CANONICAL FORMS FOR CRITICAL POINTS

by

Kenneth R. Meyer
University of Minnesota

1. Introduction: The nature of the local local structure of a differential equation near a critical point is a time honored problem in the theory of ordinary differential equation. The present paper is hoped to be the first in a series which extends many of the known linearization and canonical form theorems to certain critical cases which naturally occur when the system admits an integral.

A standard problem concerns itself with the question of when can an equation of the form

(1) $\dot{\xi} = A\xi + F(\xi)$

be reduced by a change of variables

(2) $\xi = \eta + G(\eta)$

to an equation of the form

(3) $\dot{\eta} = A\eta + H(\eta)$

where H is in some simple canonical form. In the above ξ and η are m-vectors, A is an $m \times m$ constant matrix and F,G,H are smooth and second order in some sense. The simplest canonical form would be $H \equiv 0$ but this is not always possible even under strong smoothness requirements on F. Thus in general one must seek simple canonical forms for H so that the local geometry of 3) is easy to analyse.

In most of the known results concerning this question it is assumed, among other things, that the eigen-values of A have nonzero real parts. We shall call this the noncritical case and refer the reader to the historical remarks following chapter 9 of Ordinary Differential Equations by P. Hartman [1] for a complete survey of the rich literature of the noncritical case. Moser [2] considered a critical case when 1) is an analytic Hamiltonian system. He showed that a convergent

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for all non-negative integers $\gamma_1, \ldots, \gamma_n, \alpha, \beta$ such that $\alpha + \beta + \gamma_1 + \ldots + \gamma_n \geq 2$ and $\alpha \neq b$ when $\gamma_\ell = 0$ for $\ell \neq j$ and $\gamma_j = 1$.

Under these assumptions we shall prove that there exists an analytic change of variables of the form

$$(6) \qquad \begin{aligned} x &= u + a(u,v,w) \\ y &= v + b(u,v,w) \\ z &= w + c(u,v,w) \end{aligned}$$

which reduces 4) to the form

$$(7) \qquad \begin{aligned} \dot{u} &= i\omega u(1 + k(uv)) \\ \dot{v} &= i\omega v(1 + k(uv)) \\ \dot{w} &= (B + M(uv))w \end{aligned}$$

where $M(\mu) = \text{diag}(m(\mu), \ldots, m_n(\mu))$; u,v,a,b,k,m_j are scalars; w and c are in-vectors; a,b,c are analytic near $u = v = w = 0$ and have power series expansions which begin with terms at least two; $k, m_1, \ldots, m_n$ are analytic functions of the single variable $\mu = uv$ for μ small and are zero for $\mu = 0$.

In 7) note that u and v are conjugate variables and that uv is an integral. The $w = 0$ plane is invariant and filled with periodic solution of period $2\pi\{\omega(1 + k(uv))\}^{-1}$. The characteristic exponents of these periodic solutions are

$$0, 0\ 2\pi\{\lambda_j + m_j(uv)\}\{\omega(1 + k(uv))\}^{-1}.$$

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2. Preliminary Reductions: Since we have assumed that 4) admits an invariant analytic surface, $z = s(x,y)$, we can make a change of variables $x' = x$, $y' = y$, $z' = z - s(x,y)$ so that 4) is reduced to the equation

$$\begin{aligned} \dot{x}' &= i\omega x' + f'(x',y',z') \\ \dot{y}' &= i\omega y' + g'(x',y',z') \\ \dot{z}' &= Bz + h'(x',y',z') \end{aligned}$$

Since $z' = 0$ defines an invariant plane, $h'(x', y', 0) \equiv 0$. By assumption the symplectic transformation exists which reduces 1) to 3) where 3) is in Birkhoff's normal form even in some cases when A has a single pair of pure imaginary eigen-values. We shall not give a precise statement of Moser's theorem here.

