- the ellipses denote terms of order greater than m in the variables $\sqrt{I_1}, \sqrt{I_2}$, and
- H is an analytic function of the variables $\sqrt{I_1}, \sqrt{I_2}, \phi_1, \phi_2$ and 2π periodic in ϕ_1 and ϕ_2.

That H_m is a function of the single angle $a\phi_1 + b\phi_2$ is equivalent to the fact that H_m is constant along the solutions of the linear equations whose Hamiltonian is H_2, i.e., H_m is constant along the solutions of

$$\dot\phi_1 = -\omega_1, \qquad \dot\phi_2 = \omega_2, \qquad \dot I_1 = 0, \qquad \dot I_2 = 0.$$

Let

$$\Psi(\phi) = H_m(\omega_2, \omega_1, a\phi),$$

where

$$\phi = \phi_1 + \frac{b}{a}\phi_2.$$

Let $D_{2j} = H_{2j}(\omega_2, \omega_1)$. If, for some $j = 2, \ldots, l - 1$, we have $D_{2j} \neq 0$, then Arnold's stability theorem [1, 11] guarantees the stability of the equilibrium solution $q_i = p_i = 0$. Therefore, we assume in addition that

$$D_{2j} = 0, \qquad \text{for} \quad j = 2, \ldots, l - 1,$$

and so H_m is the term that will decide the stability or instability of the equilibrium.

Theorem 4.1. *If $\Psi(\phi) \neq 0$, for all ϕ, then the equilibrium solution $q_i = p_i = 0$ is stable. If Ψ has a simple zero, that is, if there exists ϕ^* such that $\Psi(\phi^*) = 0$ and $\Psi'(\phi^*) \neq 0$, then the equilibrium solution is unstable.*

Remark. *For the stability statement we do not need the resonance condition and so H_m could be independent of an angle. Thus, this theorem includes Arnold's theorem.*

Proof. We follow the ideas in the proof of Arnold's stability theorem as given in [7, 11]. Since $D_{2j} = 0$, the homogeneous polynomial H_{2j} has H_2 as a factor, that is, $H_{2j} = H_2 F_{2j-2}$, where the second factor is a homogeneous polynomial of degree $j - 1$ in I_1, I_2. We have

$$H = H_2 F + H_m(I, \phi) + \cdots,$$

with $F = 1 + F_2 + \cdots + F_{2l-4}$. Near the origin the values of F_{2j} are small and we can take the reciprocal of the function F,

$$F^{-1} = 1 + \cdots,$$

where the ellipses represent terms of degree at least 1 in I_1, I_2. Therefore, $\tilde{H} = F^{-1} H$ can be written as

$$\tilde{H} = H_2 + H_m(I_1, I_2, a\phi_1 + b\phi_2) + \cdots,$$

where the ellipsis represents terms of degree at least $m + 1$ in $\sqrt{I_1}, \sqrt{I_2}$.

Since $H = F\tilde{H}$ the equations of motion are of the form $\dot{z} = J\nabla H = FJ\nabla\tilde{H} + \tilde{H}\nabla F$ where $z = (I_1, I_2, \phi_1, \phi_2)$ and J is the usual 4×4 skew-symmetric matrix of mechanics. If we change time by $\mathrm{d}\tau = F\,\mathrm{d}t$ and let $' = /\mathrm{d}\tau$ the equations of motion on the set $H = 0$ (or $\tilde{H} = 0$) are

$$z' = J\nabla\tilde{H}.$$

Thus, near the equilibrium the flow defined by $\tilde{H}$ on $\tilde{H} = 0$ is a reparametrization of the flow defined by H on $H = 0$.

It suffices to prove instability on the surface $H = 0$ or, equivalently, on $\tilde{H} = 0$. Solving the equation

$$0 = \tilde{H} = \omega_1 I_1 - \omega_2 I_2 + H_m(I_1, I_2, a\phi_1 + b\phi_2) + \cdots, \tag{5}$$

for I_2, we get

$$I_2 = \frac{\omega_1}{\omega_2} I_1 + \frac{1}{\omega_2} H_m\left(I_1, \frac{\omega_1}{\omega_2} I_1, a\phi_1 + b\phi_2\right) + \mathrm{O}(I_1^{\frac{m+1}{2}})$$

or,

$$I_2 = \frac{b}{a} I_1 + \frac{1}{\omega_2^{\frac{m+2}{2}}} H_m(\omega_2, \omega_1, a\phi_1 + b\phi_2) I_1^{\frac{m}{2}} + \mathrm{O}(I_1^{\frac{m+1}{2}}).$$

The right-hand sides of these equations are analytic functions of $\sqrt{I_1}, \phi_1, \phi_2$.

