A GENERIC PHENOMENON IN CONSERVATIVE HAMILTONIAN SYSTEMS

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In a conservative Hamiltonian system two of the characteristic multipliers of any periodic solution must be $+1$. It has been conjectured that generically all periodic solutions in such a system must have the other characteristic multipliers not equal to $+1$. (See [**1**], p. 182.) We wish to propose an example where this is not the case.

Consider a conservative Hamiltonian system of two degrees of freedom (i.e. a four dimensional system). If a periodic solution has two characteristic multipliers different from $+1$, a classical theorem asserts that this periodic solution lies locally in a smooth cylinder filled with periodic solutions. The energy manifold at an energy h (i.e. the manifold defined by taking the Hamiltonian $H = h$ to be a constant) intersects this cylinder in a circle which is a periodic solution. See Figure 1. Thus there exists a smooth one parameter family of periodic solutions

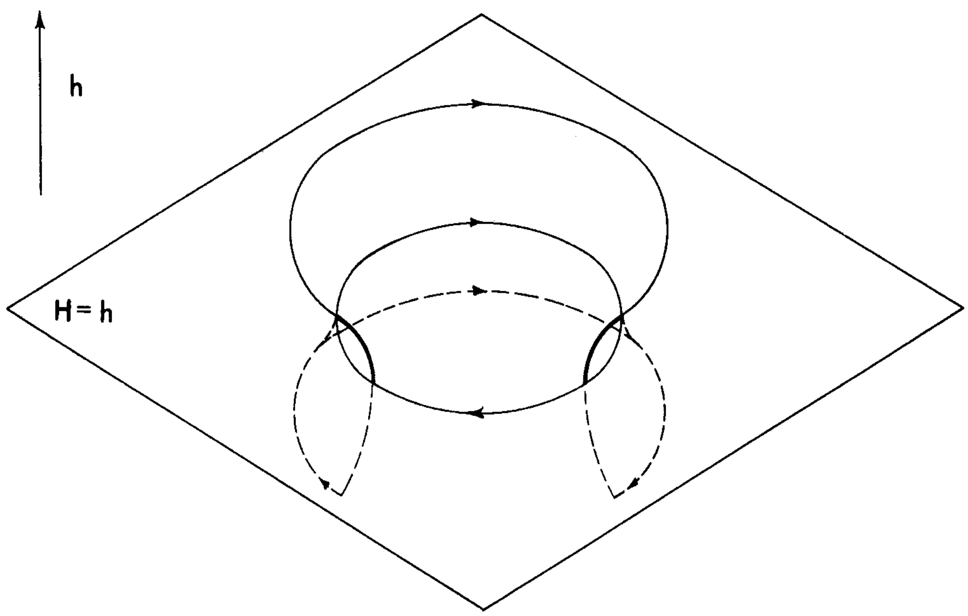

FIGURE 1

[1] This research was supported in part by NONR 3776(00), School of Mathematics, University of Minnesota.

[2] This research was supported in part by NGR 24–005–063, Center for Control Sciences, University of Minnesota.

near any periodic solution with two characteristic multipliers not equal to $+1$ and the parameter may be taken as energy. The period and characteristic multipliers are smooth functions of the parameter. If the period remains bounded the family can be extended until it reaches an equilibrium point or a periodic orbit γ_1 which has all characteristic multipliers equal to $+1$. If the least periods also converge to the least period of γ_1, then one has generically the following picture, Figure 2.

