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The Condition of Regular Degeneration for Singularly Perturbed Systems of Linear Differential-Difference Equations*

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INTRODUCTION

This paper is a continuation of the work presented in the first author's paper [1] in which the problem of singular perturbation was treated for a linear scalar constant coefficient differential-difference equation with a single retardation. The above paper contains references to the previous work on this problem. The present paper considers a system of linear constant coefficient differential-difference equations with several retardations.

In order to keep the length of this paper to a minimum the initial conditions, forcing terms, and some of the coefficients have been assumed independent of the perturbing parameter. These question are thoroughly discussed in the above paper.

The paper is divided into three parts. In the first part the existence of the solution to the degenerate equation is discussed. Also explicit formulas are found for the solution.

The second part of the paper contains a working definition of complete regularity and the main convergence theorems.

In the third section the concept of complete regularity is discussed and an equivalent (usable) formulation is given to this concept.

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1. Existence of Solutions and Formulas for Solutions

We shall consider the following system of equations

$$\frac{d}{dt}\{E(\epsilon)x(t)\} = \sum_{k=0}^{h} B_k x(t - \omega_k) + f(t) \tag{1}$$

where $0 = \omega_0 < \omega_1 < \cdots < \omega_h$ and

$$B_k = \begin{pmatrix} B_{k11} & B_{k12} \\ B_{k21} & B_{k22} \end{pmatrix}, \qquad E(\epsilon) = \begin{pmatrix} I_n & 0 \\ 0 & \epsilon I_m \end{pmatrix}$$

and

B_{k11} are $n \times n$ real constant matrices for $k = 0, ..., h$

B_{k12} are $n \times m$ real constant matrices for $k = 0, ..., h$

B_{k21} are $m \times n$ real constant matrices for $k = 0, ..., h$

B_{k22} are $m \times m$ real constant matrices for $k = 0, ..., h$

I_n is the $n \times n$ identity matrix

I_m is the $m \times m$ identity matrix.

Thus B_k and $E(\epsilon)$ are $(n + m) \times (n + m)$ matrices. Both $x(t)$ and $f(t)$ are $(n + m)$-vector functions of the real variable t. We shall take $(d/dt)\{E(\epsilon)x(t)\}$ to mean the right hand derivative of $E(\epsilon)x(t)$. Thus when $\epsilon = 0$ the right hand derivative of $x(t)$ may not in general exist but it will for $E(0)x(t)$. The scalar ϵ is to be nonnegative. We shall write $x(\epsilon, t)$ in place of $x(t)$ when we wish to show the dependence of the solutions of (1) on the parameter ϵ. Also we shall take $x(t) = (y(t)', z(t)')'$, $x(\epsilon, t) = (y(\epsilon, t)', z(\epsilon, t)')'$ and $f(t) = (f_1(t)', f_2(t)')'$ where $y(t)$, $y(\epsilon, t)$ and $f_1(t)$ are n-vectors and $z(t)$, $z(\epsilon, t)$ and $f_2(t)$ are m-vectors. (The prime denotes the transpose).

The convergence of the solutions of (1) is to be discussed as $\epsilon \to 0^+$. In particular does the solution of (1) when $\epsilon \neq 0$ tend uniformly as $\epsilon \to 0$ to the solution of (1) when $\epsilon = 0$? Is this convergence uniform over a finite or infinite interval of t?

Before treating these questions it will be necessary first to discuss the existence of solutions to the above equations and to find explicit formulas for these solutions. The methods for treating these questions are slight extensions of those found in [1]. In particular it is found that analogs of the kernel $K(t)$ as found in [1] cannot be defined. But one can define the analog of the integral of $K(t)$ and thus obtain expressions for the solutions as integral operators on the initial conditions. In this case though, the integrals are Stieltjes integrals.

Let us now consider the existence of solutions. When $\epsilon \neq 0$ we may apply Theorem 6.2 of Bellman and Cooke [2] to insure existence and uniqueness to the initial value problem since $\det E(\epsilon) = \epsilon^m \neq 0$. When $\epsilon = 0$ we require the following

THEOREM 1. *Consider the system* (1) *when* $\epsilon = 0$. *Let* $x_0(t)$ *be an* $(n + m)$-*vector function that is continuous and of bounded variation for* $t \in [0, \omega_n]$ *and let* $f(t)$ *be an* $(n + m)$-*vector function that is continuous and of bounded variation for* $t \in [\omega_h, \infty)$. *Assume that*

$$\det B_{022} \neq 0. \tag{2}$$

Then there exists a unique $(n + m)$-*vector function* $x(t)$ (*or* $x(0, t)$) *defined for all* $t \geqslant 0$ *such that*

(a) $x(t) = x_0(t)$ for $t \in [0, \omega_h]$

(b) $x(t)$ *satisfies* (1) *for all* $t \in (\omega_h, \infty)$

(c) $y(t)$ *is continuous and* $z(t)$ *is continuous from the left.*

Moreover $x(t)$ *and* $\dot{y}(t)$ *are of bounded variation for* $t \in [0, R]$ *for any* $0 < R < \infty$ *and the set of discontinuities of* $x(t)$ (*properly just of* $z(t)$) *is contained in the set* $Q = \{t^* \mid t^* = \sum_{k=1}^{h} j_k \omega_k$, *where* j_k *is a nonnegative integer for each* $k = 1, 2, \ldots, h\}$. *If*

$$\frac{d}{dt}\{E(0)x_0(t)\}\Big|_{t=\omega_h-0} = \sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h) \tag{3}$$

then $\dot{y}(t)$ *and* $z(t)$ *are continuous for all* $t \geqslant 0$.

PROOF: When $\epsilon = 0$ the system (1) can be written

$$\dot{y}(t) = \sum_{k=0}^{h} \{B_{k11} y(t - \omega_k) + B_{k12} z(t - \omega_k)\} + f_1(t)$$

$$0 = \sum_{k=0}^{h} \{B_{k21} y(t - \omega_k) + B_{k22} z(t - \omega_k)\} + f_2(t).$$

By (2) the matrix B_{022} is nonsingular and so the second equation above can be solved for $z(t)$ and this expression substituted into the first equation to obtain

$$\dot{y}(t) - (B_{011} - B_{012}B_{022}^{-1}B_{021})y(t) = \sum_{k=1}^{h} \{B_{k11} - B_{012}B_{022}^{-1}B_{k21})y(t - \omega_k)$$

$$+ (B_{k12} - B_{012}B_{022}^{-1}B_{k22})z(t - \omega_k)\} + f_2(t) - B_{012}B_{022}^{-1}f_1(t) \tag{4a}$$

$$z(t) = - B_{022}^{-1}\left\{\sum_{k=1}^{h} (B_{k21} y(t - \omega_k) + B_{k22} z(t - \omega_k)) + B_{021} y(t) + f_2(t)\right\}. \tag{4b}$$

Let the coefficient matrix for $y(t)$ in (4a) be A and let the right hand side of (4a) be $v(t)$. Then (4a) may be written

$$\frac{d}{dt} e^{-At} y(t) = e^{-At} v(t).$$

Since $v(t)$ contains y and z only with arguments $t - \omega_k$, $k \neq 0$, it is clear that $v(t)$ is defined and continuous for $t \in [\omega_h, \omega_h + \omega_1]$ and so $y(t)$ can be continued to $[\omega_h, \omega_h + \omega_1]$ continuously. By (4b) $z(t)$ can be continued to $(\omega_h, \omega_h + \omega_1]$ uniquely and such that $z(t)$ is continuous save possibly for a jump at $t = \omega_h$. Clearly the theorem follows by induction on this process.

Henceforth we shall assume that (2) holds and that $x_0(t)$ and $f(t)$ are continuous and of bounded variation for t in $[0, \omega_h]$ and t in $[\omega_h, \infty)$ respectively. Throughout we shall denote by $x(\epsilon, t)$ the unique function defined for all $t \geqslant 0$ that satisfies (1) for all $t > \omega_h$, has initial value $x_0(t)$ and whose existence is assured by Theorem 6.2 of Bellman and Cooke [2] for $\epsilon \neq 0$ and by Theorem 1 for $\epsilon = 0$.

It is well known that when $\epsilon \neq 0$ the solution $x(\epsilon, t)$ is exponentially bounded provided $f(t)$ is exponentially bounded. By a simple but lengthy inductive argument the same is true when $\epsilon = 0$. Thus the LaPlace Stieltjes transform of $x(t, \epsilon)$ exists in some right hand plane and satisfies

$$\begin{aligned} H(\epsilon, s) \int_{\omega_n}^{\infty} e^{-st}\, dx(\epsilon, t) &= e^{-\omega_h s} \frac{d}{dt} E(t) x(\epsilon, t) \Big|_{t=\omega_h} \\ &\quad + \sum_{k=1}^{h} e^{-\omega_k s} B_k \int_{\omega_h - \omega_k}^{\omega_h} e^{-st}\, dx_0(t) + \int_{\omega_h}^{\infty} e^{-st}\, df(t) \\ &= e^{-\omega_h s} \left\{ \sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h) \right\} \\ &\quad + \sum_{k=1}^{h} e^{-\omega_k s} B_k \int_{\omega_h - \omega_k}^{\omega_h} e^{-st}\, dx_0(t) + \int_{\omega_h}^{\infty} e^{-st}\, df(t) \end{aligned} \tag{5}$$

where

$$H(\epsilon, s) = sE(\epsilon) - \sum_{k=0}^{h} e^{-s\omega_k} B_k . \tag{6}$$

In a manner similar to that found in [1] we wish to find a matrix function $W(\epsilon, t)$ whose LaPlace Stieltjes transform is $H(\epsilon, s)^{-1}$.