Let $H^\dagger(I_1, \phi_1, \phi_2)$ be the negative of the right-hand side of this expression for I_2. From the equations of motion we see that ϕ_2 is an increasing function of τ and so we can take it as the new independent variable (time). The function $H^\dagger$ then defines a time-dependent Hamiltonian with one degree of freedom, 2π-periodic in ϕ_1 and ϕ_2.

We now make the symplectic change of variables

$$\phi = \phi_1 + \frac{b}{a}\phi_2, \qquad r = I_1$$

which is generated by the function

$$S(r, \phi_1, \phi_2) = r\left(\phi_1 + \frac{b}{a}\phi_2\right).$$

Since the derivative of S with respect to the time ϕ_2 is

$$\frac{\partial S}{\partial \phi_2} = r\frac{b}{a} = \frac{b}{a}I_1,$$

the new Hamiltonian function is given by

$$K(r, \phi, \phi_2) = \Psi(\phi)r^n + \mathrm{O}(r^{n+\frac{1}{2}}),$$

where $n = m/2$ and

$$\Psi(\phi) = -\frac{1}{\omega_2^{n+1}} H_m(\omega_2, \omega_1, a\phi).$$

We notice that K is 2π-periodic in ϕ and $2a\pi$-periodic in ϕ_2. By hypothesis, $\Psi(\phi)$ has a simple zero. Therefore, lemma 2.1 implies that $r = 0$ is an unstable equilibrium for the Hamiltonian system defined by K. Consequently, the equilibrium $q_i = p_i = 0$ is unstable.

If we just want to prove the stability of the equilibrium point on the level set $H = 0$ we could simply apply lemma 2.1, but with a little extra effort we can get the full stability statement. First we scale the action variables $I_i = \epsilon^2 J_i$, where ϵ is a small scale variable. This is a symplectic change of coordinates with multiplier ϵ^{-2}; so, the Hamiltonian (4) becomes (in the variables J_1, J_2, ϕ)

$$H = H_2 F + \epsilon^{m-2} H_m + \mathrm{O}(\epsilon^{m-1}),$$

where, now,

$$F = 1 + \epsilon^2 F_2 + \cdots + \epsilon^{2l-4} F_{2l-4}.$$

We fix a bounded neighbourhood of the origin, say, $|J_i| \leqslant 4$ so that the remainder term is uniformly $\mathrm{O}(\epsilon^{m-1})$ in it and henceforth restrict our attention to this neighbourhood. Let h be a new parameter in the interval $[-1, 1]$. Since $F = 1 + \cdots$, we have

$$H - \epsilon^{m-1}h = KF,$$

where

$$K = H_2 + \epsilon^m H_{m-2} + \mathrm{O}(\epsilon^{m-1}). \tag{6}$$

For sufficiently small ϵ, the function F is positive in the neighbourhood under consideration and so the level set $H = \epsilon^{m-1}h$ is the same as the level set $K = 0$. Let $z = (J_1, J_2, \phi_1, \phi_2)$ and let ∇ be the gradient operator with respect to these variables. The equations of motion are

$$\dot{z} = J\nabla H = F(J\nabla K) + K(J\nabla F).$$

On the level set $K = 0$, the equations become

$$\dot{z} = F(J\nabla K).$$

For small ϵ, as we noticed, F is positive. So the reparametrization $\mathrm{d}\tau = F\,\mathrm{d}t$ transforms this equation to

$$z' = J\nabla K(z), \tag{7}$$

where the prime denotes a derivative with respect to τ.

We have thus shown that in the considered neighbourhood, and for small ϵ, the flow defined by H on each level set $H = \epsilon^{m-1}h$ is a reparametrization of the flow defined by K on the level set $K = 0$. Now, the stability of the equilibrium on each level set $H = \epsilon^{m-1}h$ guarantees, by varying the parameter h, the stability of the equilibrium. Thus, it suffices to prove stability of the origin for the system (7), on the level set $K = 0$.

