On the Existence of Solutions to Linear Differential-Difference Equations

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This paper discusses the existence of solutions to linear differential-difference equations in certain degenerate cases. Questions of this type occur in connection with singular perturbations of differential-difference equations (see [2]). In singular perturbation theory, one is interested in relating the behavior of a perturbed equation with the behavior of an unperturbed equation that is degenerate in some sense. It is the existence of solutions to the degenerate equations of singular perturbations that is investigated here.

Consider the equation

$$\sum_{k=0}^{h} \{A^{2k}x^{(1)}(t-\omega_k) + A^{2k+1}x(t-\omega_k)\} = f(t), \tag{*}$$

where A^j, $j = 0, 1, ..., 2h+1$ are $m \times m$ constant matrices; ω_j, $j = 0, 1, ..., h$ are real scalars; $0 = \omega_0 < \omega_1 < \cdots < \omega_h$, and $f(t)$ is a real m-vector function of the scalar t. The m-vector function $x(t)$ is to be determined so as to satisfy (*) and certain initial conditions. $x^{(1)}(t) = dx(t)/dt$.

The problem of the existence of solutions to (*), when det $A^0 \neq 0$ is adequately discussed in [1]. The case of det $A^0 = 0$ will be the primary one in this treatment.

The first section gives a simple procedure for reducing Eq. (*) to one of the same form with $\det(A^0 + \lambda A^1) \not\equiv 0$, i.e., $A^0 + \lambda A^1$ is a regular pencil (see [3]). Thus there is no loss of generality in assuming that det $(A^0 + \lambda A^1) \not\equiv 0$.

The second section shows that solutions to (*) always exist, provided the initial conditions and forcing functions are suitably restricted. The third section shows that solutions to (*) exist under much weaker assumptions on the initial conditions and forcing functions, provided the spectrum of the associated linear operator is suitably well behaved. This last result

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establishes a close connection between the phenomenon of "loss of derivatives" and the existence of advanced chains of zeros to the characteristic equation.

I

In order to discuss the existence of solutions to (*), it is convenient to assume that the pencil $A^0 + \lambda A^1$ is regular. This section will show that there is no loss of generality in making this assumption, in the sense that the question of the existence of a solution to Eq. (*) can always be reduced to the question of the existence of solutions of an equation of the same form but with $A^0 + \lambda A^1$, a regular pencil.

The reduction will consist of a finite sequence of row and column operations plus a finite sequence of one additional operation.

Consider the characteristic matrix

$$H(s) = \sum_{k=0}^{h} \{s \exp(-\omega_k s)\, A^{2k} + \exp(-\omega_k s)\, A^{2k+1}\}. \tag{I.1}$$

The entries in (I. 1) are exponential polynomials of the form

$$p_{ij}(s) = sa_{ij}^1 + a_{ij}^2 + \cdots + \exp(-\omega_h s)\, a_{ij}^{2h+1} .$$

We shall use $p_{ij}(s)$ as a generic symbol representing the i, jth element of $H(s)$ at each stage of the reduction. These exponential polynomials can be ordered in the following way

$$p_{ij}(s) \geqq p_{kl}(s), \qquad (p_{ij}(s) > p_{kl}(s)).$$

provided $a_{ij}^1 = \cdots = a_{ij}^s = 0$; $a_{ij}^{s+1} \neq 0$; then $a_{kl}^1 = \cdots = a_{kl}^s = 0$ (and $a_{kl}^{s+1} = 0$).

A finite sequence of row and column operations on the matrix (I.1) is equivalent to changing A^j to PA^jQ for $j = 0, 1, \ldots, 2h + 1$, where P and Q are nonsingular $m \times m$ matrices. These operations correspond to multiplying Eq. (*) by P from the left and making the substitution $x(t) = Qy(t)$.

By standard row and column operations, the matrix (I.1) can be reduced to a matrix of the same form with the following properties:

$$\begin{aligned} &p_{ii} \geqq p_{jj} \text{ for } i > j, \\ &p_{ii} > p_{ji}\,; \qquad p_{ii} > p_{ij} \text{ for all } j \neq i \text{ if } p_{ii} \neq 0, \\ &p_{ij} = p_{ji} = 0 \text{ for all } j \text{ if } p_{ii} = 0. \end{aligned} \tag{I.2}$$

To make this reduction one simply takes one of the polynomials that is maximal with respect to this ordering to the 1,1th position. Then to every row (column) except the first, one adds a scalar multiple of the first row (column) so that $p_{11} > p_{j1}(p_{11} > p_{1j})$ for all $j \neq 1$. By repeating this process in the standard way, one arrives at a matrix of the same form as (I.1) such that (I.2) holds.

