Stability of a Lurie Type Equation

K. R. Meyer

In their study of nonlinear electrical circuits Brayton and Moser [1] investigated the asymptotic behavior of a system of nonlinear differential equations that describe the state of an electrical network. The aim was to give conditions that insure nonoscillating solutions. The criterion obtained in [1] was very restrictive and Moser in [2] obtained more general criteria by using the method of Popov of automatic control theory. The method of Popov has been very successful in the study of the stability properties of the Lurie equations (see [3] for a detailed discussion).

At first glance the equations of Brayton and Moser bear no resemblance to the usual Lurie equations but this note will show that by a change of variables the equations take a form similar to the Lurie equations. Once the equations are written in this new form it is then clear how to use the methods developed in control theory to study their stability properties. In particular it is clear that Popov's method would yield a stability criterion. It is also clear how to construct a Liapunov function for these equations. We choose the latter to reprove Moser's theorem in a straightforward way.

The system considered in [1, 2] is of the form

$$\begin{aligned} \dot{x} &= -Ax + By \\ \dot{y} &= Cx - f(y) \end{aligned} \tag{1}$$

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where x is an n-vector, y an m vector, A, B and C are constant matrices of appropriate dimensions, A nonsingular, and f is an m vector valued function of the m vector y.

One wishes to find conditions on A, B, C and f so that all solutions of (1) approach a finite number of equilibrium states and hence rule out oscillatory behavior. The fundamental assumption on f is that it can be written in the form $f(y) = \nabla G(y) - cA^{-1}By$ where G is a scalar function and ∇ stands for gradient. It is also assumed that G tends to infinity as y tends to ∞ and G has a finite number of critical points $(y_1, \ldots, y_k)$.

Moser then obtains conditions on the coefficients A, B and C such that all solutions of (1) tend to $x = 0$, $y = y_j$, $j = 1, \ldots, k$.

If one makes the change of variables $u=x$, $v=-y - cA^{-1}x$ and lets $K = -(A+BCA^{-1})$, $D=-CA^{-1}$ then the Equations (1) become

$$(2) \qquad \begin{aligned} \dot{u} &= Ku - Bv \\ \dot{v} &= \nabla G(y) \\ y &= Du - v \end{aligned}$$

If y is a scalar and $G(y) = \int_0^y \varphi(\tau)\, d\tau$ then Equations (2) reduce to the usual indirect control equations of Lurie.

Thus it is natural to make the Lefschetz change of variables $z = Ku - Bv$, $\sigma = Du - v$ which is nonsingular provided

$$(3) \qquad \begin{vmatrix} K & -B \\ D & -I \end{vmatrix} = |K|\,|-I - DK^{-1}B| \neq 0 \quad .$$

This condition is clearly necessary for isolated equilbrium points. Under this change of variables the Equations (2) become

$$(4) \qquad \begin{aligned} \dot{z} &= Kz - B\nabla G(\sigma) \\ \dot{\sigma} &= Dz - \nabla G(\sigma) \end{aligned}$$

For Equations (4) the natural Liapunov function is of the form

(5) $$V = z'Pz + G(\sigma)$$

where P is a positive definite symmetric matrix

We shall show that one can give conditions on the coefficients of (4) such that one can find a P that makes V a Liapunov function for (4). The existence of such a P is the result of a lemma by Anderson [4]. Anderson's lemma is a natural generalization of the Kalman-Yarnkovich lemma discussed in [3]. Henceforth we shall assume that $(K, B, D,)$ is a completely controllable, completely observable triple. This assumption is necessary for the application of the lemma of Anderson but one could dispense with this assumption by using the methods developed in [5].

The condition on the coefficients of (4) are stated in terms of the transfer function

(6) $$T(\lambda) = I + D\{\lambda I - K\}^{-1} B.$$

An $m \times m$ matrix function Z of a complex variable λ is called <u>positive real</u> if

i) the elements of Z are rational functions with no poles for $\mathrm{Re}\lambda > 0$

ii) $\overline{Z(\lambda)} = Z(\overline{\lambda})$

iii) $Z(\overline{\lambda})^T + Z(\lambda)$ is nonnegative definite for $\mathrm{Re}\lambda > 0$. Z is called <u>strictly positive real</u> if i) holds for $\mathrm{Re}\lambda \geq 0$, ii) holds and $Z(\overline{\lambda})^T + Z(\lambda)$ is positive definite for $\mathrm{Re}\lambda \geq 0$.

The main theorem is then

<u>Theorem 1</u>. <u>If</u> $T(\lambda) = I + D\{\lambda I - K\}^{-1} B$ <u>is positive real then all solutions of</u> (4) <u>are bounded and if it is strictly positive real all solutions of</u> (4) <u>approach one of the equilibrium points</u> $(0, \sigma_i)$ <u>where</u> σ_i <u>is such that</u> $\nabla G(\sigma_i) = 0$.