Now, from (6), we have

$$K = \omega_1 J_1 - \omega_2 J_2 + \epsilon^{m-2} H_m(J_1, J_2, a\phi) + \mathrm{O}(\epsilon^{m-1}).$$

From this point on we proceed to compute the Hamiltonian in the $K = 0$ set just as in the instability case to get (with $n = m/2$)

$$K(r, \phi, \phi_2) = \epsilon^{m-2}\Psi(\phi) r^n + \mathrm{O}(\epsilon^{m-1}).$$

The difference is that K is analytic for $\frac{1}{2} \leqslant r \leqslant 3$ for all small ϵ, and so by lemma 2.2 there exist invariant tori which separate the $r = 1$ torus from the $r = 2$ torus for all small ϵ, say $0 \leqslant \epsilon \leqslant \epsilon_0$. For all $0 \leqslant \epsilon \leqslant \epsilon_0$ all solutions which start with $r \leqslant 1$ must have $r \leqslant 2$ for all τ. Since on $K = 0$ we have $J_2 = (\omega_1/\omega_2)J_1 + \cdots$ and a bound on $r = J_1$ implies a bound on J_2. Thus there are constants, c and k, such that if $J_1(\tau)$, $J_2(\tau)$ satisfy the system for $0 \leqslant \epsilon \leqslant \epsilon_0$, start on $K = 0$ and satisfy $|J_1(0)|, |J_2(0)| \leqslant c$ then $|J_1(\tau)|, |J_2(\tau)| \leqslant k$ for all τ and $0 \leqslant \epsilon \leqslant \epsilon_0$.

Returning to the original unscaled variables with the original Hamiltonian H this means that for $0 \leqslant \epsilon \leqslant \epsilon_0$ all solutions which start on $H = \epsilon^m h$ and satisfy $|I_1(0)|, |I_2(0)| \leqslant \epsilon^2 c$ must satisfy $|I_1(\tau)|, |I_2(\tau)| \leqslant \epsilon^2 k$ for all t and all $-1 \leqslant h \leqslant 1$, $\epsilon \leqslant \epsilon_0$. Thus the equilibrium is stable. □

As applications consider the classical counter example of Cherry and theorems of Markeev [8, 9] and Alfriend [2, 3] and their applications to the Lagrange equilateral triangular libration points in the restricted three-body problem.

Cherry's counterexample. In the second edition (1917) of Whittaker's book on dynamics, the equations of motion about the Lagrange triangular libration point $\mathcal{L}_4$ of the circular three-body problem are linearized, and the assertion is made that the libration point is stable for $0 < \mu < \mu_1$ on the basis of this linear analysis where μ is the mass ratio parameter and $\mu_1 = \frac{1}{2}(1 - \sqrt{69}/9)$ is the critical mass ratio parameter of Routh. In the third edition of Whittaker [13] this assertion was dropped, and an example due to Cherry [4] was included. Cherry's example is a polynomial Hamiltonian system of two degrees of freedom, the linearized equations are two harmonic oscillators with frequencies in a ratio of 2:1. He explicitly gives the solution and thus shows that the higher-order terms can destablize the system. However, a closer look reveals that the Hamiltonian is in Birkhoff's normal form thus indicating the origin of the example.

Cherry's counterexample in action-angle coordinates is

$$H = 2I_1 - I_2 + I_1^{1/2} I_2 \cos(\phi_1 + 2\phi_2), \tag{8}$$

and by the above theorem the equilibrium is unstable.

Resonance. Consider the case where the linear system is in 1:2 resonance, i.e. when the linearized system has exponents $\pm \mathrm{i}\omega_1$ and $\pm \mathrm{i}\omega_2$ with $\omega_1 = 2\omega_2$. Let $\omega = \omega_2$. The normal form for the Hamiltonian is a function of I_1, I_2 and the single angle $\theta_1 + 2\theta_2$. Assume the system has been normalized through terms of degree three, i.e. assume the Hamiltonian is of the form

$$H = 2\omega I_1 - \omega I_2 + \delta I_1^{1/2} I_2 \cos\psi + H^\dagger, \tag{9}$$

where $\psi = \theta_1 + 2\theta_2$, $H^\dagger(I_1, I_2, \theta_1, \theta_2) = \mathrm{O}((I_1 + I_2)^2)$.

