JOURNAL OF DIFFERENTIAL EQUATIONS **154**, 140–156 (1999)
Article ID jdeq.1998.3565, available online at http://www.idealibrary.com on IDEAL®

Bridges between the Generalized Sitnikov Family and the Lyapunov Family of Periodic Orbits*

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Received January 20, 1998; accepted July 1, 1998

The linearization of the spatial restricted three–body problem at the collinear equilibrium point $\mathscr{L}_2$ has two pairs of pure imaginary eigenvalues and one pair of real eigenvalues so the center manifold is four dimensional. By the classical Lyapunov center theorem there are two families of periodic solutions emanating from this equilibrium point. Using normal form techniques we investigate the existence of bridges of periodic solutions connecting these two Lyapunov families. A bridge is a third family of periodic solutions which bifurcates from both the Lyapunov families. We show that for the mass ratio parameter μ near 1/2 and near 0 there are many bridges of periodic solutions. © 1999 Academic Press

Key Words: periodic solutions; restricted three–body problem; KAM tori; Hill's lunar problem; normal form.

1. INTRODUCTION

In general periodic solutions of Hamiltonian systems are not isolated. They are typically found in families parameterized by the Hamiltonian

* The research of the first and third authors was partially supported by a DGICYT grant number PB96–1153. The research of the second author was partially supported by grants from the National Science Foundation and the Taft Foundation.

0022-0396/99 $30.00

and/or the period and often as the parameter is varied a multitude of other families of periodic solutions bifurcate from the original family. The evolution of these families can be quite complicated. The classic numerical investigation of the families of periodic solutions connected to $\mathscr{L}_4$ in the planar restricted three–body problem by Deprit and Henrard [8] is a case in point.

For this paper the families of periodic solutions will be parameterized by h the value of the Hamiltonian. If $\mathscr{F}_1(h)$ and $\mathscr{F}_2(h)$ are two one–parameter families of periodic solutions of a Hamiltonian system then a *bridge between* $\mathscr{F}_1$ *and* $\mathscr{F}_2$ is a third one-parameter family $\mathscr{F}_3(h)$ connecting the two, i.e. there exist h_1 and h_2 such that $\mathscr{F}_3(h)$ exists for $h_1 < h < h_2$ (or $h_2 < h < h_1$) and $\mathscr{F}_3(h)$ bifurcates from $\mathscr{F}_1(h)$ at $h = h_1$ and from $\mathscr{F}_2(h)$ at $h = h_2$. In general the period of the solutions in the bridge will be much longer than the periods in either of the other two families.

We will investigate bridges of periodic solutions in the Hamiltonian system of three degrees of freedom defined by the spatial circular restricted three–body problem near the $\mathscr{L}_2$ equilibrium point for values of the mass parameter μ near 1/2 and near 0. The linear system at $\mathscr{L}_2$ contains three two–dimensional invariant planes, two of them containing harmonic oscillators, and the third one having a hyperbolic saddle.

The spatial circular restricted three–body problem is defined by two positive masses $1-\mu$ and μ (called the *primaries*) which move in circular orbits around their center of mass, and of a massless particle (the *infinitesimal*) which is attracted by the gravitational force of the primaries but the infinitesimal does not perturb the primaries motion. The position and momentum of the infinitesimal in the usual rotating coordinates will be denoted by $(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3)$. In these coordinates the primaries are fixed on the q_1 axis—see (3.7).

It is known that for the planar circular restricted three–body problem a family of periodic orbits emerges from the equilibrium point $\mathscr{L}_2$, this family is usually called the Lyapunov family at $\mathscr{L}_2$. Additional information on this family can be found in Siegel and Moser [15] or Szebehely [17]. For the planar case the linear system at $\mathscr{L}_2$ has eigenvalues $\pm\omega_1 i$ and $\pm\lambda$. The Lyapunov family of periodic solutions is associated to the imaginary eigenvalues $\pm\omega_1 i$.