We shall follow this lead and consider the case when A has only one pair of pure imaginery eigen-values and all the other eigen-values have nonzero real parts. We shall replace the Hamiltonian assumption with the assumption that 1) has a local invariant surface filled by periodic solutions. Such an invariant surface might be given by the Liapunov center theorem [3,4] if 1) admits a first integral. Past these assumptions there are various combinations of smoothness conditions and further eigen-value assumptions which one might try in order to parallel the known theorems in the noncritical case. The present paper deals with a critical case which parallels the Poincaré linearization theorem which states that if F is analytic and second order and the eigen-values λ_i of A satisfy $\lambda_j \neq \Sigma k_i \lambda_i$ for all non-negative integers k_i, $\Sigma k_i \geq 2$ and $\text{Re}\ \lambda_i < 0$ then there exists an analytic transformation 2) which reduces 1) to 3) with $H \equiv 0$. [5].

Specifically we shall prove the following. Let $A = \text{diag}(i\omega, -i\omega, \lambda_1, \ldots, \lambda_n)$ where $\omega \neq 0$, $\text{Re}\ \lambda_i < 0$ and $n = m-2$. If $\xi = (x,y,z)$ where x and y are scalars and z is an n-vector then equation 1) can be written in the form

$$
4) \qquad \begin{aligned} \dot{x} &= i\omega x + f(x,y,z) \\ \dot{y} &= i\omega y + g(x,y,z) \\ \dot{z} &= Bz + h(x,y,z) \end{aligned}
$$

where $B = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, f and g are scalars and h is an n-vector. Let f, g, and h be analytic in x,y,z which have power series expansions at $x = y = z = 0$ which begin with terms of degree at least 2. We shall assume that there exists an analytic invariant surface given by $z = s(x,y)$ for 4) where s has a convergent series expansion about $x = y = 0$ which begins with terms at least of degree 2 such that all solutions of 4) on this invariant surface are periodic. Furthermore we shall assume that

$$
5) \qquad \lambda_j \neq \Sigma \gamma_\ell \lambda_\ell + (\alpha - \beta)\omega i
$$

flow on $z' = 0$ is locally a center and so by Poincare's center theorem [4,6] there is an analytic change of variables of the form $x'' = x' + \xi(x',y')$, $y'' = y' + \eta(x',y')$, $z'' = z'$ where ξ and η are scalar analytic functions of second order so that the above equation is of the form

$$\begin{aligned}
\dot{x}'' &= i\omega x'' + f''(x'',y'',z'') \\
\dot{y}'' &= i\omega y'' + g''(x'',y'',z'') \\
\dot{z}'' &= Bz'' + h''(x'',y'',z'')
\end{aligned}$$

In the above $f''(x'',y'',0) = i\omega k(x''y'')x'$ and $g''(x'',y'',0) = i\omega k(x''y'')y''$ where k is an analytic function of the single variable $\mu = x''y''$ which is zero when $\mu = x''y'' = 0$. Thus the equations in the $z'' = 0$ plane are

$$\begin{aligned}
\dot{x}'' &= i\omega(1 + k(x''y''))x'' \\
\dot{y}'' &= i\omega(1 + l(x''y''))y'' .
\end{aligned}$$

Now we can change parameters from t to τ by

$$\frac{dt}{d\tau} = \omega(1 + k(x''y''))$$

and so the equations in the $z'' = 0$ plane are linear. For the full equations we have

$$\begin{aligned}
\frac{dx''}{d\tau} &= i\omega x'' + f'''(x'',y'',z'') \\
\frac{dy''}{d\tau} &= i\omega y'' + g'''(x'',y'',z'') \\
\frac{dz''}{d\tau} &= B'z + h'''(x'',y'',z'')
\end{aligned}$$

where now $B' = w^{-1}B$ and f''', g''' and h''' are all zero when $z'' = 0$.

These equations are of the same form as 4) except we have $\omega = 1$ and the nonlinear functions are all zero when $z'' = 0$. Henceforth we shall drop the primes and return to the parameter t but assume that in 4) we have

(8)
$$\begin{aligned}
f(x,y,0) &= 0 \\
g(x,y,0) &= 0 \\
h(x,y,0) &= 0 \\
\omega &= 1 .
\end{aligned}$$

As we have just seen we can assume 8) without any loss in generality. We note that the change of parameters is not a change of coordinates in the usual sense. Thus after we have performed the change of variables described in the remainder of this paper it would be necessary to return to the original parameter to obtain the change of coordinates asserted in the first section. We note that this change of parameters will not alter the form of the final equations.