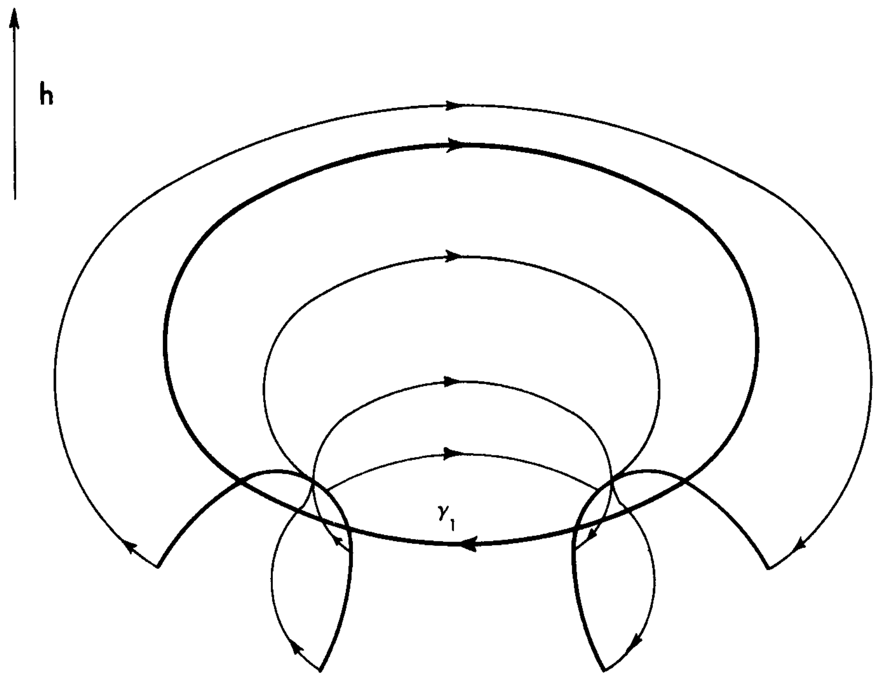

FIGURE 2

The orbit γ_1 is not the termination of the family but merely the termination of the region where energy can be used as the parameter. Generically, there must exist another smooth family parameterized by energy that smoothly meets the original family at γ_1. These two families must be of different stability types, that is, one family must consist of orbits with nontrivial multipliers of elliptic type and the other must consist of orbits with multipliers of hyperbolic type. It can be shown that the period can be used as a parameter near γ_1. Thus the periodic solutions lie in a one parameter family where the parameter can be taken as either energy or period and the family achieves a maximum (or minimum) of energy at an orbit having all characteristic multipliers equal to $+1$.

Once this picture is understood it is natural to suspect that one could construct a one parameter family that does not terminate. See Figure 3 below. This figure shows a torus filled with periodic solutions. An energy level meets the torus in two