We can state Theorem 1 for the original system of Equations (1) by tracing back the coordinate changes. In terms of the original matrices

$$
\begin{aligned}
T(\lambda) &= I + (-CA^{-1})(\lambda I + A + BCA^{-1})^{-1}B \\
&= I - C\{\lambda A + A^2 + BC\}^{-1}B \\
&= I - C\{\lambda A + A^2\}^{-1}B\,\{I + C(\lambda A + A^2)^{-1}B\}^{-1} \qquad (7)\\
&= \{I + C(\lambda A + A^2)^{-1}B\}^{-1}
\end{aligned}
$$

thus

Corollary 1. If $T(\lambda)^{-1} = \{I + C(\lambda A + A^2)^{-1}B\}$ is strictly positive real and (3) holds them all solutions of (1) approach one of the equilibrium points $(0, y_i)$ where y_i is a critical points of G.

Remark. Moser does not assume that (3) hold explicitly but one can easily show that (3) is equivalent to the condition that the residue at ∞ of $T(\lambda) + T(\bar{\lambda})^*$ is nonsingular. This is an easy consequence of Moser's condition.

Proof of Theorem 1.

We prove Theorem 1 by using the lemma given below to show that there exists a Liapunov function of the form (5) for (4).

Anderson's Lemma: If $T(\lambda) = I + D(\lambda I + K)^{-1}B$ is positive real then there exists a positive definite $n \times n$ matrix P and an $m \times n$ matrix L such that

$$
\begin{aligned}
PK + K'P &= -LL' \\
PB &= L - \tfrac{1}{2}D
\end{aligned} \qquad (8)
$$

In the proof of this lemma one has also the following matrix identity

$$
(m^*(i\omega)L - I)(L'm(i\omega) - I) = I + \frac{1}{2}\{C'm(i\omega) + m^*(i\omega)C\} \qquad (9)
$$

where $m(i\omega) = (i\omega I - K)^{-1}B$ and $*$ denotes conjugate transpose. One sees at once that the right hand side of (8) is $\frac{1}{2}$ $\{T(i\omega) + T'(-i\omega)\}$ and so if T is strictly positive real then

(10) $$I - L'(i\omega I - K)^{-1}B = (I - L'm(i\omega))$$

is nonsingular for all real ω. This fact is useful in the analysis of set where $\dot{V}$ is identically zero.

Let the P in (5) be as given by Anderson's lemma then the derivative $\dot{V}$ of V along the trajectories of (4) is given by

$$-\dot{V} = -z^T\{K'P + PK\}z + 2z'\{PB - \tfrac{1}{2}D'\}\nabla G + \nabla G'\nabla G = \|\nabla G + L'z\|^2$$

Since $V \to \infty$ as z and $\sigma \to \infty$ and $\dot{V} \leq 0$ it follows by the well-known Liapunov theorems that all solutions of (4) are bounded.

In order to conclude that all solutions of (4) tend to $(0, \sigma_0)$ we must use the theorem of LaSalle [6] that states that all solutions tend to the largest invariant set of (4) that is contained in the set where $\dot{V} \equiv 0$. Thus Theorem 1 is established once one shows that the largest invariant set contained in the set where $\dot{V} \equiv 0$ is the set $\{(0, \sigma_1), \ldots, (0, \sigma_k)\}$.

Let $z(t)$, $\sigma(t)$ be a solution of (4) that is such that $z(t) \not\equiv 0$ and $\dot{V}(z(t), \sigma(t)) \equiv 0$. Then $G(\sigma(t)) = -Lz(t)$ and so $z(t)$ satisfies

$$\dot{z} = \{K + BL'\}z$$

But $\dot{z}$ is bounded for all t and so the matrix $K + BL'$ must have an eigenvalue on the imaginary axis. The characteristic equation for $K + BL'$ is

$$|\lambda I - K - BL'| = |\lambda I - K|\,|I - (\lambda I - K)^{-1}BL'| = |\lambda I - K|\,|I - L'(\lambda I - K)^{-1}B|$$

But we have seen that if $T(\lambda)$ is strictly positive real the matrix $I - L'(\lambda I - K)^{-1}B$ is nonsingular for $\lambda = i\omega$, ω real. Hence $z(t) \equiv 0$, $\sigma(t) \equiv 0$.

References

1. R. K. Brayton and J. K. Moser, "A theory of nonlinear networks I," Quart, of Appl. Math., 22 (1964) pp. 1-33.

2. J. K. Moser, "On nonoscillating networks," Quart. of Appl. Math., 25 (1967) pp. 1-9.

3. S. Lefschetz, Stability of nonlinear control systems, Academic Press, New York, 1965.

4. B. D. O. Anderson, "A system theory criterion for positive real matrices," SIAM Journal on Control, 5 (1967) pp. 171-182.

5. K. R. Meyer, "On the existence of Lyapunov functions for the problem of Lurie," J. SIAM Control, Ser. A, 3 (1966) pp. 373-383.

6. J. P. LaSalle, "The extent of asymptotic stability," Proc. Math. Acad. Sci., 46 (1954) pp. 363-365.