By the same procedure as found in the proof of Theorem 1 we find *that there exists a unique* $(n + m) \times (n + m)$ *matrix function* $W(\epsilon, t)$ *defined for all* t *and* $\epsilon \geqslant 0$ *that satisfies the following conditions*:

(a) $W(\epsilon, t) = 0$ *for* $t \leqslant 0, \epsilon \geqslant 0$;

(b) $W(\epsilon, t)$ *is continuous for all* t *if* $\epsilon \neq 0$;

(c) $W(0, t)$ *has its first n rows continuous and the last m rows are continuous from the left for all* t;

(d) $$\frac{d}{dt}\{E(\epsilon)W(\epsilon, t)\} - \sum_{k=0}^{h} B_k W(\epsilon, t - \omega_k) = \begin{cases} 0 & \text{for } t \leqslant 0 \\ I_{n+m} & \text{for } t > 0 \end{cases}$$

where I_{n+m} *is the* $(n + m)$ *identity matrix and the above derivative is a right hand derivative when* $\epsilon = 0$. Moreover $W(\epsilon, t)$ and $(d/dt)\{E(\epsilon)W(\epsilon, t)\}$ are of bounded variation for t in $[0, R]$ when $\epsilon \geqslant 0$ and $R \geqslant 0$ are fixed.

Again by a simple but lengthy argument it can be shown that the variation of $W(\epsilon, t)$ is exponentially bounded. Thus the LaPlace Stieltjes transform of $W(\epsilon, t)$ converges absolutely in some right hand plane and furthermore it is clear that

$$H(\epsilon, s)^{-1} = \int_0^\infty e^{-st}\, dW(\epsilon, t) \tag{7}$$

for those values of s for which the integral converges.

Now we shall use (5) and (7) to show that the solution $x(\epsilon, t)$ can be expressed as an integral operator on the initial conditions. This will be done by showing that each term in (5) is the LaPlace Stieltjes transform of a certain function and then applying the uniqueness theorem. First

$$H(\epsilon, s)^{-1}e^{-s\omega_k} = \int_{\omega_h}^\infty e^{-st}\, dW(\epsilon, t - \omega_k).$$

Now let

$$\mu(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

Then

$$\int_{\omega_h-\omega_k}^{\omega_h} e^{-st}\, dx_0(t) = \int_0^\infty e^{-st}\, d\{x_0(t)\mu(\omega_h - t)\mu(t - \omega_h + \omega_k)\}$$
$$- e^{-s(\omega_h-\omega_k)}x_0(\omega_h - \omega_k) + e^{-s\omega_h}x_0(\omega_h)$$

and hence by the convolution theorem

$$H(\epsilon, s)^{-1}e^{-\omega_k s} \int_{\omega_h-\omega_k}^{\omega_h} e^{-st}\, dB_k x_0(t)$$
$$= \int_{\omega_h}^\infty e^{-st}\, d_t \int_{\omega_h-\omega_k}^{\omega_h} W(\epsilon, t - \omega_k - \alpha)\, d_\alpha B_k x_0(\alpha).$$

In the same manner

$$H(\epsilon, s)^{-1} \int_{\omega_h}^{\infty} e^{-st}\, df(t) = \int_{\omega_h}^{\infty} e^{-st}\, d_t \int_{\omega_h}^{t} W(\epsilon, t-\alpha)\, df(\alpha)$$

and so by the uniqueness theorem for Laplace Stieltjes transforms

$$x(\epsilon, t) = x_0(\omega_h) + W(\epsilon, t-\omega_h) \left\{ \sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h) \right\}$$

$$+ \sum_{k=1}^{h} \int_{\omega_h-\omega_k}^{\omega_h} W(\epsilon, t-\omega_k-\alpha)\, dB_k x_0(\alpha) + \int_{\omega_h}^{t} W(\epsilon, t-\alpha)\, df(\alpha). \tag{8}$$

In general the uniqueness theorem would state that (8) holds only at points of continuity but since both sides of (8) are continuous from the left it holds for all $t > \omega_h$ and all $\epsilon \geqslant 0$.

We have derived (8) under the assumption that $f(t)$ is exponentially bounded. If $f(t)$ is not exponentially bounded we can define

$$f_T(t) = \begin{cases} f(t) & \text{for} \quad t \in [\omega_h, T] \\ f(T) & \text{for} \quad t \in [T, \infty) \end{cases}$$

for any $T > \omega_h$. Then the formula (8) is valid for the solution $x_T(\epsilon, t)$ of (1) with $f(t)$ replaced by $f_T(t)$. For $t \in [\omega_h, T]$, $x_T(\epsilon, t) = x(\epsilon, t)$ and so (8) holds for $x(\epsilon, t)$ if $t \in [\omega_h, T]$. But T is arbitrary and so (8) holds in general.

2. The Regularity Conditions and the Main Convergence Theorems

As was pointed out in Cooke [1] the degeneration problem can be discussed in terms of the new condition of complete regularity. This condition is useful in the discussion of the convergence of the roots of det $H(\epsilon, s)$ as $\epsilon \to 0^+$ and thus gives the needed information about the convergence of $W(\epsilon, t)$ as $\epsilon \to 0^+$. The following is our working definition of σ-regularity and σ-complete regularity. In the following section an equivalent formulation is given that is easier to check in examples.

Definition 1. Let σ_0 and σ_1 be real numbers, $\sigma_1 \geqslant \sigma_0$. The equation (1) will be said to be $[\sigma_0, \sigma_1]$-*regular* (as $\epsilon \to 0^+$) if the following condition holds.

Condition A. There exist positive numbers ϵ_1 and γ_1 such that

$$|\theta(\epsilon, s)| \geqslant \gamma_1 \tag{9}$$

for all s, ϵ satisfying $0 \leqslant \epsilon \leqslant \epsilon_1$ and $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$ where

$$\theta(\epsilon, s) = \det \left\{ \epsilon s I_m - \sum_{k=0}^{h} e^{-\omega_k s} B_{k22} \right\}.$$

The equation (1) will be said to be σ_0-*completely regular* if there exist positive numbers ϵ_1 and γ_1 such that (9) holds for $0 \leqslant \epsilon \leqslant \epsilon_1$ and $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$ for all $\sigma_1 \geqslant \sigma_0$, ϵ_1 and γ_1 are independent of σ_1, and all the characteristic roots of B_{022} have negative real parts.

NOTE. In comparing the above with [1] it should be noted that the condition A stated above corresponds to the condition AC of [1]. There is no condition needed here that corresponds to condition B of [1], since in the present paper the singularly perturbed derivative is evaluated only at t.

The usefulness of the concept of $[\sigma_0, \sigma_1]$-regularity can readily be seen from the following theorem. This theorem states that $[\sigma_0, \sigma_1]$-regularity insures a uniform lower bound on $|\det H(\epsilon, s)|$ in a certain strip.

THEOREM 2. *Let σ_0 and σ_1 be finite real numbers, $\sigma_0 \leqslant \sigma_1$. Let $S[\sigma_0, \sigma_1]$ be a closed region obtained by removing from the strip $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$ circles of any fixed radius centered at the zeros of* $\det H(0, s)$. *Assume the equation* (1) *is $[\sigma_0, \sigma_1]$-regular. Then there exist positive numbers ϵ_2, γ_2 such that*

$$|\det H(\epsilon, s)| \geqslant \gamma_2 |s|^n \tag{10}$$

for $0 \leqslant \epsilon \leqslant \epsilon_2$, $s \in S[\sigma_0, \sigma_1]$. In particular, for $0 \leqslant \epsilon \leqslant \epsilon_2$, any zeros of $\det H(\epsilon, s)$ *in the strip $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$ must lie within the circles about the zeros of* $\det H(0, s)$.

Furthermore, if the equation (1) *is σ_0-completely regular, and if $S[\sigma_0]$ denotes a closed region obtained by removing from the half plane* $\text{Re}(s) \geqslant \sigma_0$ *circles of fixed radius about the zeros of* $\det H(0, s)$, *then there exist postive numbers ϵ_2 and γ_2 such that* (10) *holds for* $0 \leqslant \epsilon \leqslant \epsilon_2$, $s \in S[\sigma_0]$. *Any zeros of* $\det H(\epsilon, s)$ *in* $\text{Re}(s) \geqslant \sigma_0$ *must lie within the circles about the zeros of* $\det H(0, s)$.