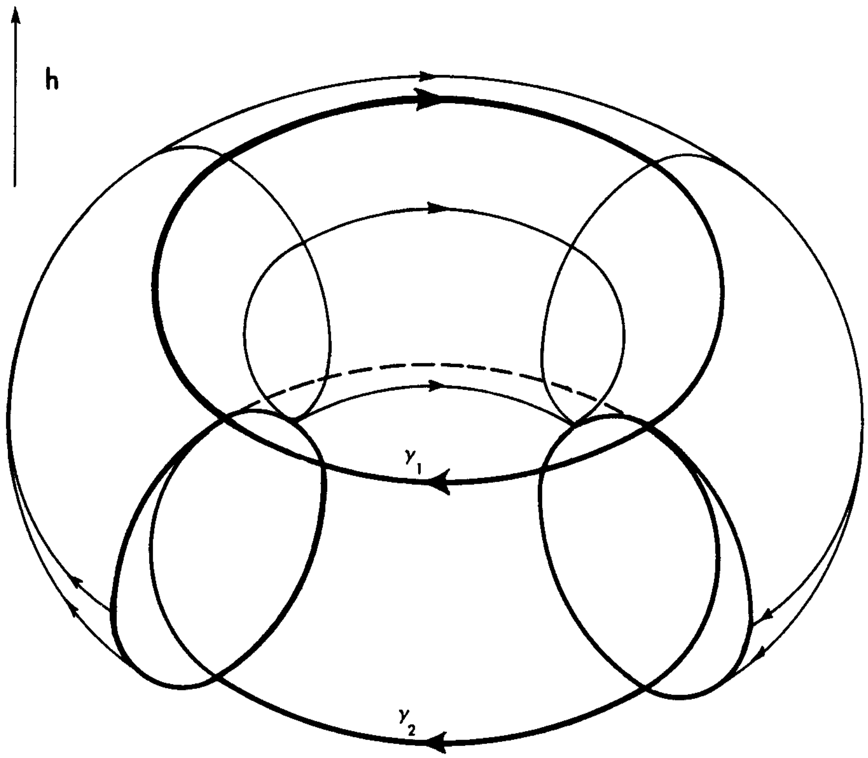

FIGURE 3

periodic orbits, one of elliptic type and one of hyperbolic type, except at the two orbits γ_1 and γ_2 where energy takes its maximum and minimum values. The orbits γ_1 and γ_2 have all characteristic multipliers $+1$. We propose that such an example exists and cannot be destroyed by a small perturbation.

In order to see how to construct such an example consider a three dimensional local cross section to the flow at some periodic solution. The intersection of an energy level with the cross section is a two dimensional manifold which can be taken as a disk. The flow defines an area preserving diffeomorphism of the disk into itself. Thus the problem of studying the local behavior near a periodic solution in a Hamiltonian system reduces to studying a one parameter family of area preserving diffeomorphisms of the disk. The figures on the right in Figure 4 indicate the changes in the map as the parameter is varied. The cross section indicated by this sequence is taken near the orbit γ_1. In the first figure there are two fixed points corresponding to the periodic solutions of elliptic and hyperbolic type. The second figure is similar except that the two fixed points are closer together. The bottom figure shows the local behavior of the cross section in the