3. <u>The Functional Equations</u>: Since by the previous section we may assume that, f, g, and h = 0 when $z = 0$ we shall transform 4) to 7) with $k \equiv 0$ and $\omega = 1$ then a,b,c satisfy the equations

$$\{iua_u - iva_v - ia + a_w Bw\} = f(u+a,\ v+b,\ w+c) - a_w Mw \tag{9}$$

$$\{iub_v - ivb_v + ib + b_w Bw\} = g(u+a,\ v+b,\ w+c) - b_x Mw \tag{10}$$

$$\{iuC_u - ivC_v + C_w Bw - B_c + Mw\} = h(u+a,\ v+b,\ w+c) - c_w Mw \tag{11}$$

Conversely it is enough to find solutions a, b, and c of equations 9), 10), and 11). In order to solve the above equations we shall require that a,b,c satisfy the further requirements that

$$a(u,v,0) \equiv b(u,v,0) \equiv 0\ ,\quad c(u,v,0) \equiv 0, \tag{12}$$

and that the j^{th} component of c does not contain any terms of the form $(uv)^{\ell} w_j$ where ℓ is a positive integer.

Condition 12) states that the transformation 6) carries the $z = 0$ plane onto the $w = 0$ plane and acts as the identity on this plane.

4. <u>The Function Spaces</u>: It will be necessary to introduce some function spaces so that equations 9), 10), and 11) can be solved by the contracting principle.

Let $\delta > 0$ be a constant to be chosen later. Let $\mathfrak{a}^o$ and $\mathfrak{a}^1$ be the space of all power series of the form

$$a(u,v,w) = \Sigma a_{\alpha\beta\gamma_1 \cdots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \cdots w_n^{\gamma_n}$$

such that $a(u,v,0) \equiv 0$ and $a \in \mathcal{A}^o$ provided

$$\|a\|_o = \Sigma \quad \delta^{\alpha+\beta+\gamma_1+ \cdots + \gamma_n} < \infty$$

or $a \in \mathcal{A}^1$ provided

$$\|a\|_1 = \Sigma \mid a_{\alpha\beta\gamma_1 \cdots \gamma_n} \mid (\Sigma \gamma_i) \, \delta^{\alpha+\beta+\gamma_1+ \cdots + \gamma_n} < \infty .$$

In the above summations $\alpha + \beta + \gamma_1 + \cdots + \gamma_n \geq 2$. Note that $\mathcal{A}^o$ and $\mathcal{A}^1$ are Banach spaces, $\mathcal{A}^1 \subset \mathcal{A}^o$ and $\|a\|_o \leq \|a\|_1$.

Let C^o and C^1 be the space of all series of the form

$$c(u,v,w) = \Sigma c_{\alpha\beta\gamma_1 \cdots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \cdots w_n^{\gamma_n}$$

where $c_{\alpha\beta\gamma_1 \cdots \gamma_n}$ are n vectors, $c(u,v,0) \equiv 0$ and $c \in C^o$ provided

$$\|c\|_o = \Sigma \mid c_{\alpha\beta\gamma_1 \cdots \gamma_n} \mid \delta^{\alpha+\beta+\gamma_1+ \cdots + \gamma_n} < \infty$$

and $c \in C^1$ provided

$$\|c\|_1 = \Sigma \mid c_{\alpha\beta\gamma_1 \cdots \gamma_n} \mid (\Sigma \gamma_i) \, \delta^{\alpha+\beta+\gamma_1+ \cdots + \gamma_n} < \infty .$$

Here again $\alpha+\beta+\gamma_1+ \cdots + \gamma_n \geq 2$, the above spaces are Banach spaces, $C^1 \subseteq C^o$ and $\|c\|_o \leq \|c\|$.

Let $\tilde{C}^o$ and $\tilde{C}^1$ be the linear subspaces of C^o and C^1 respectively such that if $c \in \tilde{C}^o$ or $c \in \tilde{C}^1$ then the j^{th} component of c does not contain a term of the form $(uv)^\ell w_j$.

Let $\mathcal{M}^o$ be the space of all $M = \mathrm{diag}(M_1, \ldots, M_n)$ where

$$M_j(uv) = \Sigma_{\alpha \geq 1} m_{j\alpha} (uv)^\alpha$$

and

$$\|M\|_o = \Sigma_j \Sigma_{\alpha \geq 1} |m_{j\alpha}| \, \delta^{2\alpha} < \infty .$$

$\mathcal{M}^o$ is a Banach space also.

