Stability of a Hamiltonian System in a Limiting case

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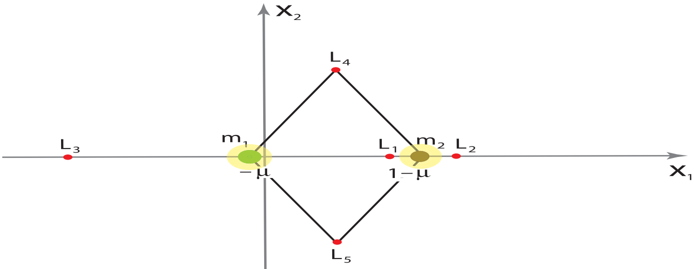

The Hamiltonian of the restricted three-body problem:

$$\mathcal{H} = \frac{1}{2}(y_1^2+y_2^2)-(x_1y_2-x_2y_1)-\frac{\mu}{\sqrt{(x_1-1+\mu)^2+x_2^2}}-\frac{1-\mu}{\sqrt{(x_1+\mu)^2+x_2^2}}\,.$$

Equations: $\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \qquad x \in \mathbb{R}^2,\ y \in \mathbb{R}^2$

Hamiltonian equation in $\mathbb{R}^4$ with one parameter μ the mass ratio parameter, $0 < \mu \leq \frac{1}{2}$.

There are five equilibrium (libration) points
$\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ the unstable collinear points of Eulerian,
$\mathcal{L}_4, \mathcal{L}_5$ the equilateral triangle points of Lagrange.

$\mu_R = \frac{1}{2}(1 - \sqrt{69}/9) \sim 0.0385$ the Routh value.
$\mathcal{L}_4, \mathcal{L}_5$ stable for $0 < \mu < \mu_R$ with 2 known exceptions.
$\mathcal{L}_4, \mathcal{L}_5$ unstable for $\mu_R < \mu \leq \frac{1}{2}$.

For $\mu = \mu_R$ there are many false proofs of the stability of $\mathcal{L}_4, \mathcal{L}_5$ dating back to the 1960s and one recent long complicated proof that maybe correct.

We give a relatively simple proof of the stability of $\mathcal{L}_4, \mathcal{L}_5$ when $\mu = \mu_R$ based on four simple ideas.

- An approbate scaling
- Understanding the local geometry
- Coordinates adapted to the geometry
- Special polar coordinates for a section map

Shift origin so $z = 0 \in \mathbb{R}^4$ is $\mathcal{L}_4$.

Equations $\dot{z} = J\nabla\mathcal{H}(z, \nu_R) = Az + \cdots$.

A is a 4×4 Hamiltonian matrix.

Eigenvalues double $\pm i$ and not diagonalizable.

Invariants:

$$\Gamma_1 = x_2 y_1 - x_1 y_2, \qquad \Gamma_2 = \tfrac{1}{2}(x_1^2 + x_2^2),$$
$$\Gamma_3 = \tfrac{1}{2}(y_1^2 + y_2^2), \qquad \Gamma_4 = x_1 y_1 + x_2 y_2, \tag{1}$$

where $z = (x_1, x_2, y_1, y_2)$. Sokol'skii's normal form

$$H = \Gamma_1 + \delta\Gamma_2 + H^{\dagger}(\Gamma_1, \Gamma_3, \nu), \tag{2}$$

where $\delta = \pm 1$.

Quadratic Hamiltonian or linear equations

$$H_2 = \Gamma_1 + \delta \Gamma_2.$$

Coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & -1 & 0 \end{bmatrix}, \qquad (3)$$

with eigenvalues $\lambda = \pm i$.

(This is the Hamiltonian equivalent of a Jordan form.)

Scale the variables by

$$\begin{array}{ll} x_1 \rightarrow \epsilon^2 x_1, & x_2 \rightarrow \epsilon^2 x_2, \\ y_1 \rightarrow \epsilon y_1, & y_2 \rightarrow \epsilon y_2, \\ H \rightarrow \epsilon^{-3} H & \end{array} \quad (4)$$

(Meyer & Schmidt 1971)

The Hamiltonian becomes

$$H = \Gamma_1 + \epsilon\{\delta\Gamma_2 + \eta\delta\Gamma_3^2\} + O(\epsilon^2). \quad (5)$$

(This uneven scaling has found the important terms.)

Theorem: If $\eta > 0$ then the origin is stable.

Corollary: $\mathcal{L}_4$ is stable when $\mu = \mu_R$.