PROOF: Let (1) be $[\sigma_0 . \sigma_1]$-regular. The highest degree term in $\det H(\epsilon, s)$ in s is $\epsilon^m s^{n+m}$. Choose τ_0 so large that $|\det H(\epsilon, s)| > \frac{1}{2}\epsilon^m |s|^{n+m}$ for $|s| > \tau_0/\epsilon$, $\text{Re}\, s \geqslant \sigma_0$; then $|\det H(\epsilon, s)| > \frac{1}{2}\tau_0{}^m |s|^n$ for all s such that $|s| > \tau_0/\epsilon$ and $s \in S[\sigma_0, \sigma_1]$ and for ϵ in any fixed interval $0 < \epsilon \leqslant \epsilon_0$. Now consider $|s| \leqslant \tau_0/\epsilon$, $\epsilon \neq 0$. In general $\det H(\epsilon, s) = \sum_{j=0}^{n} p_j(\epsilon, s) s^j$ where $p_j(\epsilon, s)$ is a polynomial is ϵs with coefficients of the form $\sum a_i e^{-\beta_i s}$, a_i and β_i are real numbers. For $|s| \leqslant \tau_0/\epsilon$ each $p_j(\epsilon, s)$ is bounded with a bound that is independent of ϵ. By condition A there exists $\epsilon_1 > 0$ such that $|p_n(\epsilon, s)| = |\theta(\epsilon, s)| \geqslant \gamma_1$ for all $0 \leqslant \epsilon \leqslant \epsilon_1$. Thus for $s \in S[\sigma_0, \sigma_1]$ and $\tau_0/\epsilon \geqslant |s| \geqslant M$ for $\epsilon \neq 0$ and $|s| \geqslant M$ for $\epsilon = 0$ where M is some fixed number we have $|\det H(\epsilon, s)| \geqslant \frac{1}{2}\gamma_1 |s|^n$ for $0 \leqslant \epsilon \leqslant \epsilon_1$. Since there are no zeros of $\det H(0, s)$

in the set $\tilde{S} = S[\sigma_0, \sigma_1] \cap \{s \mid |s| \leqslant M\}$ then by Rouche's theorem there is an ϵ_3 such that for $0 \leqslant \epsilon \leqslant \epsilon_3$ there are no zeros of $\det H(\epsilon, s)$ in $\tilde{S}$. Since $\tilde{S} \times [0, \epsilon_3]$ is a compact set, $|\det H(\epsilon, s)|$ has a positive lower bound on $\tilde{S} \times [0, \epsilon_3]$ and so clearly we may pick a $\gamma_3 > 0$ small enough so that $|\det H(\epsilon, s)| \geqslant \gamma_3 |s|^n$ for all $s \in \tilde{S}$ and $0 \leqslant \epsilon \leqslant \epsilon_3$. Thus the theorem is proved by taking $\gamma_2 = \min[\frac{1}{2}\tau_0^m, \frac{1}{2}\gamma_1, \gamma_3]$ and $\epsilon_2 = \min[\epsilon_1, \epsilon_3]$. The second part of the theorem follows in the same way.

The above theorem is useful in the discussion of the convergence of $W(\epsilon, t)$ as $\epsilon \to 0^+$ since by the inverse theorem for Laplace Stieltjes transforms (see Widder [3])

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H(\epsilon, s)^{-1} \frac{e^{st}}{s}\, ds = \begin{cases} W(\epsilon, t); & c > 0, c > \sigma_i \\ W(\epsilon, t) - W(\epsilon, \infty); & \sigma_i < c < 0 \end{cases} \tag{11}$$

where σ_i is the abscissa of convergence for the transform of $W(\epsilon, t)$.

Since $H(\epsilon, s)^{-1} = (\det H(\epsilon, s))^{-1} \operatorname{adj} H(\epsilon, s)$, a typical element of $H(\epsilon, s)^{-1}$ will be a quasi-polynomial in s and e^{-s} with coefficients that depend on ϵ, divided by $\det H(\epsilon, s)$. A typical term is then of the form

$$\frac{b\epsilon^i s^j e^{-\alpha s}}{\det H(\epsilon, s)}$$

where $\alpha = \sum_{k=0}^{h} j_k \omega_k$, j_k are nonnegative integers and b is a real number. It is therefore necessary to investigate integrals of the following form

$$\int_{(c)} \frac{\epsilon^i s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)}\, ds \tag{12}$$

as $\epsilon \to 0^+$. In the above and henceforth $\int_{(c)} = \lim_{T\to\infty} \int_{c-iT}^{c+iT}$. The following lemmas will cover the convergence properties of (12) for all cases we shall find necessary to discuss. Since some of the lemmas are proved the same way as in Cooke [1] we shall only give the reference to the corresponding lemmas in [1] where the proofs can be found.

It is easy to see from (11) and (6) that $i - j + n \geqslant 0$, $m \geqslant i \geqslant 0$, and $n + m \geqslant j \geqslant 0$.

LEMMA 1. *Let* $i - j + n > 0$, $m \geqslant i \geqslant 0$ *and* $n + m \geqslant j \geqslant 0$. *Assume that Eq.* (1) *is σ-completely regular and that all zeros of* $\det H(0, s)$ *lie in the half plane* $R(s) \leqslant \sigma_1$ *where* $\sigma_1 < \sigma$. *Then*

$$\lim_{\epsilon\to 0^+} \int_{(\sigma)} \frac{s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)}\, ds = \int_{(\sigma)} \frac{s^{j-1} e^{s(t-\alpha)}}{\det H(0, s)}\, ds \tag{13}$$

if $i = 0$ *and*

$$\lim_{\epsilon\to 0^+} \int_{(\sigma)} \frac{\epsilon^i s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)}\, ds = 0 \tag{14}$$

for $i \neq 0$. The convergence in (13) *and* (14) *is uniform for t in any finite interval* $[t_0, t_1]$ *and if* $\sigma < 0$ *the convergence is uniform for* $t \in [t_0, \infty)$. *Moreover*

$$\left| \int_{(\sigma)} \frac{\epsilon^i s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)} ds \right| \leqslant \gamma_0 e^{\sigma t}, \qquad 0 \leqslant \epsilon \leqslant \epsilon_0, t \geqslant t_0 \tag{15}$$

for some $\epsilon_0 > 0$ *where* γ_0 *depends on* σ *but not on* ϵ *or* t.

The proof is similar to the proof of Lemmas 11.1 and 11.7 in [1].

LEMMA 2. *Let* $i - j + n = 0$, $m \geqslant i \geqslant 0$ *and* $n + m \geqslant j \geqslant 0$. *Assume that equation* (1) *is* σ*-completely regular and that all zeros of* $\det H(0, s)$ *lie in the half plane* $\mathrm{Re}(s) < \sigma_1$ *where* $\sigma_1 < \sigma$. *Then there exist constants* γ_0 *and* ϵ_0 *such that*

$$\left| \int_{(\sigma)} \frac{\epsilon^i s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)} ds \right| \leqslant \gamma_0 e^{\sigma t}, \qquad 0 \leqslant \epsilon \leqslant \epsilon_0 . \tag{15}$$

Thus the above integral is bounded as $\epsilon \to 0^+$ *for t in any finite interval* $[t_0, t_1]$ *and if* $\sigma < 0$ *for t in* $[t_0, \infty)$.

The proof is similar to the proofs of Lemmas 11.2 and 11.7 of [1].

LEMMA 3. *Let* $i - j + n = 0$, $m \geqslant i \geqslant 0$ *and* $n + m \geqslant j \geqslant 0$. *Assume that Eq.* (1) *is* σ*-completely regular and that all zeros of* $\det H(0, s)$ *lie in a plane* $\mathrm{Re}(s) < \sigma_1$ *where* $\sigma_1 < \sigma$. *Then*

$$\lim_{\epsilon \to 0^+} \int_{(\sigma)} \frac{s^{n-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)} ds = \int_{(\sigma)} \frac{s^{n-1} e^{s(t-\alpha)}}{\det H(0, s)} ds, \qquad \mathrm{i} = 0 \tag{17}$$

and

$$\lim_{\epsilon \to 0^+} \int_{(\sigma)} \frac{\epsilon^i s^{j-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)} ds = 0, \qquad \mathrm{i} \neq 0 \tag{18}$$

for each $t \in CQ$ *where* $CQ = \{t^* \mid t^* \geqslant 0,\ t^* \neq \alpha + \sum_{k=1}^{h} j_k \omega_k$ *where* j_k *are nonnegative integers*$\}$. *The convergence is uniform in any compact subset of* CQ.

The following proof of this lemma is slightly different from that found in Cooke [1] so it will be included.