5. <u>The Linearized Functional Equations</u>: The following lemma follows at once from our assumptions on the eigen-values, $\pm i, \lambda_1, \ldots, \lambda_n$.

Lemma 1. <u>There exists a constant</u> $s > 0$ <u>such that</u>

(13) $$|i\alpha - i\beta \pm i + \Sigma\, \gamma_i\lambda_i| \geq s(\Sigma\, \gamma_i)$$

and

(14) $$|i\alpha - i\beta + \Sigma\, \gamma_i\lambda_i - \lambda_i - \lambda_j| \geq s(\Sigma\, \gamma_i)$$

<u>where in the first inequality</u> $\alpha,\beta,\gamma_1, \ldots, \gamma_n$ <u>are non-negative integers such that</u> $\alpha+\beta+\gamma_1+ \ldots + \gamma_n \geq 2$ <u>and in the second inequality</u> $\alpha,\beta,\gamma_1, \ldots, \gamma_n$ <u>are non-negative integers such that</u> $\alpha+\beta+\gamma_1+ \ldots +\gamma_n \geq 2$ <u>and are never such that</u> $\alpha = \beta = 0$, $\gamma_j = 1$ <u>and</u> $\gamma_i = 0$ for $i \neq j$.

In order to solve the functional equations 9), 10), and 11) we shall first solve the linearized equations. Namely.

Lemma 2. <u>Given</u> $\bar{a}, \bar{b}, \in \mathcal{A}^o$ <u>and</u> $\bar{c} \in \tilde{C}^o$ <u>there exists unique</u> $a,b \in \mathcal{A}^1$, $c \in C^1$ <u>and</u> $M \in \mathfrak{M}^o$ <u>such that</u>

(15) $$\{iua_u - iva_v - ia + a_w Aw\} = \bar{a}$$

(16) $$\{iub_u - ivb_v + ib + b_w Aw\} = \bar{b}$$

(17) $$\{iuC_u - ivC_v + c_w Aw - A_c + Mw\} = \bar{c}$$

Moreover

(18) $$s\|a\|_1 \leq \|\bar{a}\|_o$$

(19) $$s\|b\|_1 \leq \|b\|_o$$

(20) $$s\|c\|_1 \leq \|\bar{c}\|_o$$

(21) $$\|M\|_o \leq \delta\, \|\bar{c}\|_o$$

Proof: Let $\bar{a} = \Sigma \bar{a}_{\alpha\beta\gamma_1 \ldots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \ldots w_n^{\gamma_n}$ and $a = \Sigma a_{\alpha\beta\gamma_1 \ldots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \ldots w_n^{\gamma_n}$. Then if a solves 15) we must have

$$\{i\alpha - i\beta - i + \Sigma\, \gamma_i\lambda_i\}\, a_{\alpha\beta\gamma_1 \ldots \gamma_n} = \bar{a}_{\alpha\beta\gamma_1 \ldots \gamma_n}.$$

In view of our assumptions on the eigen-values and lemma 1 we have that $a_{\alpha\beta\gamma_1 \ldots \gamma_n}$ exists and $s(\Sigma\, \gamma_i)\, |a_{\alpha\beta\gamma_1 \ldots \gamma_n}| < |\bar{a}_{\alpha\beta\gamma_1 \ldots \gamma_n}|$. Thus a exists and $a \in \mathcal{A}^1$. Moreover the above inequality implies 18). Equation 16) and estimate 19) are treated in the same way.

Consider 17) now. Let the j^{th} component of $\bar{c}$ be denoted by $\bar{c}^j$ etc., and assume

$$\bar{c}^j = \Sigma\, \bar{c}^j_{\alpha\beta\gamma_1 \cdots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \cdots w_n^{\gamma_n}$$

$$c^j = \Sigma\, c^j_{\alpha\beta\gamma_1 \cdots \gamma_n} u^\alpha v^\beta w_1^{\gamma_1} \cdots w_n^{\gamma_n}$$