For the spatial problem and for $\mu = 1/2$ if we choose initial position and velocity of the infinitesimal mass on the q_3-axis its motion remains forever on this axis. The study of such a motion is called the *circular Sitnikov problem*. In fact the existence of a family of periodic orbits living on the q_3-axis is well–known, and it will be called the *Sitnikov family*. For more details, see Sitnikov [16], Alekseev [2] and Moser [12].

For the spatial case the linear system at $\mathscr{L}_2$ has eigenvalues $\pm\omega_1 i$, $\pm\omega_2 i$ and $\pm\lambda$. Now, again associated to the imaginary eigenvalues $\pm\omega_1 i$ there is the planar Lyapunov family, and associated to the imaginary eigenvalues $\pm\omega_2 i$ there is a family of periodic solutions emanating from $\mathscr{L}_2$ called the *generalized Sitnikov family* which coincides with the Sitnikov family when $\mu = 1/2$.

The main goal of this paper is to prove the existence of bridges connecting the generalized Sitnikov and Lyapunov families for values of the mass parameter μ near 1/2 and near 0. In [5], bridges of periodic orbits connecting the Sitnikov family and the Lyapunov family are found numerically. The key tools for proving this will be normalization by Lie transforms, Mathematica and a result of Meyer and Palmore [11], who proved the existence of this kind of bridges between the two Lyapunov families near the triangular equilibrium points $\mathscr{L}_4$ or $\mathscr{L}_5$ of the planar circular restricted three–body problem.

2. BRIDGES CONNECTING LYAPUNOV FAMILIES

In this section we state a variation of one of the main results of [11]. Consider an analytic Hamiltonian system of three–degrees of freedom with an equilibrium point at the origin with Hamiltonian

$$H(x, y) = \sum_{j=2}^{\infty} H_j(x, y), \tag{1}$$

where H_j is a homogeneous polynomial of degree j in $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)$. Assume that the linearized system with Hamiltonian H_2 has two pair of pure imaginary eigenvalues $\pm\omega_1 i$, $\pm\omega_2 i$ and one pair of real eigenvalues $\pm\lambda$. If $\omega_1 \neq \omega_2$, $\omega_1\omega_2 \neq 0$, and $\lambda \neq 0$ then we can assume that a linear symplectic change of coordinates has been made so that

$$H_2 = \omega_1 I_1 + \omega_2 I_2 + \lambda I_3, \tag{2}$$

where

$$I_1 = \tfrac{1}{2}(x_1^2 + y_1^2), \qquad I_2 = \tfrac{1}{2}(x_2^2 + y_2^2), \qquad I_3 = x_3 y_3.$$

If in addition $k_1\omega_1 + k_2\omega_2 \neq 0$ for all integers k_1, k_2 such that $|k_1| + |k_2| \leqslant 4$ then we can assume that a symplectic polynomial change of coordinates has been made so that

$$H_3 = 0, \qquad H_4 = \tfrac{1}{2}(AI_1^2 + 2BI_1 I_2 + CI_2^2) + D_1 I_1 I_3 + D_2 I_2 I_3 + D_3 I_3^2. \tag{3}$$

Thus, we can assume that the Hamiltonian is in Birkhoff normal form through terms of order four. If in addition the Hamiltonian H depends analytically on a parameter δ, and the above assumptions hold when $\delta = 0$ then there is a δ_0 such that the change of variables and the quantities ω_1, ω_2, λ, A, B, C, D_1, D_2, D_3 are analytic in δ for $|\delta| \leqslant \delta_0$—see [6, 10] for details.

Since two of the six eigenvalues of the linearized system have nonzero real part the system admits a four dimensional center manifold [7] and the Hamiltonian on this center manifold is

$$H = \omega_1 I_1 + \omega_2 I_2 + \tfrac{1}{2}(AI_1^2 + 2BI_1 I_2 + CI_2^2) + \text{h.o.t.} \tag{4}$$

There are many theorems about Hamiltonian systems of the form discussed above. We have applications of the following theorems in mind.