PROOF: We shall consider the case when $\sigma > 0$ (the argument for $\sigma < 0$ is almost the same). Let

$$I^i(\epsilon, t) = \int_{(\sigma)} \frac{\epsilon^i s^{i+n-1} e^{s(t-\alpha)}}{\det H(\epsilon, s)} ds$$

and write $I^i(\epsilon, t) = J_1{}^i(\epsilon, t) + J_2{}^i(\epsilon, t)$ where

$$J_1{}^i(\epsilon, t) = \int_{(\sigma)} \frac{\epsilon^i s^{i-1} s^{s(t-\alpha)}}{\theta(\epsilon, s)} ds$$

and

$$J_2^i(\epsilon, t) = \int_{(\sigma)} \epsilon^i s^{i+n-1} e^{s(t-\alpha)} \frac{\theta(\epsilon, s)s^n - \det H(\epsilon, s)}{\theta(\epsilon, s)s^n \det H(\epsilon, s)} ds$$

where as before $\theta(\epsilon, s) = \det\{\epsilon s I_m - \sum_{k=0}^{h} e^{-\omega_k s} B_{k22}\}$. By the same method as used to prove Lemma 1 it follows that $\lim_{\epsilon \to 0^+} J_2^i(\epsilon, t) = J_2^i(0, t)$ uniformly for t in any compact subset of $[0, \infty)$. For the treatment of $J_1^i(\epsilon, t)$ we must be more careful. First let $z = \epsilon s$ and then

$$J_1^i(\epsilon, t) = \int_{(\epsilon\sigma)} \frac{z^{i-1} e^{(z/\epsilon)(t-\alpha)}}{\theta(\epsilon, z/\epsilon)} dz.$$

Now let $\theta(\epsilon, z/\epsilon) = \theta_0(z) + \theta_1(z)e^{-\alpha_1 z/\epsilon} + \cdots + \theta_r(z)e^{-\alpha_r z/\epsilon}$ where $\theta_k(z)$ is a polynomial in z and $\alpha_i = \sum_{k=1}^{h} j_k \omega_k$ where j_k are nonnegative integers. (Note that $\theta_0(z)$ is the characteristic polynomial for the matrix B_{022}.) Clearly we can pick σ, sufficiently large such that for $\operatorname{Re} z \geqslant \sigma_1$ we have

$$|\theta_1(z)e^{-\alpha_1 z/\epsilon} + \cdots + \theta_r(z)e^{-\alpha_r z/\epsilon}| < \tfrac{1}{2} |\theta_0(z)|$$

for a fixed $\epsilon \neq 0$. Thus

$$J_1^i = \sum_{k=0}^{\infty} (-1)^k \int_{(\sigma_1)} \frac{z^{i-1} e^{(z/\epsilon)(t-\alpha)}}{\theta_0(z)} \left\{ \frac{\theta_1(z)e^{-\alpha_1 z/\epsilon} + \cdots + \theta_r(z)e^{-\alpha_r z/\epsilon}}{\theta_0(z)} \right\}^k dz.$$

A typical term after the expression in the bracket has been expanded by the binomial theorem is

$$L^i = \int_{(\sigma_1)} \frac{z^{i-1} p(z) e^{z(t-\beta)/\epsilon}}{\theta_0(z)^{k+1}} dz,$$

where $p(z)$ is a polynomial and $\beta = \alpha + \sum_{s=1}^{h} j_s \omega_s$ where j_s are nonnegative integers. If $t < \beta$ we may move the contour to $+\infty$ and show $L^i = 0$. If $t > \beta$ we may move the contour to the left to prove that L^i is the sum of the residues of the integrand. The residues are of two types. The first comes from the zeros $\{z_\gamma\}$ of $\theta_0(z)$. The sum of the residues due to these poles is of the form

$$\sum P_\gamma \left(\frac{t-\beta}{\epsilon} \right) e^{z_\gamma (t-\beta)/\epsilon}$$

where P_γ is a polynomial of degree less than the multiplicity of z_γ in $\theta_0(z)^{k+1}$. These terms clearly tend to zero as $\epsilon \to 0^+$ except where $t = \beta$ and uniformly if t is in a compact set that does not contain $t = \beta$. This proves the second part of the lemma since when $i \neq 0$ there are no other poles of the integrand.

Let us consider the case when $i = 0$ further. In this case the pole at $z = 0$ has the residue $p(0)\{\theta_0(0)\}^{-k-1}$. Thus the pole at $z = 0$ is independent of ϵ and it is clear that the sum of these residues is $J_1^0(0, t)$ since

$$J_1^0(0, t) = \int_{(\sigma_1)} \frac{e^{s(t-\alpha)}}{\theta(0, s)s}\, ds$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_{(\sigma_1)} \frac{e^{s(t-\alpha)}}{s\theta_0(0)} \left\{ \frac{\theta_1(0)e^{-\alpha_1 s} + \cdots + \theta_r(0)e^{-\alpha_r s}}{\theta_0(0)} \right\}^k ds.$$

Thus $J_1^0(\epsilon, t) \to J_1^0(0, t)$ for all $t \in CQ$ and uniformly if t is any compact subset of CQ.

We are now ready to state the first main theorem.

THEOREM 3. (The Convergence Theorem). *Let $x(\epsilon, t)$ be the solution of* (1) *corresponding to the initial function $x_0(t)$ where $x_0(t)$ is continuous and of bounded variation in $[0, \omega_h]$ and where $f(t)$ is continuous and of bounded variation in $[\omega_h, \infty)$. Assume that for some sufficiently large σ_0 the equation* (1) *is σ_0-completely regular.*[1] *Let $Q = \{t^* \mid t^* = \sum_{k=1}^{h} j_k \omega_k$ where j_k is a nonnegative integer for $k = 1, 2, ..., h\}$. Let $x(\epsilon, t) = (y(\epsilon, t)', z(\epsilon, t)')'$ where y and z are as before. Then*

$$\lim_{\epsilon \to 0^+} y(\epsilon, t) = y(0, t) \tag{19}$$

where the convergence is uniform in t for any bounded subset of $[\omega_h, \infty)$. If $\dot{x}_0(t)$ exists, is continuous and is of bounded variation in $[0, \omega_h]$ (where $\dot{x}(0)$ is the right hand derivative and $\dot{x}(\omega_h)$ is the left hand derivative) and if $\dot{f}(t)$ exists, is continuous and is of bounded variation in $[\omega_h, \infty)$ (where $\dot{f}(\omega_h)$ is the left hand derivative) then

$$\lim_{\epsilon \to 0^+} z(\epsilon, t) = z(0, t) \tag{20}$$

where the convergence is bounded in any bounded subset of $CQ = [\omega_h, \infty) - Q$ and uniform in any compact subset of CQ. Moreover if (3) *holds then the convergence in* (20) *is uniform for t in any compact subset of $[\omega_h, \infty)$.*

PROOF: Let

$$W(\epsilon, t) = \begin{pmatrix} W_{11}(\epsilon, t) & W_{12}(\epsilon, t) \\ W_{21}(\epsilon, t) & W_{22}(\epsilon, t) \end{pmatrix} \tag{21}$$

and

$$H(\epsilon, s)^{-1} = \begin{pmatrix} \tilde{H}_{11}(\epsilon, s) & \tilde{H}_{12}(\epsilon, s) \\ \tilde{H}_{21}(\epsilon, s) & \tilde{H}_{22}(\epsilon, s) \end{pmatrix} \tag{22}$$

[1] See the remark following Theorem 6.

where W_{11} and $\tilde{H}_{11}$ are $n \times n$ submatrices, W_{12} and $\tilde{H}_{12}$ are $n \times m$ submatrices, etc. By the inverse theorem for Laplace Stieltjes transforms

$$W_{lp}(\epsilon, t) = \frac{1}{2\pi i} \int_{(c)} \tilde{H}_{lp}(\epsilon, s) \frac{e^{st}}{s}\, ds; \qquad l, p = 1, 2 \tag{23}$$

where c is greater than the abscissa of convergence of $W(\epsilon, t)$ and c is positive. By considering (6) it is clear that each term of each element of W_{11}, W_{12} and W_{21} is of the form (12) with $i - j + n > 0$ and so $\lim_{\epsilon\to 0} W_{lp}(\epsilon, t) = W_{lp}(0, t)$ uniformly for t in any bounded subset of $[0, \infty)$ for $l = 1, p = 1$; $l = 1, p = 2$, and $l = 2, p = 1$ by Lemma 1. Thus (19) follows at once from Lemma 1 and (8).

As for the second part of the theorem we must integrate by parts all the integrals in (8). First we define a new function $V(\epsilon, t)$ such that $(d/dt)V(\epsilon, t) = W(\epsilon, t)$ and $V(\epsilon, 0) = 0$. It is easy to verify that $\int_0^\infty e^{-st}\, dV(\epsilon, t) = s^{-1}H(\epsilon, s)^{-1}$ and thus $V(\epsilon, t) = \int_{(c)} H(\epsilon, s)^{-1}\, e^{st}/s^2\, ds$. By Lemma 1 we have $\lim_{\epsilon\to 0^+} V(\epsilon, t) = V(0, t)$ where the convergence is uniform for t in any compact subset of $[0, \infty)$.

Now

$$\begin{aligned} \int_{\omega_h}^{t} W(\epsilon, t - \alpha)\, df(\alpha) &= \int_{\omega_h}^{t} W(\epsilon, t - \alpha) f'(\alpha)\, d\alpha \\ &= -\int_{\omega_h}^{t} \{dV(\epsilon, t - \alpha)\} f'(\alpha) \\ &= V(\epsilon, t - \omega_h) f'(\omega_h) + \int_{\omega_h}^{t} V(\epsilon, t - \alpha)\, df'(\alpha) \end{aligned}$$

and

$$\begin{aligned} \int_{\omega_h-\omega_k}^{\omega_h} W(\epsilon, t - \omega_k - \alpha)\, dB_k x_0(\alpha) &= V(\epsilon, t - \omega_h) B_k x_0'(\omega_h - \omega_k) \\ &- V(\epsilon, t - \omega_k - \omega_h) B_k x_0'(\omega_h) + \int_{\omega_h-\omega_k}^{\omega_h} V(\epsilon, t - \omega_k - \alpha)\, dB_k x_0'(\alpha). \end{aligned}$$

Substituting the above in (8) we have

$$\begin{aligned} x(\epsilon, t) &= x_0(\omega_n) + W(\epsilon, t - \omega_h) \Big\{ \sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h) \Big\} \\ &+ \sum_{k=1}^{h} \{ V(\epsilon, t - \omega_h) B_k x_0'(\omega_h - \omega_k) - V(\epsilon, t - \omega_h - \omega_k) B_k x_0'(\omega_h) \} \\ &+ V(\epsilon, t - \omega_h) f'(\omega_h) + \sum_{k=1}^{h} \int_{\omega_h-\omega_k}^{\omega_h} V(\epsilon, t - \omega_k - \alpha)\, dB_k x'_0(\alpha) \\ &+ \int_{\omega_h}^{t} V(\epsilon, t - \alpha)\, df'(\alpha). \end{aligned} \tag{24}$$

All terms that do not contain W converge uniformly for t in any bounded subset of $[\omega_h, \infty)$. The term that does contain W is not under the integral sign. By Lemma 1 and 3 $\lim_{\epsilon\to 0^+} W_{22}(\epsilon, t) = W_{22}(0, t)$ for all $t \in CQ$ and uniformly for t in a compact subset of CQ, and by Lemmas 1 and 2 the convergence is bounded. Thus the last m rows converge as stated in the theorem.