(It was known and easy that $\eta < 0$ implies instability.)

$$H = \Gamma_1 + \epsilon\{\delta\Gamma_2 + \eta\delta\Gamma_3^2\} + \cdots ,$$

and when $\epsilon = 0$

$$H_0 = \Gamma_1 = \frac{1}{2}(-u_1^2 + u_2^2 - v_1^2 + v_2^2).$$

This is two harmonic oscillators, so use action-angle variables $(I_1, I_2, \theta_1, \theta_2)$ and

$$H_0 = I_2 - I_1$$

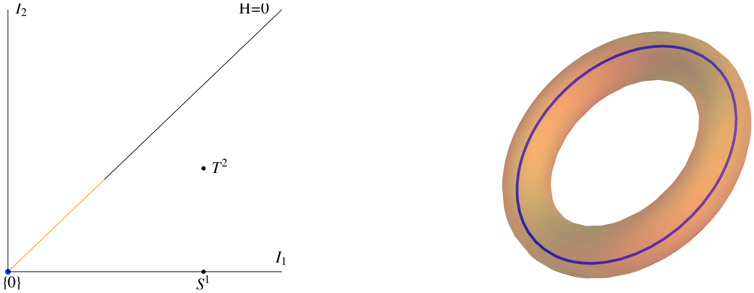

Figure: Tori Coordinates

Figure: Cross Section in a Energy Level

Oops: $F = H$.
Σ is cross section; Σ_e is cross section in energy level.

Take a cross section in the $H = 0$ level set.

Use the scaled normalized equations and compute the Poincaré map

$$q \to q + \epsilon 2p^3 + \cdots, \qquad p \to p - \epsilon q + \cdots$$

where (q, p) are rectangular coordinates in the section.

This looks like the time ϵ map defined by the sin lemniscate function. (C. F. Gauss, January 8, 1797)

The sin lemniscate function, denoted sl α, is the solution of

$$\xi'' + 2\xi^3 = 0, \qquad \xi(0) = 0, \quad \xi'(0) = 1 \tag{6}$$

where $' = d/d\alpha$.

This equation has an integral given by

$$\eta^2(\alpha) + \xi^4(\alpha) \equiv 1 \quad \text{with} \quad \eta(\alpha) = \xi'(\alpha) \tag{7}$$

which implies $\xi(\alpha)$ is periodic — see Figure 3a.

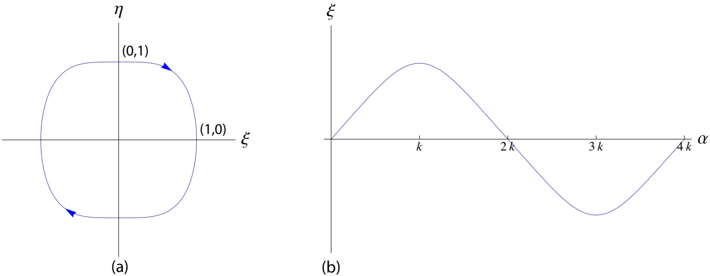

Figure: The sin lemn function: (a) its integral and (b) its graph

Let κ be the least positive value such that $\xi(\kappa) = 1$.

Solving for $\eta = d\xi/d\alpha$ and then separating variables one finds

$$\kappa = \int_0^1 \frac{d\tau}{\sqrt{1-\tau^4}} = \frac{1}{4}\,\mathbf{B}\left(\frac{1}{4}, \frac{1}{2}\right) \approx 1.311028777\ldots,$$

where **B** is the classical Beta function.

By symmetry arguments $\xi(\alpha)$ is odd and even about $\alpha = \kappa$, i.e., $\xi(\kappa + \alpha) \equiv \xi(\kappa - \alpha)$, and therefore $\xi(\alpha)$ is 4κ–periodic — see Figure 3b.

Recall that the Poincaré map looks like

$$q \to q + \epsilon 2p^3 + \cdots, \qquad p \to p - \epsilon q + \cdots$$

Change to “polar coordinates” (ρ, α) by

$$q = \rho^2 \eta(\alpha), \qquad p = \rho \xi(\alpha),$$

so the Poincaré map becomes

$$\rho \to \rho + O(\epsilon^2), \qquad \alpha \to \alpha - 2\epsilon\rho + O(\epsilon^2).$$

It is a twist map!

So Moser's invariant curve theorem implies the existence of a big set of invariant circles encircling the fixed point and thus the fixed point is stable.

This in turn implies the equation is stable in $H = 0$ and a slight variation of the argument implies stability for H very small. Which implies the origin is stable and the Theorem is proved.

Since it is known that $\eta > 0$ in the restricted three body problem, $\mathcal{L}_4$ is stable for $\mu = \mu_R$.