THEOREM 2.1 (Lypunov's Center Theorem [9]). *If $\omega_1/\omega_2 \neq k$, $1/k$ for all nonzero integers k then there are two families of periodic solutions of the system whose Hamiltonian is* (1) *with H_2 as in* (2) *emanating from the origin with periods which limit to $2\pi/\omega_1$ and $2\pi/\omega_2$ at the origin.*

THEOREM 2.2 (Arnold's Theorem [3, 4]). *Assume H is of the form* (1) *with H_2, H_3, H_4 as in* (2) *and* (3) *and*

$$\Delta = A\omega_2^2 - 2B\omega_2\omega_1 + C\omega_1^2 \neq 0. \tag{5}$$

Then the flow admits smooth invariant two–tori on the center manifold and the flow on the center manifold has the origin as a stable equilibrium point.

If in addition H depends analytically on a parameter δ for $\delta < \delta_0$ and

$$\omega_1(\delta) = \omega_1(0) + \omega_1'(0)\delta + \text{h.o.t.}, \qquad \omega_2(\delta) = \omega_2(0) + \omega_2'(0)\delta + \text{h.o.t.},$$

then we define

$$M_1 = \frac{\omega_1(0)\,\omega_2'(0) - \omega_2(0)\,\omega_1'(0)}{A\omega_2(0) - B\omega_1(0)}, \qquad M_2 = \frac{\omega_1(0)\,\omega_2'(0) - \omega_2(0)\,\omega_1'(0)}{B\omega_2(0) - C\omega_1(0)}. \tag{6}$$

THEOREM 2.3 (Meyer–Palmore Theorem [11]). *If in addition to the hypothesis of Theorems 2.5 and 2.5 the quantities M_1 and M_2 are defined and nonzero with opposite sign then there exist bridges of periodic solutions between the two Lyapunov families.*

The conditions in Theorem 2.3 need explanation. Since the quantities M_1, M_2 are nonzero the ratio of the frequencies $\omega_1(\delta)/\omega_2(\delta)$ has nonzero derivative at $\delta = 0$ and so as the parameter δ varies the frequency ratio sweeps through rational values. Let $\delta_{a/b}$ denote the value of δ when the frequency ratio is rational, a/b. (In this informal discussion we assume that a and b are not small integers.) When $\delta = \delta_{a/b}$ the linearized system has the two normal mode families of periodic solutions (the linearized Lyapunov families) of period $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2$ and all the other solutions are periodic with the common period $T_c = 2a\pi/\omega_1 = 2b\pi/\omega_2$. The characteristic multipliers of the periodic solutions with period T_1 (resp. T_2) are a^{th} (resp. b^{th}) roots of unity.

The fact that M_1 and M_2 are defined implies that the periods vary as one moves along a Lyapunov family for fixed δ. The fact that they are of different sign implies that for δ near $\delta_{a/b}$ there is a unique periodic solution on one of the Lyapunov families whose multipliers are a^{th} roots of unity and there is a unique periodic solution on the other the Lyapunov families whose multipliers are b^{th} roots of unity. These are the candidates for the bifurcation orbits.

$\Delta \neq 0$ is a condition on the nonlinear terms. It is not only the twist condition of KAM theory but the twist condition needed to apply one of the variants of the Poincaré–Birkhoff fixed point theorem. It is this condition that implies that bifurcations actually occur at the candidates for bifurcating orbits in the previous paragraph and that these bifurcated orbits actually form a bridge between the two Lyapunov families.

This theorem has its strengths and its weakness. Its strength lies in the fact that hypothesis are only on the quadratic and quartic terms in the normalized Hamiltonian no matter what the frequency ratio is. Its weakness come from the fact that the existence is established by a fixed point theorem and so there is no uniqueness information. All the theorem says is that for each value of h there are at least two periodic solutions in the bridge family. If you wish to compute many more terms in the normal form then Schmidt [14] proves that generically there are two periodic solutions in the bridge one elliptic and one hyperbolic for each value of h. His theorem requires a different computation of the normal form for each ratio a/b.