Now if (3) holds, the submatrix $W_{22}(\epsilon, t)$ multiples a zero vector and so does not appear in the expression for $x(t)$. Thus the convergence in (20) will be uniform for t in any compact set of $[\omega_h, \infty)$.

For the next theorem we shall use the following notation. Let $\| x \| = \sup\{| x_i | \mid$ where x_i is the ith component of the vector $x\}$ and $\| A \| = \sup\{\| Ax \| \mid$ for all x such that $\| x \| = 1\}$. Let $Vf(t)$ be the $m + n$ vector whose ith component is the variation of the ith component of $f(t)$ from ω_h to t and let $Vx_0(t)$ be the $n + m$ vector whose ith component is the variation of the ith component of $x_0(t)$ from 0 to t.

THEOREM 4. (Asymptotic Stability Theorem.) *Let $x(\epsilon, t)$ be the solution of* (1) *corresponding to the initial function $x_0(t)$ where $x_0(t)$ is continuous and of bounded variation for $t \in [0, \omega_h]$ and where $f(t)$ is continuous and of bounded variation for $t \in [\omega_h, \infty)$. Assume that Eq.* (1) *is σ_0-completely regular and that all zeros of* det $H(0, s)$ *lie in a half plane* $\mathrm{Re}(s) \leqslant \sigma_2$ *where $\sigma_2 < \sigma_0$. Suppose also that*

$$\int_{\omega_h}^{\infty} e^{-\sigma_0 t}\, dVf(t) \leqslant \gamma_1$$

and if $\sigma_0 \leqslant 0$ that $\|f(t)\| \leqslant \gamma_1 e^{-\sigma_0 t}$ for $t \geqslant \omega_h$. Then there exist positive constants ϵ_2 and γ_2 such that

$$\| x(\epsilon, t)\| \leqslant \gamma_2 e^{\sigma_0 t}; \qquad 0 < \epsilon \leqslant \epsilon_2, \qquad t \geqslant \omega_h. \tag{25}$$

PROOF: By the hypothesis we may apply Lemmas 1 and 2 to obtain $\| W(\epsilon, t)\| \leqslant \gamma_3 e^{\sigma_0 t}$ if $\sigma_0 > 0$ and $\| W(\epsilon, t) - W(\epsilon, \infty)\| \leqslant \gamma_3 e^{\sigma_0 t}$ if $\sigma_0 \leqslant 0$ for all $0 < \epsilon \leqslant \epsilon_2$ where γ_3 is independent of ϵ. If $\sigma_0 > 0$ it follows from (8) that

$$\begin{aligned} \| x(\epsilon, t)\| \leqslant \| x_0(\omega_h)\| &+ \gamma_3 e^{-\omega_h \sigma} \left\| \sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h) \right\| e^{\sigma_0 t} \\ &+ \left\| \sum_{k=1}^{h} \gamma_3 e^{-(\omega_k + \alpha)\sigma_0}\, dVB_k x_0(\alpha) \right\| e^{\sigma_0 t} \\ &+ \left\| \int_{\omega_h}^{\infty} \gamma_3 e^{-\sigma_0 \alpha}\, dVf(\alpha) \right\| e^{\sigma_0 t}. \end{aligned}$$

Hence (25) holds if $\sigma_0 > 0$. Let $\sigma_0 \leqslant 0$ then we can rewrite (8) as

$$x(\epsilon, t) = x_0(\omega_h) + \sum_{k=0}^{k} W(\epsilon, \infty)B_k x_0(\omega_h) + W(\epsilon, \infty)f(t)$$

$$+ \{W(\epsilon, t - \omega_h) - W(\epsilon, \infty)\} \left\{\sum_{k=0}^{h} B_k x_0(\omega_h - \omega_k) + f(\omega_h)\right\}$$

$$+ \sum_{k=1}^{h} \int_{\omega_h - \omega_k}^{\omega_h} \{W(\epsilon, t - \omega_k - \alpha) - W(\epsilon, \infty)\}\, dB_k x_0(\alpha)$$

$$+ \int_{\omega_h}^{t} \{W(\epsilon, t - \omega_h) - W(\epsilon, \infty)\}\, df(\alpha).$$

The theorem will follow in the same way as in the above case if it can be shown that the first two terms are zero. If $\sigma_0 = 0$ we can shift all the contour integrals to the left and so there is no loss in generality in assuming $\sigma_0 < 0$.

Consider

$$R(a) = \frac{-1}{2\pi i} \int_{(a)} H(\epsilon, s)^{-1} \sum_{k=0}^{h} B_k e^{-\omega_k s} \frac{ds}{s}, \qquad \sigma_0 < a.$$

From the fact that $W(\epsilon, t) = 0$ for $t \leqslant 0$, $\epsilon \neq 0$ and from (11) it follows that

$$R(a) = \begin{cases} 0 & \text{if } a > 0 \\ \sum_{k=0}^{h} W(\epsilon, \infty)B_k & \text{if } \sigma_0 < a < 0. \end{cases}$$

Since $H(\epsilon, s) = s\, E(\epsilon) - \sum_{k=0}^{h} e^{-s\omega_k} B_k$

$$R(a) = \frac{1}{2\pi i} \int_{(a)} I \frac{ds}{s} - \frac{1}{2\pi i} \int_{(a)} H(\epsilon, s)^{-1} E(\epsilon)\, ds$$

where I is the identity matrix. By direct computation

$$\int_{(a)} I \frac{ds}{s} = \begin{cases} -\frac{1}{2}I & \text{if } a < 0 \\ \frac{1}{2}I & \text{if } a > 0. \end{cases}$$

and since $H(\epsilon, s)^{-1}E(s)$ has no poles for $s > \sigma_0$ the integral

$$\frac{1}{2\pi i} \int_{(a)} H(\epsilon, s)^{-1} E(\epsilon)\, ds$$

is independent of a for $a > \sigma_0$ and so equal to $\frac{1}{2}I$. Hence $R(a) = -I$ for $a < 0$.

Thus

$$x_0(\omega_h) + \sum_{k=0}^{h} W(\epsilon, \infty) B_k x_0(\omega_h) = 0$$

and so the second part of the theorem (i.e., with $\sigma_0 \leqslant 0$) follows.

3. Necessary and Sufficient Conditions for Regularity

Definitions of $[\sigma_0, \sigma_1]$-regularity and of σ_0-complete regularity (as $\epsilon \to 0+$) of Eq. (1) were given in Definition 1 of Section 2, in terms of Condition A. In this section we shall give several other characterizations of regularity, including one which can be used as a practical test in at least some examples. One such example is treated at the end of the section. The methods used are similar to those in Cooke [1], Section 8. A theorem on Diophantine approximation plays an important role.

As observed in the proof of Lemma 3, the function $\theta(\epsilon, s)$ defined by

$$\theta(\epsilon, s) = \det \left\{ \epsilon s I_m - \sum_{k=0}^{h} e^{-\omega_k s} B_{k22} \right\}$$

can be written in the form

$$\theta(\epsilon, s) = \theta_0(\epsilon s) + \sum_{\nu=1}^{r} e^{-\alpha_\nu s} \theta_\nu(\epsilon s),$$

where each $\theta_\nu(z)$ is a polynomial in z and

$$\theta_0(z) = \det(z I_m - B_{022})$$

is a polynomial of degree m, the characteristic polynomial of the matrix B_{022}. The numbers α_ν are combinations of the ω_k with integer coefficients,

$$\alpha_\nu = \sum_{k=0}^{h} j_k \omega_k, \qquad j_k \geqslant 0,$$

which can be supposed arranged so that

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r .$$

The polynomials $\theta_\nu(z)$, $1 \leqslant \nu \leqslant r$, have degree $m - 1$ at most.

Along with the above condition A, we now consider a new *Condition A'*. There exists a postive number δ such that

$$\left| \theta_0(iy) + \sum_{\nu=1}^{r} e^{-\alpha_\nu s}\theta_\nu(iy) \right| > 0$$

for $-\infty < y < \infty$, $\sigma_0 - \delta \leqslant \mathrm{Re}(s) \leqslant \sigma_1 + \delta$.

That is, for every y the exponential polynomial

$$\sum_{\nu=0}^{r} \theta_\nu(iy) e^{-\alpha_\nu s}$$

(which has constant coefficients for fixed y) has no zero for s in a strip enclosing the strip $\sigma_0 \leqslant \mathrm{Re}(s) \leqslant \sigma_1$.