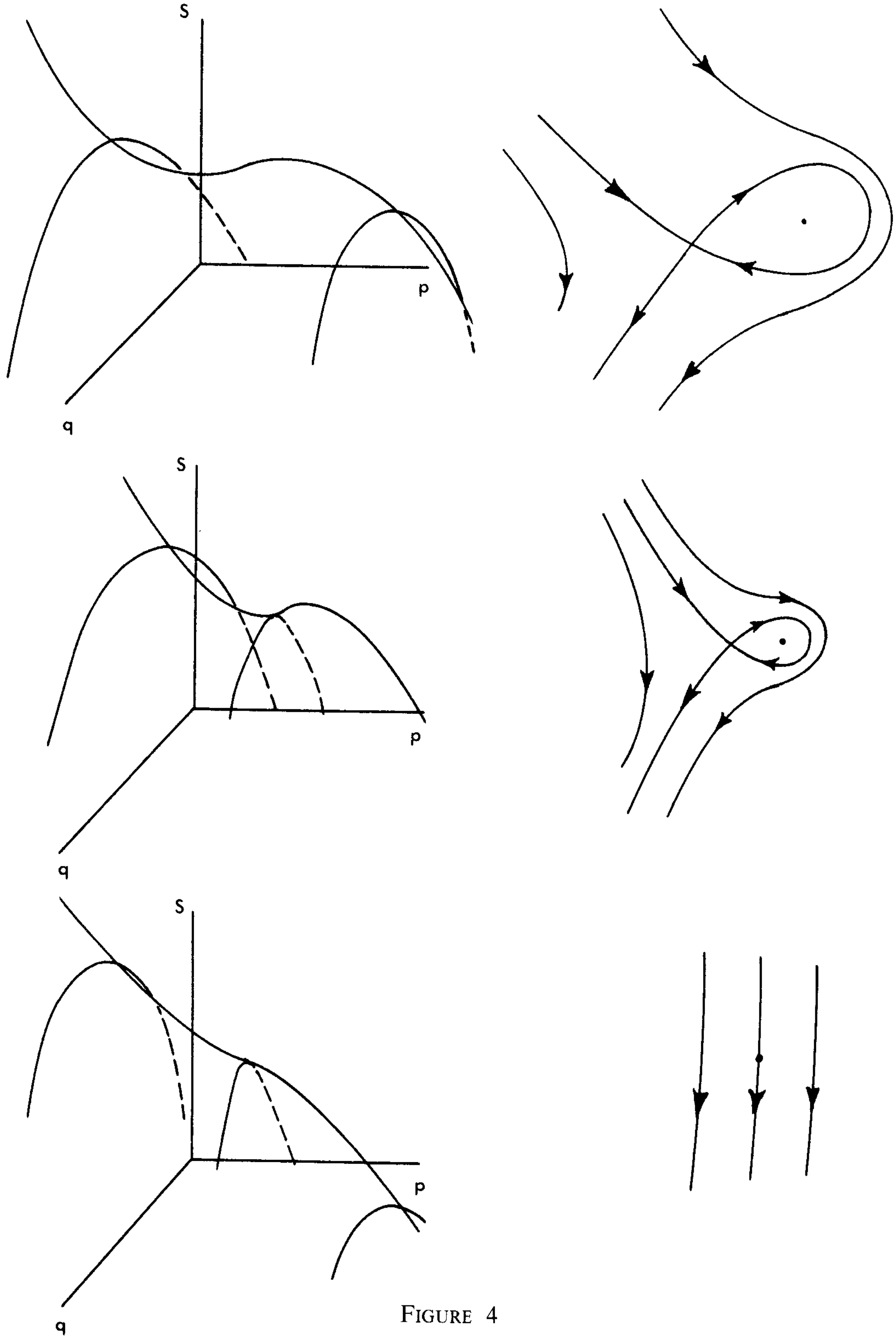

FIGURE 4

energy level of γ_1. From this point of view we see that the two families come together at γ_1 and then disappear as the parameter energy is increased.

Poincaré has given a simple method for studying area preserving maps (see [2], Chapter XXX, Vol. III). If $T:(q, p) \to (Q, P)$ defines an area preserving mapping of the plane into itself with the origin as a fixed point then the form (in q and p) given by

$$\Omega = (Q - q)d(P + p) - (P - p)d(Q + q)$$

is exact. Thus there exists a function $S(q, p)$ such that $dS = \Omega$. If $P + p$ and $Q + q$ can be used as local coordinates, that is if -1 is not an eigenvalue of the Jacobian matrix of T, then one sees that a critical point of S corresponds to a fixed point of T and vice versa. By some elementary algebra one can show that saddle points of S correspond to hyperbolic fixed points of T and maxima and minima of S correspond to elliptic fixed points of T. One can construct T from S in the obvious way except in the degenerate case where the Hessian determinant of S is $-1/4$. Thus in order to construct the example one needs to construct a function S with extrema as shown on the left in Figure 4 and then suspend the map in a flow. The function $S(q, p; h) = q^2 + ph - p^3$ is sufficient. The usual methods of jet transversality can be used to show that the above phenomenon persists under perturbation. (See [3] for details.)

The behavior near a maximum in energy of a family of periodic solutions is well known to people working in numerical computations in the restricted three body problem [4]. The example of a torus has been found only recently in the restricted problem near the Lagrangian triangular equilibrium by numerical experimentation. (See [5].)

References

1. R. Abraham, *Foundations of mechanics*, Benjamin, New York, 1967.
2. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 3, Dover, New York, 1957.
3. H. I. Levine, *Singularities of differential mappings*. I, Mathematisches Institut der Universität Bonn, 1959.
4. A. Deprit and J. Henrard, "A manifold of periodic orbits," in *Advances in astronomy and astrophysics*, Vol. 6, Academic Press, New York, 1968.
5. J. Palmore, *Bridges and natural centers in the restricted three body problem*, University of Minnesota Report, Center for Control Sciences, 1968.

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