3. THE HAMILTONIAN AT $\mathscr{L}_2$ FOR μ NEAR $\frac{1}{2}$

In this section we will apply Theorems 2.1, 2.2, and 2.3 to the spatial circular restricted problem at $\mathscr{L}_4$ at $\mu = 1/2$. If the masses of primaries $m_1 = 1-\mu$ and $m_2 = \mu$ are fixed at $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$, then the restricted three–body problem is defined by the Hamiltonian function

$$H(q_1, q_2, q_3, p_1, p_2, p_3, v) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - p_2 q_1 + p_1 q_2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{7}$$

where $q=(q_1, q_2, q_3)$ and $p=(p_1, p_2, p_3)$ are the position and generalized momentum of the infinitesimal and

$$r_1^2=(q_1+\mu)^2+q_2^2+q_3^2, \qquad r_2^2=(q_1-1+\mu)^2+q_2^2+q_3^2.$$

In this section we replace $1-\mu$ by $\frac{1}{2}+\nu$ and μ by $\frac{1}{2}-\nu$ so that ν small means μ is near 1/2. For $\nu=0$ the equilibrium point $\mathscr{L}_2$ is the origin of coordinates. Expanding the Hamiltonian H at the origin up to terms of order 4 we obtain $H=H_2+H_4+\cdots$ where

$$\begin{aligned} H_2 &= \tfrac{1}{2}(p_1^2+p_2^2+p_3^2)-p_2q_1+p_1q_2-8q_1^2+4q_2^2+4q_3^2 \\ H_4 &= -32q_1^4-12q_2^4-12q_3^4+96q_1^2q_2^2-24q_2^2q_3^2+96q_3^2q_1^2, \end{aligned}$$

where we have dropped the constant term.

The linearized system is $\dot{Q}=RQ=J\nabla H_2$ where $Q=(q^T, p^T)^T$ and

$$\begin{aligned} R &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & 0 & -1 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 \end{pmatrix}, \\ J &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{8}$$

The eigenvalues of R are $\pm\omega_1 i$, $\pm\omega_2 i$, and $\pm\lambda$ where

$$\omega_1=\sqrt{8\sqrt{2}-3}, \qquad \omega_2=2\sqrt{2}, \quad \lambda=\sqrt{8\sqrt{2}+3}.$$

Since the ratio ω_1/ω_2 is irrational Lyapunov's Center Theorem applies not only for $\nu=0$ but for a small range of ν. Even though the existence of these two families is a consequence of Lyapunov's Center Theorem we shall call the one associated to ω_1 the planar Lyapunov family and the one associated to ω_2 the generalized Sitnikov family.

Next we must put the quadratic part of the Hamiltonian when $\nu=0$ into the form (2) by a linear symplectic change of variables. A basis for $\mathbb{R}^6$ consisting of real and imaginary parts of eigenvectors of R is

$$
\begin{aligned}
v_1 &= (1, 0, 0, 0, -4\sqrt{2}-6, 0)^T,\\
v_2 &= (0, -\tfrac{1}{7}(4\sqrt{2}+5)\,\omega_1, 0, \tfrac{1}{7}(4\sqrt{2}-2)\,\omega_1, 0, 0)^T,\\
v_3 &= (0, 0, 1, 0, 0, 0)^T,\\
v_4 &= (0, 0, 0, 0, 0, -\omega_2)^T,\\
v_5 &= (1, -\tfrac{1}{7}(4\sqrt{2}-5)\,\lambda, 0, \tfrac{1}{7}(4\sqrt{2}+2)\,\lambda, 4\sqrt{2}-6, 0)^T,\\
v_6 &= (1, \tfrac{1}{7}(4\sqrt{2}-5)\,\lambda, 0, -\tfrac{1}{7}(4\sqrt{2}+2)\,\lambda, 4\sqrt{2}-6, 0)^T.
\end{aligned}
$$