We shall now establish the equivalence of conditions A and A'. The proof will be given in detail, since it is more difficult than the corresponding proof in [1] (Theorem 4), although similar in basic approach.

THEOREM 5. *Let σ_0 and σ_1 be finite real numbers, $\sigma_0 \leqslant \sigma_1$. Then Condition A and Condition A' are equivalent.*

PROOF: We shall first show that the negation of A' implies the negation of A. If A' fails, then there are sequences $\{\delta_j\}$, $\{\sigma_j\}$, $\{\tau_j\}$, and $\{y_j\}$ such that $\delta_j \to 0+$, $\sigma_0 - \delta_j \leqslant \sigma_j \leqslant \sigma_1 + \delta_j$, and

$$\theta_0(iy_j) + \sum_{\nu=1}^{r} e^{-\alpha_\nu(\sigma_j + i\tau_j)}\theta_\nu(iy_j) = 0. \tag{26}$$

If $\{y_j\}$ has zero as a limit point, then on a subsequence we have

$$\lim_{j\to\infty} \left[\theta_0(iy_j) + \sum_{\nu=1}^{r} e^{-\alpha_\nu(\sigma_j+i\tau_j)}\theta_\nu(iy_j)\right] = \lim_{j\to\infty} \left[\theta_0(0) + \sum_{\nu=1}^{r} e^{-\alpha_\nu(\sigma_j+i\tau_j)}\theta_\nu(0)\right]$$

$$= \lim_{j\to\infty} \theta(0, \sigma_j + i\tau_j) = 0.$$

Since the sequence $\{\sigma_j\}$ has all its limit points on the interval $[\sigma_0, \sigma_1]$, this contradicts Condition A.

The sequence $\{y_j\}$ for which (26) holds cannot be unbounded, since on an unbounded sequence $\theta_0(iy_j)$ would grow more rapidly than $\theta_\nu(iy_j)$ $(1 \leqslant \nu \leqslant r)$, because θ_0 is a polynomial of higher degree than θ_ν.

We are left with the case in which the sequence $\{y_j\}$ has only finite nonzero limit points. Let y be one of them, and choose a subsequence on which

$\sigma_j \to \sigma, y_j \to y$, where $\sigma_0 \leqslant \sigma \leqslant \sigma_1$. From (26) we get

$$\theta_0(iy) + \sum_{\nu=1}^{r} e^{-\alpha_\nu(\sigma+i\tau_j)}\theta_\nu(iy) = o(1), j \to \infty. \tag{27}$$

Suppose that the numbers $\alpha_1, \alpha_2, ..., \alpha_r$ satisfy exactly $r - \rho$ linearly independent relations

$$\sum_{\nu=1}^{r} c_{\mu\nu}\alpha_\nu = 0 \qquad (\mu = 1, 2, ..., r - \rho) \tag{28}$$

with integral coefficients $c_{\mu\nu}$. The numbers are rationally independent if and only if $r - \rho = 0$. It follows from (28) that

$$\sum_{\mu=1}^{r} c_{\mu\nu}\left(\frac{\alpha_\nu\tau_j}{2\pi}\right) = 0,$$

for $\mu = 1, 2, ..., r - \rho$ and $j = 1, 2, 3, ...$ Therefore, according to a known theorem on rational approximation,[2] given any positive number η_j, there are integers $N_{j1}, ..., N_{jr}$ and real numbers w_j such that

$$\left| \frac{\alpha_\nu}{2\pi} w_j - N_{j\nu} - \frac{\alpha_\nu\tau_j}{2\pi} \right| < \eta_j, \qquad (\nu = 1, ..., r; \ \ j = 1, 2, ...). \tag{29}$$

Moreover, given η_j, the number w_j can be taken positive or negative and of absolute value as large as desired.[3] We can therefore suppose that $\eta_j \to 0$ and $y/w_j \to 0+$ as $j \to \infty$.

Now let $\epsilon_j = y/w_j$, $z_j = \epsilon_j\sigma + iy$. Then as $j \to \infty$, we have $\epsilon_j \to 0+$, $z_j \to iy$, and $\theta_\nu(z_j) \to \theta_\nu(iy)$. Also using (29) we obtain

$$\begin{aligned} \alpha_\nu \frac{z_j}{\epsilon_j} &= \alpha_\nu\sigma + i\alpha_\nu w_j \\ &= \alpha_\nu\sigma + i(2\pi N_{j\nu} + \alpha_\nu\tau_j + 2\pi k_{\nu j}\eta_j), \end{aligned}$$

where $| k_{\nu j} | < 1$. Therefore

$$e^{-\alpha_\nu z_j/\epsilon_j} = e^{-\alpha_\nu(\sigma+i\tau_j)}e^{-2\pi i k_{\nu j}\eta_j} = e^{-\alpha_\nu(\sigma+i\tau_j)}[1 + o(1)], \ \ j \to \infty,$$

and consequently

$$\theta_0(z_j) + \sum_{\nu=1}^{r} \theta_\nu(z_j)e^{-\alpha_\nu z_j/\epsilon_j} = \theta_0(iy) + \sum_{\nu=1}^{r} \theta_\nu(iy)e^{-\alpha_\nu(\sigma+i\tau_j)} + o(1), \qquad j \to \infty.$$

[2] Cf. Perron, [4], Satz 65. The condition of Satz 65 is satisfied with all $g^\nu = 0$.

[3] See the argument in Perron, *op. cit.*, §43, p. 163.

By Equation (27), this is $o(1)$ as $j \to \infty$. Putting $\epsilon_j s_j = z_j$, we get $\theta(\epsilon_j, s_j) = o(1)$ as $j \to \infty$, which contradicts Condition A since $\sigma_0 \leqslant \mathrm{Re}(s_j) \leqslant \sigma_1$.

Now it has to be shown that the negation of A implies the negation of A'. If A fails, there exist sequences $\{\epsilon_j\}$ and $\{s_j\}$ such that $\epsilon_j \to 0+$, $\sigma_0 \leqslant \mathrm{Re}(s_j) \leqslant \sigma_1$, and $\theta(\epsilon_j, s_j) \to 0$ as $j \to \infty$. If $\epsilon_j = 0$ for $j \geqslant j_0$, then $\theta(0, s_j) \to 0$, that is

$$\theta_0(0) + \sum_{\nu=1}^{r} \theta_\nu(0) e^{-\alpha_\nu s_j} = o(1), \qquad j \to \infty. \tag{30}$$

If not, then on a subsequence we have $\epsilon_j > 0$, $\epsilon_j \to 0+$. Let $z_j = s_j \epsilon_j = \epsilon_j x_j + i y_j$. Then

$$\theta_0(z_j) + \sum_{\nu=1}^{r} \theta_\nu(z_j) e^{-\alpha_\nu z_j/\epsilon_j} = o(1), \qquad j \to \infty. \tag{31}$$

Again the sequence $\{y_j\}$ cannot be unbounded, since θ_0 has higher degree than than θ_ν ($\nu = 1, \ldots, r$). Let y be a limit point, and choose a subsequence for which $y_j \to y$, $z_j \to iy$. Let $f(y; s)$ be defined by

$$f(y; s) = \theta_0(iy) + \sum_{\nu=1}^{r} \theta_\nu(iy) e^{-\alpha_\nu s}$$

It follows that in the case $\epsilon_j = 0$ for $j \geqslant j_0$ we have

$$f(0; s_j) = o(1), \qquad j \to \infty,$$

and in the contrary case

$$f(y; s_j) = o(1), \qquad j \to \infty.$$

In either case, s_j cannot be bounded away from the set of zeros of the exponential polynomial $f(\cdot\,; s)$, hence $f(\cdot\,; s)$ has a zero in every strip $\sigma_0 - \delta \leqslant \mathrm{Re}(s) \leqslant \sigma_1 + \delta$ ($\delta > 0$). This contradicts Condition A'.

THEOREM 6. *In order that Eq.* (1) *be σ_0-completely regular, it is sufficient that it be $[\sigma_0, \sigma_1]$-regular for each $\sigma_1 \geqslant \sigma_0$, and that all characteristic roots of the matrix B_{022} have negative real parts.*

PROOF: For any $\epsilon > 0$, consider the regions

$$\text{I:} \quad |s| \geqslant \tau_0/\epsilon, \qquad \mathrm{Re}(s) \geqslant \sigma,$$

$$\text{II:} \quad |s| \leqslant \tau_0/\epsilon, \qquad \mathrm{Re}(s) \geqslant \sigma,$$

where we shall choose τ_0 and σ large and independent of ϵ.

Select τ_0 so large that in region I, the polynomial $\theta_0(\epsilon s)$ is dominated by its leading term, that is, $|\theta_0(\epsilon s)| \geqslant \frac{1}{2}|\epsilon s|^m \geqslant \frac{1}{2}\tau_0{}^m$. Since $|\theta_\nu(\epsilon s)| \leqslant c|\epsilon s|^{m-1}$ ($\nu = 1, ..., r$), τ_0 can be selected large enough that $|\theta(\epsilon, s)| \geqslant \frac{1}{4}|\epsilon s|^m \geqslant \frac{1}{4}\tau_0{}^m$ for s in I, $\sigma \geqslant 0$, and any $\epsilon > 0$.