$$M^j = \Sigma\, m^j_k\, (uv)^k$$

Then to solve 17) we must have

$$\{i\alpha - i\beta + \Sigma\, \gamma_i\lambda_i - \lambda_j\}\, c^j_{\alpha\beta\gamma_1 \cdots \gamma_n} + \eta\, m^j_k = \bar{c}^j_{\alpha\beta\gamma_1 \cdots \gamma_n}$$

where $\eta = 0$ unless $\alpha = \beta$, $\gamma_i = 0$ for $i \neq j$ and $\gamma_j = 1$ and $\eta = 1$ otherwise. If $\alpha = \beta$, $\gamma_i = 0$, $i \neq j$ and $\gamma_j = 1$ then take $c_{\alpha\beta\gamma_1 \cdots \gamma_n} = 0$ and $m^j_k = \bar{c}_{\alpha\beta\gamma_1 \cdots \gamma_n}$. Thus $\|M\|_o \leq \delta\|\bar{c}\|_o$. Otherwise take $m^j_k = 0$ and let

$$c^j_{\alpha\beta\gamma_1 \cdots \gamma_n} = \{i\alpha - i\beta + \Sigma\, \gamma_i\lambda_j - \lambda_j\}^{-1}\, \bar{c}^j_{\alpha\beta\gamma_1 \cdots \gamma_n}\,.$$

In view of lemma 1 we have

$$(\Sigma\, \gamma_i)s\, \left| c^j_{\alpha\beta\gamma_1 \cdots \gamma_n} \right| \leq \left| \bar{c}^j_{\alpha\beta\gamma_1 \cdots \gamma_n} \right|.$$

and so 17) is solvable and 20) and 21) follow.

6. The Solution of the Functional Equations:

We shall now indicate how to solve equations 9), 10), and 11) by the contracting mapping principle. Let $\mathcal{D} = G^o \times G^o \times \tilde{C}^o$ where

$$\|(a,b,c)\|_o = \|a\|_o + \|b\|_o + \|c\|_o \quad \text{and let} \quad \mathcal{E} = G^1 \times G^1 \times C^1 \times \mathcal{M}^o$$

where $\|(a,b,c,M)\|_1 = \|a\|_1 + \|b\|_1 + \|c\|_1 + \|M\|_o$. We note that equations 15), 16), and 17) define a linear operator $L: \mathcal{E} \to \mathcal{D}: (a,b,c,M) \to (\bar{a},\bar{b},\bar{c})$ and that lemma 2) asserts that L has a bounded inverse $L^{-1}: \mathcal{D} \to \mathcal{E}$.

Let F denote the nonlinear map $F: \mathcal{E} \to \mathcal{D}: (a,b,c,M) \to (\bar{a},\bar{b},\bar{c})$ where

$$\bar{a}(u,v,w) = f(u+a,\ v+b,\ w+c) - a_w Mw$$

$$\bar{b}(u,v,w) = g(u+a,\ v+b,\ w+c) - b_w Mw$$

$$\bar{c}(u,v,w) = h(u+a,\ v+b,\ w+c) - c_w Mw.$$

(see the right hand sides of 9), 10), and 11). We note that equations 9), 10), and 11) are equivalent to

$$L(a,b,c,M) = F(a,b,c,M)$$

or

$$(a,b,c,M) = L^{-1}F(a,b,c,M).$$

We define a subset $\mathcal{K} \subset \mathcal{S}$ such that if $\rho \in \mathcal{K}$ then $\|\rho\|_1 \leq K$ where K is some constant. We define a map $\mathfrak{F}: \mathcal{K} \to \mathcal{S} : \rho \to L^{-1}F(\rho)$. It is easy to see that if K and δ are small enough

$$\|F(\rho)\|_0 \leq \varepsilon \|\rho\|_1 \qquad \text{for } \rho \in \mathcal{K}$$

and so $\mathfrak{F}: \mathcal{K} \to \mathcal{K}$ provided K and δ are small enough. Also it is easy to see that if K and δ are small enough $\|F(\rho) - F(\rho')\|_0 \leq \varepsilon \| \rho - \rho' \|_1$ for $\rho, \rho' \in \mathcal{K}$ and so $\mathfrak{F}$ is contracting if K and δ are small enough.

Thus the contracting mapping principle yield a fixed point of $\mathfrak{F}$ and thus a solution of 9), 10), and 11).

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