Now we make a linear change of variables to bring the Hamiltonian H_2 into the form (2) by

$$(q_1, q_2, q_3, p_1, p_2, p_3)^T = MZ = M(x_1, x_2, x_3, y_1, y_2, y_3)^T, \tag{9}$$

where M is the symplectic matrix

$$\left(\frac{v_2}{\sqrt{-v_1^T J v_2}}, \frac{v_4}{\sqrt{-v_3^T J v_4}}, \frac{v_6}{\sqrt{-v_5^T J v_6}}, \frac{v_1}{\sqrt{-v_1^T J v_2}}, \frac{v_3}{\sqrt{-v_3^T J v_4}}, \frac{v_5}{\sqrt{-v_5^T J v_6}}\right).$$

The linearized equations become $\dot{Z} = SZ$ where

$$S = \begin{pmatrix} 0 & 0 & 0 & \omega_1 & 0 & 0\\ 0 & 0 & 0 & 0 & \omega_2 & 0\\ 0 & 0 & -\lambda & 0 & 0 & 0\\ -\omega_1 & 0 & 0 & 0 & 0 & 0\\ 0 & -\omega_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In the new variables

$$H_2 = \omega_1 I_1 + \omega_2 I_2 - \lambda I_3,$$

with

$$I_1 = \tfrac{1}{2}(x_1^2 + y_1^2), \qquad I_2 = \tfrac{1}{2}(x_2^2 + y_2^2), \quad I_3 = x_3 y_3.$$

Since $H_3 = 0$ to put the quartic terms into Birkhoff normal form we must write H_4 in the new x, y-variables (Appendix (25)) and then keep only the terms in H_4 which are functions of I_1, I_2, and I_3. This can be done by changing to action-angle coordinates I_1, $\theta_1 = \tan^{-1} y_1/x_1$, I_2, $\theta_1 = \tan^{-1} y_2/x_2$ and then ignoring all terms that contain an angle θ_1, θ_2 or contain x_3 and y_3 in any form other than as a power of $x_3 y_3$. See [6, 10] for a discussion of the normalization procedure. With the help of Mathematica we find that the normalized quartic terms are

$$H_4 = \frac{1}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) - \left(\frac{117 - 72\omega_2}{\alpha^2\omega_2} I_2 - \frac{11151}{8\alpha^2\beta^2} I_1\right) I_3$$

$$- \frac{71793 - 24912\omega_2}{32\alpha^4} I_3^2,$$

where

$$A = -\frac{71793 + 24912\omega_2}{16\beta^4}, \qquad B = -\frac{117 + 72\omega_2}{\beta^2\omega_2}, \quad C = -\frac{9}{2},$$

and

$$\alpha = \sqrt{(8 - 5\sqrt{2})\,\lambda}, \qquad \beta = \sqrt{(8 + 5\sqrt{2})\,\omega_2}.$$

Now we compute

$$\Delta = A\omega_2^2 - 2B\omega_2\omega_1 + C\omega_1^2 \approx -33.1785 \neq 0,$$

so Arnold's Theorem applies and there are invariant two-dimensional tori in the center manifold of $\mathscr{L}_2$ for $\mu \approx 1/2$.

In order to apply Theorem 2.3 to prove the existence of bridges connecting the generalized Sitnikov family with the Lyapunov family of periodic orbits near $\mathscr{L}_2(v)$, we need to compute the derivative of the eigenvalues of the equilibrium point $\mathscr{L}_2(v)$ with respect to v. To do that we write the Hamiltonian (1) in power series of v up to order 4 of the form

$$H = \sum_{k=0}^{4} H^k(q, p)\, v^k + O(v^5),$$

where we have expanded each $H^k(q, p)$ up to order 4 in the variables q_i and p_i—see (26). Finally we compute the equations of motion associated to H up to order 3 in q_i and p_i, and up to order 4 in v—see (27). With these expansions we compute the position of the equilibrium point $\mathscr{L}_2(v)$ of this system, and we obtain

$$q_1 = p_2 = \frac{24}{17} v + \frac{36064}{83521} v^3 + O(v^5),$$

$$q_2 = q_3 = p_1 = p_3 = 0.$$

The eigenvalues of the Hamiltonian system at $\mathscr{L}_2(v)$ are $\pm\omega_1(v)\,i$, $\pm\omega_2(v)\,i$, and $\pm\lambda(v)$ where