For any $\sigma \geqslant 0$, $\epsilon > 0$, $|\theta_0(\epsilon s)|$ has a positive lower bound γ in region II, since all characteristic roots of B_{022}, that is, zeros of $\theta_0(z)$, have negative real parts. Since $\theta_\nu(\epsilon s)$ is bounded in II ($\nu = 1, ..., r$), there is *a* σ^* independent of ϵ such that $|\theta(\epsilon, s)| \geqslant \frac{1}{2}\gamma$ for s in II and $\sigma \geqslant \sigma^*$.

Combining these conclusions, we have $|\theta(\epsilon, s)| \geqslant \gamma_1$ for $\operatorname{Re}(s) \geqslant \sigma^*$, $\epsilon > 0$, where γ_1 is a fixed positive constant. If σ^* is large enough the same relation holds also for $\epsilon = 0$ and $\operatorname{Re}(s) \geqslant \sigma^*$, since $\theta(0, s) = \theta_0(0) + \sum e^{-\alpha_\nu s}\theta_\nu(0)$ and $|\theta_0(0)| = |\det(-B_{022})| > 0$. Finally for the σ^* chosen, Eq. (1) is $[\sigma_0, \sigma^*]$-regular. Condition A shows that there exist positive numbers ϵ_2 and γ_2 such that $|\theta(\epsilon, s)| \geqslant \gamma_2$, $(0 \leqslant \epsilon \leqslant \epsilon_2, \sigma_0 \leqslant \operatorname{Re}(s) \leqslant \sigma^*)$. Let $\gamma_3 = \min(\gamma_1, \gamma_2)$. Then we have $|\theta(\epsilon, s)| \geqslant \gamma_3$, $(0 \leqslant \epsilon \leqslant \epsilon_2, \sigma_0 \leqslant \operatorname{Re}(s) < \infty)$. This relation implies σ_0-complete regularity in the sense defined at the beginning of Part II.

Corollary. *If all characteristic roots of B_{022} have negative real parts, then Eq.* (1) *is σ_0-completely regular for every large real number σ_0.*

The proof is similar to that of the analogous result in [1], and is omitted. The following consequence of the Corollary is noteworthy.

Remark. *In the Convergence Theorem, Theorem* 3 *in Section* 2, *the hypothesis that Eq.* (1) *be σ_0-completely regular for some σ_0 can be replaced by the hypothesis that all characteristic roots of B_{022} have negative real parts.* In other words, a purely algebraic criterion is available for the convergence question.

The next theorem is a converse of Theorem 2. In statement and proof it differs somewhat from Theorem 6 in [1].

Theorem 7. *Suppose that* $\det H(0, s)$ *has at most a finite number of zeros in* $\sigma_0 \leqslant \operatorname{Re}(s) \leqslant \sigma_1$ *and there exist positive numbers ϵ_1 and γ_1 such that*

$$|\det H(\epsilon, s)| \geqslant \gamma_1 |s|^n \tag{32}$$

for $0 \leqslant \epsilon \leqslant \epsilon_1$, $s \in S[\sigma_0, \sigma_1]$. *Then Eq.* (1) *is $[\sigma_0, \sigma_1]$-regular. Moreover, if for every $\sigma_1 \geqslant \sigma_0$ there are numbers ϵ_1 and ν_1 for which* (32) *holds, and if all characteristic roots of B_{022} have negative real parts, then Eq.* (1) *is σ_0-completely regular.*

Proof: We argue by contradiction. If Eq. (1) is not $[\sigma_0, \sigma_1]$-regular, Condition A fails, and there are sequences $\{\epsilon_j\}$ and $\{s_j\}$ such that $\epsilon_j \to 0+$, $\sigma_0 \leqslant \operatorname{Re}(s_j) \leqslant \sigma_1$, and $\theta(\epsilon_j, s_j) \to 0$ as $j \to \infty$. Either Eq. (30) or Eq. (31)

holds. Recall from the proof of Theorem 2 that det $H(\epsilon, s) = \sum_{k=0}^{n} p_k(\epsilon, s)s^k$ where $p_k(\epsilon, s)$ is a polynomial in ϵs with coefficients which are exponential polynomials, and $p_n(\epsilon, s) = \theta(\epsilon, s)$. We consider several cases.

(1) It is impossible that $z_j = \epsilon_j s_j$ be unbounded as $j \to \infty$, since $\theta_0(z)$ is of higher degree than $\theta_\nu(z)$.

(2) If $\epsilon_j s_j$ is bounded but s_j is unbounded, that is, $\text{Im}(s_j)$ is unbounded, then $p_k(\epsilon_j, s_j)$ is bounded, and

$$| \det H(\epsilon_j, s_j)| = o(| s_j |^n), \qquad | s_j | \to \infty.$$

Since det $H(0, s)$ has at most a finite number of zeros in $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$, $_js$ cannot for large j lie in one of the circles surrounding these zeros, hence $s_j \in S[\sigma_0, \sigma_1]$ for j large. The above estimate therefore contradicts the hypothesis (32).

(3) If $\epsilon_j s_j$ and s_j are bounded, then $\epsilon_j s_j \to 0$; and for a suitable subsequence $s_j \to s$ where $\sigma_0 \leqslant \text{Re}(s) \leqslant \sigma_1$. From (30) or (31) we derive

$$\theta(0, s) = \theta_0(0) + \sum_{\nu=1}^{r} \theta_\nu(0)e^{-\alpha_\nu s} = 0.$$

Let $s = \sigma + i\tau$, and assume that Eq. (28) holds. Then, as in the proof of Theorem 5, given any positive number η_k, there are integers $N_{k\nu}$ and there is a real number w_k such that

$$\left| \frac{\alpha_\nu}{2\pi} w_k - N_{k\nu} - \frac{\alpha_\nu \tau}{2\pi} \right| < \eta_k, \qquad \nu = 1, ..., r; \quad k = 1, 2, ...$$

Moreover we can arrange that $\eta_k \to 0$, $w_k \to \infty$ as $k \to \infty$. Then

$$e^{-\alpha_\nu s} = e^{-\alpha_\nu(\sigma + i\tau)} = e^{-\alpha_\nu(\sigma + iw_k)}e^{2\pi i l_{k\nu}\eta_k}$$

where $| l_{k\nu} | < 1$. Since $\eta_k \to 0$ it follows that

$$\theta_0(0) + \sum_{\nu=1}^{r} \theta_\nu(0)e^{-\alpha_\nu(\sigma + iw_k)} = o(1), \qquad k \to \infty.$$

That is, $\theta(0, s_k) = o(1)$ as $k \to \infty$, where $s_k = \sigma + iw_k$, and this is impossible for the same reason as in (1).

The second part of Theorem 7 follows what has just been proved and from Theorem 6.

The next theorem shows that the uniform absence of zeros is of itself enough to imply regularity.

THEOREM 8. *Let* $\sigma_0 \leqslant \sigma_1$ *and suppose that* $\det H(0, s)$ *has at most a finite number of zeros in* $\sigma_0 \leqslant \mathrm{Re}(s) \leqslant \sigma_1$. *Also suppose that for some positive numbers* δ, ϵ_0, *the function* $\det H(\epsilon, s)$ *has no zeros in* $S[\sigma_0 - \delta, \sigma_1 + \delta]$, *for* $0 \leqslant \epsilon \leqslant \epsilon_0$. *Assume that all the characteristic roots of the matrix* B_{022} *have negative real parts. Then there are positive numbers* ϵ_1, γ_1 *such that*

$$| \det H(\epsilon, s)| \geqslant \gamma_1 | s |^n$$

for $0 \leqslant \epsilon \leqslant \epsilon_1$, $s \in S[\sigma_0, \sigma_1]$. *In particular, Eq.* (1) *is* $[\sigma_0, \sigma_1]$*-regular.*

PROOF: If there are no such numbers ϵ_1, γ_1, there are sequences $\{\epsilon_j\}$ and $\{s_j\}$ such that $\epsilon_j \geqslant 0$, $\epsilon_j \to 0+$, s_j is in $S[\sigma_0, \sigma_1]$, and

$$\lim_{j\to\infty} | \det H(\epsilon_j, s_j)| \, | s_j |^{-n} = 0. \tag{33}$$

We examine the various possible cases.

First, if $\{s_j\}$ is bounded, there is a subsequence such that $s_j \to s_0$ where $s_0 \in S[\sigma_0, \sigma_1]$. Then $\det H(\epsilon_j, s_j) \to \det H(0, s_0)$, and from (33) it follows that $\det H(0, s_0) = 0$. This is impossible, since $\det H(0, s)$ has no zeros in $S[\sigma_0, \sigma_1]$.

If $\{s_j\}$ is unbounded, and $\epsilon_j = 0$ for $j \geqslant j_0$, $| \det H(0, s_j)| | s_j |^{-n} \to 0$. Now if M is large enough, $\det H(0, s)$ has no zeroes in the intersection of $\sigma_0 \leqslant \mathrm{Re}(s) \leqslant \sigma_1$ and $| \mathrm{Im}(s)| \geqslant M$, and so $| \det H(0, s)| \geqslant \gamma_1 | s |^n$ in that region. This contradicts the previous relation.