Corollary 4.1 (Alfriend–Markeev theorem). *If in the presence of 1:2 resonance, the Hamiltonian system is in the normal form (9) with $\delta \neq 0$ then the equilibrium is unstable.*

Consider the circular restricted three-body problem with mass ratio parameter μ [7]. When $\mu = \mu_2 = \frac{1}{2} - \frac{1}{30}\sqrt{\frac{611}{3}} \approx 0.024\,2939$ the exponents of the Lagrange equilateral triangle libration point $\mathcal{L}_4$ are $\pm 2\sqrt{5}\mathrm{i}/5, \pm\sqrt{5}\mathrm{i}/5$ and so the ratio of the frequencies ω_1/ω_2 is 2. Expanding the Hamiltonian about $\mathcal{L}_4$ when $\mu = \mu_2$ in a Taylor series through cubic terms gives

$$H = \tfrac{1}{14}\left\{5x_1^2 - 2\sqrt{611}x_1x_2 - 25x_2^2 - 40x_1y_2 + 40x_2y_1 + 20y_1^2 + 20y_2^2\right\}$$
$$\times \frac{1}{240\sqrt{3}}\left\{-7\sqrt{611}x_1^3 + 135x_1^2x_2 + 33\sqrt{611}x_1x_2^2 + 135x_2^3\right\} + \cdots.$$

Using Mathematica we can put this Hamiltonian into the normal form (9) with

$$\omega = \frac{\sqrt{5}}{5} \approx 0.447\,213, \qquad \delta = \frac{11\sqrt{11}}{18\sqrt[4]{5}} \approx 1.355\,42,$$

and so we have: *the libration point $\mathcal{L}_4$ of the restricted three-body problem is unstable when $\mu = \mu_2$.*

4.0.2. 1:3 Resonance. Now consider the system in the case when the linear system is in 1:3 resonance, i.e. $\omega_1 = 3\omega_2$. Let $\omega = \omega_2$. The normal form for the Hamiltonian is a function of I_1, I_2 and the single angle $\theta_1 + 3\theta_2$. Assume the system has been normalized through terms of degree four, i.e. assume the Hamiltonian is of the form

$$H = 3\omega I_1 - \omega I_2 + \delta I_1^{1/2} I_2^{3/2}\cos\psi + \tfrac{1}{2}\{AI_1^2 + 2BI_1I_2 + CI_2^2\} + H^\dagger, \qquad (10)$$

where $\psi = \theta_1 + 3\theta_2$, $H^\dagger = \mathrm{O}((I_1 + I_2)^{5/2})$. Let

$$D = A + 6B + 9C, \qquad (11)$$

and recall from Arnold's theorem the important quantity $D_4 = \frac{1}{2}D\omega^2$.

Corollary 4.2 (Alfriend–Markeev theorem). *If in the presence of 1:3 resonance, the Hamiltonian system is in the normal form (10) and if $6\sqrt{3}|\delta| > |D|$ then the equilibrium is unstable, whereas, if $6\sqrt{3}|\delta| < |D|$ then the equilibrium is stable.*

When $\mu = \mu_3 = \frac{1}{2} - \frac{\sqrt{213}}{30} \approx 0.013\,5160$ the exponents of the Lagrange equilateral triangle libration point $\mathcal{L}_4$ of the restricted three-body problem are $\pm 3\sqrt{10}\mathrm{i}/10, \pm\sqrt{10}\mathrm{i}/10$ and so the ratio of the frequencies ω_1/ω_2 is 3.

Using Mathematica we can put this Hamiltonian into the normal form (10) with

$$\omega = \frac{\sqrt{10}}{10} \approx 0.316\,228, \qquad \delta = \frac{3\sqrt{14\,277}}{80} \approx 4.480\,74$$
$$A = \frac{309}{1120}, \qquad B = -\frac{1219}{560}, \qquad C = \frac{79}{560}.$$

From this we compute

$$6\sqrt{3}|\delta| \approx 46.5652 > |D| \approx 8.341\,07,$$

and so we have: *the libration point $\mathcal{L}_4$ of the restricted three-body problem is unstable when $\mu = \mu_3$.*

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