$$\omega_1(v)=\omega_1+\frac{63(8-17\sqrt{2})\,\lambda}{289\sqrt{119}}\,v^2+O(v^4),$$

$$\omega_2(v)=\omega_2\left(1-\frac{126}{289}\,v^2\right)+O(v^4),$$

$$\lambda(v)=\lambda-\frac{63(8+17\sqrt{2})\,\omega_1}{289\sqrt{119}}\,v^2+O(v^4).$$

This is an expansion in $\delta=v^2$ and thus

$$\omega_1'(0)=\frac{63(8-17\sqrt{2})\,\lambda}{289\sqrt{119}},\qquad \omega_2'(0)=-\frac{126}{289}\,\omega_2,$$

and so

$$\begin{aligned} M_1&=\frac{\omega_1(0)\,\omega_2'(0)-\omega_2(0)\,\omega_1'(0)}{A\omega_2(0)-B\omega_1(0)}\approx 0.021616,\\ M_2&=\frac{\omega_1(0)\,\omega_2'(0)-\omega_2(0)\,\omega_1'(0)}{B\omega_2(0)-C\omega_1(0)}\approx -0.022384. \end{aligned}\tag{10}$$

Since the M_1 and M_2 have opposite signs Theorem 2.3 applies and so there are bridges between the planar Lyapunov family and the generalized Sitnikov family for μ near 1/2.

4. THE HAMILTONIAN AT $\mathscr{L}_4$ FOR SMALL μ

The location of the primary of mass $1-\mu$ and the equilibrium point $\mathscr{L}_2$ of the restricted problem all tend to the origin as μ tends to zero. Thus it is useless to set $\mu=0$ in (7) when studying the equilibrium point for small μ. But Hill's lunar problem can be used since it can be considered as a limit of the restricted problem. To see this we shall make a sequence of symplectic coordinate changes and scaling.

In what follows we shall drop all constants from the Hamiltonian. In the restricted problem (7) move one primary to the origin by the change of coordinates

$$\begin{aligned} &q_1\to q_1+1-\mu, && q_2\to q_2, && q_3\to q_3,\\ &p_1\to p_1, && p_2\to p_2+1-\mu, && p_3\to p_3, \end{aligned}$$

so that the Hamiltonian becomes

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)-p_2q_1+p_1q_2-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}-(1-\mu)\,q_1, \qquad (11)$$

where

$$r_1^2=(q_1+1)^2+q_2^2+q_3^2, \qquad r_2^2=q_1^2+q_2^2+q_3^2.$$

By Newton's binomial series

$$\{1+u\}^{-1/2}=1-\tfrac{1}{2}u+\tfrac{3}{8}u^2+\cdots$$

so

$$-\frac{1-\mu}{\sqrt{(q_1+1)^2+q_2^2+q_3^2}}=-(1-\mu)\left\{1-q_1+q_1^2-\frac{1}{2}q_2^2-\frac{1}{2}q_3^2+\cdots\right\}$$

and the Hamiltonian becomes

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)-p_2q_1+p_1q_2-\frac{\mu}{r_2}-(1-\mu)\left\{q_1^2-\frac{1}{2}q_2^2-\frac{1}{2}q_3^2+\cdots\right\}. \qquad (12)$$

We consider the mass μ as a small parameter and distance to the primary to be small by scaling

$$q\to\mu^{1/3}q, \qquad p\to\mu^{1/3}p,$$

which is symplectic with multiplier $\mu^{-2/3}$ so the Hamiltonian becomes

$$H=L+O(\mu^{1/3}), \qquad (13)$$

where L is the Hamiltonian of Hill's lunar problem

$$L=\frac{1}{2}(p_1^2+p_2^2+p_3^2)-p_2q_1+p_1q_2-\frac{1}{\|q\|}-q_1^2+\frac{1}{2}(q_2^2+q_3^3). \qquad (14)$$