Now if $p_{ii}(s) = sa_{ii}^1 + \cdots + \exp(-\omega_h s)\, a_{ii}^{2h+1}$ and $a_{ii}^1 = a_{ii}^2 = \cdots = a_{ii}^{2k-1} = 0$ and either $a_{ii}^{2k} \neq 0$ or $a_{ii}^{2k+1} \neq 0$, then multiply the ith row by $(a_{ii}^{2k})^{-1} \exp(\omega_k s)$ or $(a_{ii}^{2k+1})^{-1} \exp(\omega_k s)$. This is equivalent to making the substitution $x_i(t) = y_i(t + \omega_k)$ in (*).

Now by rearranging rows and columns, (I.2) will hold and furthermore, if $p_{ii} \neq 0$, then either $a_{ii}^1 = 1$ or $a_{ii}^1 = 0$ and $a_{ii}^2 = 1$.

The last operation may introduce new exponents in the exponential polynomials, i.e., there may now exist terms in the exponential polynomial of the form $s \exp(\omega_i - \omega_j)s$. If we call these new exponents η_k, $k = 0, \ldots, l$, where $0 = \eta_0 < \eta_1 < \cdots < \eta_l$, then the matrix (I.2) is reduced to one of the form

$$\sum_{k=0}^{l} \{s \exp(-\eta_k s)\, B^{2k} + \exp(-\eta_k s)\, B^{2k+1}\},$$

where $B^0 = \operatorname{diag}\{I_p, 0_q, 0_s\}$, $B^1 = \operatorname{diag}\{C^1, I_q, 0_s\}$ and $B^k = \operatorname{diag}\{D^k, 0_s\}$ for $k = 2, \ldots, 2h + 1$, where I_p and I_q are the $p \times p$ and $q \times q$ idenity matrices, 0_q and 0_s are the $q \times q$ and $s \times s$ zero matrices, and C^1 and D^k, $k = 2, \ldots, 2l + 1$ are $p \times p$ and $(p + q) \times (p + q)$ matrices. Thus the equation is reduced to the form

$$\sum_{k=0}^{l} \{B^{2k} x^{(1)}(t - \eta_k) + B^{2k+1} x(t - \eta_k)\} = \bar{f}(t),$$

where the B^j are as above. Clearly for the existence of a solution for $t \geqq 0$, one must have $\bar{f}_{p+q+1}(t) = \bar{f}_{p+q+2}(t) = \cdots = \bar{f}_m(t) = 0$ for all $t \geqq 0$. If this requirement is met, we are reduced to considering an equation of the same form as the above with $s = 0$ and $B^0 = \operatorname{diag}\{I_p, 0_q\}$ and $B^1 = \operatorname{diag}\{C^1, I_q\}$ and thus $B^0 + \lambda B^1$ is regular.

II

In this section it will be shown that a solution to (*) always exists provided the initial conditions and forcing function are sufficiently smooth.

Let $\mu(t)$ be the function that is zero for $t \leqq 0$ and one for $t > 0$ and let $A^0 + \lambda A^1$ be a regular pencil.

Lemma 1. There exists a unique $m \times m$ matrix function $U(t)$ defined for all t such that $A^0U(t)$ is differentiable and $d/dt\{A^0U(t)\} + A^1U(t) = \mu(t)I_m$.

If $f(\cdot) \in C^k[0, \infty)$ (or $f(\cdot) \in C^\infty[0, \infty)$) then

$$x(t) = x_0 - U(t)A^1x_0 - \int_0^t \{d_sU(t-s)\}f(s)$$

satisfies the equation $d/dt\{A^0x(t)\} + A^1x(t) = f(t)$ for all $t \geqq 0, x(0) = x_0$. Moreover $x(\cdot) \in C^k[0, \infty)$ (or $x(\cdot) \in C^\infty[0, \infty)$) if $Bx_0 - f(0)$ is in the range of A^0.

Proof. There exist nonsingular matrices P and Q such that $PA^0Q = \operatorname{diag}\{I_p, 0_q\}$ and $PA^1Q = \operatorname{diag}\{C, I_q\}$, where C is some $p \times p$ matrix and I_p, I_q and 0_q are as before. If A^0 and A^1 are as described above, then $U(t) = \operatorname{diag}\{\mu(t) \int_0^t \exp(-C\beta)\, d\beta, \mu(t)I_q\}$ and the rest of the lemma follows by direct verification.

Henceforth if $g(\cdot) \in C^k[a, b]$, $k \geqq 1$, $g^{(r)}(t_0)$ will denote the rth derivative of $g(\cdot)$ at t_0 if $r \leqq k$ and $t_0 \in (a, b)$, and the rth right (or left) derivative if $t_0 = a$ (or $t_0 = b$).