If $\{s_j\}$ is unbounded and $\epsilon_j > 0$, $\epsilon_j \to 0+$, we consider several subcases. Let $s_j = x_j + iy_j$, and suppose first that $\epsilon_j y_j$ is unbounded. Pick a subsequence on whcih $\epsilon_j y_j \to \infty$, $x_j \to x$. Since $| \epsilon_j s_j | \to \infty$, we have

$$| \det H(\epsilon_j, s_j)| > \tfrac{1}{2}\epsilon_j^m | s_j |^{n+m},$$

as in the proof of Theorem 2. This contradicts (33).

Next suppose that $y_j \to \infty$ but $\epsilon_j y_j \to 0$. Since $\epsilon_j s_j \to 0$, and each $p_k(\epsilon, s)$ is a polynomial in ϵs with coefficients which are independent of ϵ and are exponentials in s, it follows that $p_k(\epsilon_j, s_j) = p_k(0, s_j) + o(1)$ as $j \to \infty$. Therefore

$$\det H(\epsilon_j, s_j) = \det H(0, s_j) + \sum_{k=0}^{n} o(1) s_j^k.$$

Since $| s_j | \to \infty$, and $s_j \in S[\sigma_0, \sigma_1]$, we have $| \det H(0, s_j)| \geqslant \gamma_1 | s_j |^n$ and $| \det H(\epsilon_j, s_j)| \geqslant \gamma_2 | s_j |^n$, where $\gamma_1 > 0$, $\gamma_2 > 0$. This contradicts (33).

It remains only to consider the case in which $y_j \to \infty$ and $\epsilon_j y_j$ has a finite nonzero limit point. Let this limit point be z and choose a subsequence such that $x_j \to x$, $\epsilon_j y_j \to z$. Take $\delta > 0$ and, as in [1], introduce the sequence of functions

$$f_j(s) = (s + s_j)^{-n} \det H(\epsilon_j, s + s_j),$$

which are regular in $|s| < \delta$ for all sufficiently large j since $|s_j| \to \infty$. We have

$$f_j(s) = \sum_{k=0}^{n} p_k(\epsilon_j, s + s_j)(s + s_j)^{k-n}.$$

Since $\epsilon_j(s + s_j) \to iz$ as $j \to \infty$ and $|s + s_j| \to \infty$, the family $\{f_j(s)\}$ is uniformly bounded for $|s| < \delta$, and by Montel's Selection Theorem (see [1]) there is a subsequence, which we again denote by $\{f_j(s)\}$, which converges uniformly in $|s| \leqslant \delta_1 < \delta$. Let $F(s)$ be the limit of this subsequence. Clearly

$$F(s) = \lim_{j\to\infty} p_n(\epsilon_j, s + s_j) = \lim_{j\to\infty} \theta(\epsilon_j, s + s_j)$$

$$= \theta_0(iz) + \sum_{\nu=1}^{r} \theta_\nu(iz)e^{-\alpha_\nu(s+s_j)} + o(1), \qquad j \to \infty,$$

where the $o(1)$ is uniform in $|s| \leqslant \delta_1$.

If $F(s)$ is identically zero, we have

$$\theta_0(iz) + \sum_{\nu=1}^{r} \theta_\nu(iz)\, e^{-\alpha_\nu s_j} e^{-\alpha_\nu s} = o(1), \qquad j \to \infty, |s| \leqslant \delta_1.$$

It follows that the coefficients tend to zero as $j \to \infty$, that is, $\theta_0(iz) = 0$ and

$$\lim_{j\to\infty} \theta_\nu(iz)e^{-\alpha_\nu s_j} = 0.$$

However, it is impossible to have $\theta_0(iz) = 0$, since iz cannot be a characteristic root of B_{022}. Thus $F(s)$ is not identically zero, and there is a δ_2, $0 < \delta_2 < \delta$, so that $F(s)$ has no zero on $|s| = \delta_2$. It now follows from Rouché's Theorem, as in [1], that for every large j, $f_j(s)$ and $F(s)$ have the same number of zeros inside $|s| = \delta_2$. Since $F(0) = 0$, by (33), there is at least one zero, and so $\det H(\epsilon_j, s + s_j) = 0$ for some s in $|s| \leqslant \delta_2$. Since $|s + s_j| \to \infty$, $s + s_j$ is in $S[\sigma_0 - \delta, \sigma_1 + \delta]$. This shows that for arbitrarily small values of ϵ there are zeros of $\det H(\epsilon, s)$ in $S[\sigma_0 - \delta, \sigma_1 + \delta]$. This contradicts the hypothesis, and the proof of Theorem 8 is complete.

The proof of the following lemma is similar to that of Lemma 7.1 in [1], and is omitted.

LEMMA 4. *Let C be a closed contour containing v zeros λ_i of* det $H(0, s)$, *of multiplicities μ_i, respectively. Let C_i denote a circle with center at λ_i of radius* *so small that no C_i intersects C or another C_j. Then there exists an $\epsilon_0 > 0$ such that for $0 \leqslant \epsilon \leqslant \epsilon_0$ the circle C_i contains zeros of* $\det H(\epsilon, s)$ *of total multiplicity μ_i ($i = 1, ..., \nu$), and the set inside C and outside all C_i contains no zero of* $\det H(\epsilon, s)$.

For each $\epsilon \geqslant 0$, we now let $M(\epsilon)$ denote the supremum of real parts of zeroes $\lambda(\epsilon)$ of $\det H(\epsilon, s)$:

$$M(\epsilon) = \sup[\operatorname{Re}(\lambda(\epsilon)) : \det H(\epsilon, \lambda(\epsilon)) = 0]. \tag{34}$$

Also, define

$$M^* = \limsup_{\epsilon \to 0+} M(\epsilon), \tag{35}$$

$$\sigma^* = \inf \sigma_0 , \tag{36}$$

the last taken over all σ_0 for which Eq. (1) is σ_0-completely regular.

THEOREM 9. *Define $M(0)$, M^*, and σ^* as in* (34), (35), *and* (36). *Assume that all characteristic roots of the matrix B_{022} have negative real parts. Then* $M^* = \max(M(0), \sigma^*)$.

The proof is almost identical to the proof of Theorem 8 in [1], and is therefore omitted.

We shall conclude this discussion with an example of the use of Condition A' to compute the number σ^*. Let

$$\theta_0(z) = 1 + z, \qquad \theta_1(z) = a, \qquad \theta_2(z) = b,$$

that is

$$\theta(\epsilon, s) = \epsilon s + 1 + ae^{-\alpha_1 s} + be^{-\alpha_2 s},$$

where a and b are real numbers and $0 < \alpha_1 < \alpha_2$. We assume that α_1 and α_2 are rationally independent. Let σ denote the unique real solution of the equation

$$|a| e^{-\alpha_1 \sigma} + |b| e^{-\alpha_2 \sigma} = 1. \tag{37}$$

We shall prove that $\sigma^* = \sigma$. Condition A' for $[\sigma_0 , \sigma_1]$-regularity requires that for some δ the equation

$$1 + iy + ae^{-\alpha_1 s} + be^{-\alpha_2 s} = 0 \tag{38}$$

have no solution in the strip $\sigma_0 - \delta \leqslant \operatorname{Re}(s) \leqslant \sigma_1 + \delta$, for any real y. Take $\delta > 0$, $\sigma_0 > \sigma + \delta$, where σ satisfies (37). Then for $\operatorname{Re}(s) \geqslant \sigma_0 - \delta (\delta > 0)$ we have

$$|ae^{-\alpha_1 s} + be^{-\alpha_2 s}| \leqslant |a| e^{-\alpha_1(\sigma_0 - \delta)} + |b| e^{-\alpha_2(\sigma_0 - \delta)} < 1,$$

and therefore Eq. (38) cannot be satisfied. Hence we have $[\sigma_0 , \sigma_1]$-regularity for every $\sigma_1 > \sigma_0 > \sigma + \delta$, and since the root $z = -1$ of $\theta_0(z)$ is negative, it follows from Theorem 6 that our equation is σ_0-completely regular if $\sigma_0 > \sigma + \delta$. Hence $\sigma^* \leqslant \sigma$.

On the other hand, take $\sigma_0 < \sigma$, so that

$$|a|e^{-\alpha_1\sigma_0} + |b|e^{-\alpha_2\sigma_0} > 1.$$

For $s = \sigma_0 + i\tau$, we get

$$1 + \mathrm{Re}(ae^{-\alpha_1 s} + be^{-\alpha_2 s}) = 1 + ae^{-\alpha_1\sigma_0}\cos\alpha_1\tau + be^{-\alpha_2\sigma_0}\cos\alpha_2\tau.$$

Since α_1 and α_2 are rationally independent, there is a τ for which this is as nearly equal to $1 - |a|\exp(-\alpha_1\sigma_0) - |b|\exp(-\alpha_2\sigma_0)$ as desired, and another for which it is as nearly equal to 1 as desired, and therefore there is a τ for which

$$1 + \mathrm{Re}(ae^{-\alpha_1 s} + be^{-\alpha_2 s}) = 0.$$

Then if we define y by

$$y = -\mathrm{Im}(ae^{-\alpha_1 s} + be^{-\alpha_2 s}),$$

we see that $s = \sigma_0 + i\tau$ is a solution of Eq. (38) for this y. Hence Condition A' is violated and the equation is not σ_0-completely regular. Since this is true for any $\sigma_0 < \sigma$, it follows that $\sigma^* \geqslant \sigma$, which together with the previous result $\sigma^* \leqslant \sigma$ yields $\sigma^* = \sigma$.

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