The Hamiltonian L has an equilibrium point at

$$(q_1,q_2,q_3,p_1,p_2,p_3)=(-3^{-1/3},0,0,0,-3^{-1/3},0),$$

which is the limit of the equilibrium point $\mathscr{L}_2$ as $\mu\to 0$ in the scaling given above. We will call this equilibrium point $\mathscr{L}_2$ also. First shift the origin to this equilibrium point by $q_1\to q_1-3^{-1/3}$, $p_2\to p_2-3^{-1/3}$ and then expand the Hamiltonian L about this equilibrium point up to terms of order 4 to obtain $L = L_2 + L_3 + L_4 + \cdots$ where

$$L_2 = \frac{1}{2}(-8q_1^2 + 4q_2^2 + 4q_3^2 + 2q_2 p_1 - 2q_1 p_2 + p_1^2 + p_2^2 + p_3^2),$$

$$L_3 = -\frac{3^{4/3}}{2}(2q_1^3 - 3q_1 q_2^2 - 3q_1 q_3^2),$$

$$L_4 = -\frac{3^{5/3}}{8}(8q_1^4 + 3q_2^4 + 3q_3^4 - 24q_1^2 q_2^2 - 24q_1^2 q_3^2 + 6q_2^2 q_3^2).$$

The linearized system is $\dot{Q} = RQ = J\nabla L_2$ where $Q = (q^T, p^T)^T$ and

$$R = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 8 & 0 & 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

The eigenvalues of R are $\pm\omega_1 i$, $\pm\omega_2 i$ and $\pm\lambda$, where

$$\omega_1 = \sqrt{2\sqrt{7} - 1}, \qquad \omega_2 = 2, \quad \lambda = \sqrt{2\sqrt{7} + 1}.$$

Since the ratio ω_1/ω_2 is irrational Lyapunov's Center Theorem applies not only for $\mu = 0$ but for small positive values of μ also. Again we shall call the one associated to ω_1 the planar Lyapunov family and the one associated to ω_2 the generalized Sitnikov family.

Since in this case $L_3 \neq 0$, the normalization is a bit harder, so instead of using real coordinates we shall use complex coordinates remembering the reality conditions. We must put the quadratic part of the Hamiltonian when $\mu = 0$ into the complex normal form by a linear symplectic change of variables. The eigenvectors of R are

$$v_1 = \bar{v}_2 = (-1 - \sqrt{7}, i(3 + \sqrt{7})\sqrt{2\sqrt{7} - 1}, 0,$$
$$-2i\sqrt{2\sqrt{7} - 1}, 4\sqrt{7} + 10, 0),$$

$$v_3 = \bar{v}_4 = (0, 0, i, 0, 0, 2),$$

$$v_5 = (1 - \sqrt{7}, (\sqrt{7} - 3)\sqrt{2\sqrt{7} + 1}, 0, 2\sqrt{2\sqrt{7} + 1}, 4\sqrt{7} - 10, 0),$$

$$v_6 = (1 - \sqrt{7}, -(\sqrt{7} - 3)\sqrt{2\sqrt{7} + 1}, 0, -2\sqrt{2\sqrt{7} + 1}, 4\sqrt{7} - 10, 0).$$

Now we make a complex linear symplectic (with multiplier i) change of variables to the Hamiltonian H_2 in the form (2.2) by

$$(q_1, q_2, q_3, p_1, p_2, p_3)^T = MZ = M(z_1, z_2, z_3, z_4, z_5, z_6)^T, \tag{16}$$

where M is the symplectic matrix

$$\left(\frac{v_2}{\sqrt{|v_1^T J v_2|}}, \frac{v_4}{\sqrt{|v_3^T J v_4|}}, \frac{v_5}{\sqrt{|v_5^T J v_6|}}, \frac{v_1}{\sqrt{|v_1^T J v_2|}}, \frac{v_3}{\sqrt{|v_3^T J v_4|}}, \frac{i v_6}{\sqrt{|v_5^T J v_6|}}\right).$$