Theorem. Let $x_0(\cdot) \in C^\infty[-\omega_k, 0]$ and $f(\cdot) \in C^\infty[0, \infty)$. If $\sum_{k=0}^h \{A^{2k}x_0^{(j+1)}(-\omega_k) + A^{2k+1}x_0^{(j)}(-\omega_k)\} = f^{(j)}(0)$ for all $j = 0, 1, 2, \ldots$, then there exists a unique m vector function $x(\cdot) \in C^\infty[-\omega_k, \infty)$, such that $x(t) = x_0(t)$ for $t \in [-\omega_k, 0]$ and $\sum_{k=0}^h \{A^{2k}x^{(1)}(t-\omega_k) + A^{2k+1}x(t-\omega_k)\} = f(t)$ for all $t \geqq 0$.

Proof. The function $x(t)$ will be defined by induction. We shall show that there exists a sequence of functions $\{x_n(t)\}$ such that:

IH: (1) $x_n(\cdot)\ \varepsilon\ C^\infty[-\omega_h, n\,\omega_1]$.

(2) $\sum_{k=0}^h \{A^{2k}x_n^{(1)}(t-\omega_k) + A^{2k+1}x_n(t-\omega_k)\} = f(t) \quad \text{for} \quad t \in [0, n\omega_1]$.

(3) $x_n(t) = x_0(t) \quad \text{for} \quad t \in [-\omega_k, 0]$.

$x_0(\cdot)$ is given by hypothesis. Let $x_n(\cdot)$ satisfy IH and define $N_n(t) = -\sum_{k=1}^h \{A^{2k}x_n^{(1)}(t-\omega_k) + A^{2k+1}x_n(t-\omega_k)\} + f(t)$.

Clearly $N_n(\cdot) \in C^\infty[0, (n+1)\omega_1]$. Define $x_{n+1}(\cdot)$ by

$$x_{n+1}(t) = x_n(t) \quad \text{for} \quad t \in [-\omega_n, n\omega_1]$$

and

$$x_{n+1}(t) = x_n(n\omega_1) - U(t - n\omega_1)A^1x_n(n\omega_1) - \int_{n\omega_1}^t \{d_sU(t-s-n\omega_1)\}\, N_n(s) \quad \text{for} \quad t \in [n\omega_1, (n+1)\omega_1].$$

Clearly $x_{n+1}(\cdot)$ is well defined and satisfies (3). Moreover $x_{n+1}(\cdot)$ satisfies (2), except possibly at $t = n\omega_1$, and hence, it only remains to show (1). By (2) there exists a vector u_n such that $A^0 u_n + A^1 x_n^{(s)}(n\omega_1) = N_n^{(s)}(n\omega_1)$ —namely $u_n = x_n^{(s+1)}(n\omega_1)$. Thus by the preceding lemma, $x_{n+1}^{(s)}(\cdot)$ is continuous for $t \in [-\omega_n, (n+1)\omega_1]$ and all s. Thus (2) is established. The function $x(t)$ is defined by $x(t) = x_j(t)$, where $j = 0$, if $t \in [-\omega_n, 0]$ and $j = n$ if $t \in ((n-1)\omega_1, n\omega_1]$.

III

In this section, it will be shown that solutions to (*) exist under much weaker conditions of initial data and forcing function if the spectrum of the associated linear operator is restricted. That is, if the roots of the characteristic equation lie to the left of some line in the complex plane, then solutions to (*) exist for all t under much weaker smoothness conditions.

The proof of the following lemma is almost identical to the proof of Theorem 12.19 of [*1*].

Lemma 2. Let $A^0 + \lambda A^1$ be a regular pencil and let there exist a real number α, such that if λ is a root of $h(s) = \det H(s) = 0$, then $Re\, \lambda < \alpha$. Let q be the nullity of A^0 and $S = \{t \mid t = \sum_{k=0}^{h} n_k \omega_k,\ n_k \text{ an integer}\}$. Then

$$W(t) = \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\exp(ts)\, H^{-1}(s)}{s^{q+1}}\, ds$$

converges for all t and uniformly for t in a compact set, and $\dot{W}(t)$ converges for all $t \in (-\infty, \infty) \cap S$ and uniformly to a continuous function for t in a compact subset of $(-\infty, \infty) \cap S$.

A complete discussion of and a simple criterion for the condition that all roots of $h(s)$ lie to the left of some line in the complex plane can be found in [*1*].

Moreover for all $t \in [0, \infty) \cap S$

$$\sum_{k=0}^{h} \{A^{2k} \dot{W}(t - \omega_k) + A^{2k+1} W(t - \omega_k)\} = \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{H(s)\, H(s)^{-1}}{s^{q+1}} = \mu(t)\, \frac{t}{q!}\, I_m$$

and

$$\sum_{k=0}^{h} \{\dot{W}(t - \omega_k) A^{2k} + W(t - \omega_k) A^{2k+1}\} = \mu(t)\, \frac{t^q}{q!}\, I_m .$$

Theorem. Let $A^0 + \lambda A^1$ be a regular pencil, q the nullity of A^0 and let there exist a real number α such that if λ is a root of $h(s) = \det H(s) = 0$, then $Re\,\lambda < \alpha$. If $x_0(\cdot) \in C^{q+1}[-\omega_h, 0], f(\cdot) \in C^q[0, \infty)$ and

$$\sum_{k=0}^{h} \{A^{2k} x_0^{(j+1)}(-\omega_k) + A^{2k+1} x_0^{(j)}(-\omega_k)\} = f^{(j)}(0) \qquad \text{(III.1)}$$

for $j = 0, 1, 2, \ldots, q$, then there exists a unique m-vector function $x(\cdot) \in C^1[-\omega, \infty)$, such that $x(t) = x_0(t)$ for $t \in [-\omega_h, 0]$ and

$$\sum_{k=0}^{h} \{A^{2k} x^{(1)}(t - \omega_k) + A^{2k+1} x(t - \omega_k)\} = f(t)$$

for all $t \in [0, \infty)$. Moreover $x(t)$ is given by

$$\begin{aligned} x(t) = \mu(t) \sum_{j=0}^{q} \frac{t^j}{j!} x_0^{(j)}(0) &+ \sum_{k=0}^{h} W(t - \omega_k) A^{2k} x_0^{(q+1)}(0) \\ &+ \int_0^t W(t - a)\, df^{(q)}(a) - \sum_{k=0}^{q} \int_{-\omega_k}^{0} W(t - a - \omega_k)\, d\{A^{2k} x_0^{(q+1)}(a) \\ &+ A^{2k+1} x_0\} + \{1 - \mu(t)\}\, x_0(t). \end{aligned} \qquad \text{(III.2)}$$

Proof. First we shall derive formula (III.1) and thus show uniqueness. The following derivation is similar to the method described in [*1*; Chapter 10]. Let x be a function satisfying the conditions of the above theorem and let W be the function given by Lemma 2. By multiplying Eq. (*) by W and integrating, one obtains for $t \geq 0$:

$$\sum_{k=0}^{h} \int_0^t \{W(t - s) A^{2k}\, dx^{(1)}(s - \omega_k) + W(t - s) A^{2k+1}\, dx(s - \omega_k)\} = \int_0^t W(t - s)\, df(s),$$

and thus

$$\begin{aligned} G(t) &= \int_0^t \sum_{k=0}^{h} \{W^{(1)}(t - \omega_k - s) A^{2k} + W(t - \omega_k - s) A^{2k+1}\}\, dx(s) \\ &= \int_0^t W(t - s)\, df(s) + \sum_{k=0}^{h} W(t - \omega_k) A^{2k} x^{(1)}(0) \\ &\quad - \sum_{k=0}^{h} \left\{ \int_{-\omega_k}^{0} W^{(1)}(t - \omega_k - s) A^{2k}\, dx^{(1)}(s) \right. \\ &\quad \left. + \int_{-\omega_k}^{0} W(t - \omega_k - s) A^{2k+1}\, dx(s) \right\}. \end{aligned}$$

By Lemma 2 and the definition of W it follows that

$$G(t) = \int_0^t \frac{(t - s)^n}{n!}\, dx(s) = \frac{t^n x(0)}{n!} + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} x(s_n)\, ds_n \cdots ds_1$$

and so

$$x(t) = x(0) + \frac{d^n}{dt^n}\left[\left\{\int_0^t W(t-s)\,df(s) + \sum_{k=0}^{h} W(t-\omega_k)A^{2k}x^{(1)}(0)\right\}\right.$$
$$- \sum_{k=0}^{h}\left\{\int_{-\omega_k}^{0} W^{(1)}(t-\omega_k-s)A^{2k}\,dx^{(1)}(s)\right.$$
$$\left.\left. + \int_{-\omega_k}^{0} W(t-\omega_k-s)A^{2k+1}\,dx(s)\right\}\right].$$

By successive differentiation, integration by parts, and use of the additional boundary conditions, one obtains the formula (III.1) for $x(t)$. Thus uniqueness has been shown. Existence can be shown by direct substitution of (III.1) into (*).

The following example illustrates that, in general, if a C^1 solution is to exist and the nullity of A^0 is q, then the initial data must be of class C^{q+1} and the forcing function of class C^q.

$$x_1^{(1)}(t) = f_1(t).$$
$$x_2(t) + x_1^{(1)}(t-1) = f_2(t).$$

Here the nullity of A^0 is one and $h(s) = s$. Clearly if $x_2(t)$ is to be C^1 for $t \in [0, 1]$, the $x_1(t)$ must be C^2 for $t \in [-1, 0]$. Higher order examples can be constructed in the same way.

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