The linearized equations become $\dot{Z} = SZ$ where S is the complex diagonal matrix,

$$S = \begin{pmatrix} i\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

In the new variables

$$L_2 = i\omega_1 I_1 + i\omega_2 I_2 - \lambda I_3, \tag{17}$$

with

$$I_1 = z_1 z_4, \qquad I_2 = z_2 z_5, \quad I_3 = z_3 z_6. \tag{18}$$

The reality conditions are

$$z_1 = \bar{z}_4, \qquad z_2 = \bar{z}_5, \quad z_3 = i z_6. \tag{19}$$

By the theory of Lie transforms $L = L_2 + L_3 + L_4 + \cdots$ can be put into complex normal form $L = L^2 + L^3 + L^4$ by finding generating functions W_1 and W_2 such that

$$0 = L^3 = L_3 + \{W_1, L_2\}, \qquad L^4 = L_4 + \{W_1, L_3\} + \{W_2, L_2\}.$$

In the normal form $L_2 = L^2$, $L_3 = 0$, and L^4 is a function of I_1, I_2, I_3 only. Let $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5, k_6)$, $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{k_4} z_5^{k_5} z_6^{k_6}$, and $\mathbf{d}_{\mathbf{k}} = (k_1 - k_4)\, i\omega_1 + (k_2 - k_5)\, i\omega_2 - (k_3 - k_6)\lambda$. If

$$L_3 = \sum \mathbf{a}_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \qquad W_1 = \sum \mathbf{b}_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \quad \text{with} \quad \mathbf{b}_{\mathbf{k}} = -\frac{\mathbf{a}_{\mathbf{k}}}{\mathbf{d}_{\mathbf{k}}},$$

then we have $0 = L^3 = L_3 + \{W_1, L_2\}$. Similarly if

$$L_2 + \{W_1, L_3\} = \sum \mathbf{c_k} \mathbf{z^k}, \qquad W_2 = \sum \mathbf{e_k} \mathbf{z^k},$$

with

$$\mathbf{e_k} = \begin{cases} -\dfrac{\mathbf{c_k}}{\mathbf{d_k}} & \text{when} \quad \mathbf{d_k} \neq 0 \\ 0 & \text{when} \quad \mathbf{d_k} = 0 \end{cases}$$

then we have $L^4 = L_4 + \{W_1, L_3\} + \{W_2, L_2\}$ and L^4 is a function of I_1, I_2, I_3 only.

Since the numeric coefficients kept as rational functions of surds become unwieldy we use Mathematica with floating point coefficients. We compute

$$\begin{aligned} L^4 = {} & 0.590564 i I_1^2 + 0.554764 i I_2^2 - 1.02741 i I_3^2 \\ & + 0.994856 i I_1 I_2 - 1.56157 I_2 I_3 - 1.7266 I_3 I_1. \end{aligned} \tag{20}$$

The terms L_2 and L_4 are not yet in the real normal form of (2) and (3), but we make yet another change of variables

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}} (x_1 - y_1 i), & z_4 &= \frac{1}{\sqrt{2}} (x_1 + y_1 i), \\ z_2 &= \frac{1}{\sqrt{2}} (x_2 - y_2 i), & z_5 &= \frac{1}{\sqrt{2}} (x_2 + y_2 i), \\ z_3 &= x_3, & z_6 &= i y_3, \end{aligned} \tag{21}$$

which is symplectic with multiplier $-i$ so that

$$\begin{aligned} L^2 &= L_2 = \omega_1 I_1 + \omega_2 I_2 - \lambda I_3 \\ L^4 &= 0.590564 I_1^2 + 0.554764 I_2^2 + 1.02741 I_3^2 \\ & \quad + 0.994856 I_1 I_2 - 1.56157 I_2 I_3 - 1.7266 I_3 I_1. \end{aligned} \tag{22}$$

where now

$$I_1 = \tfrac{1}{2}(x_1^2 + y_1^2), \qquad I_2 = \tfrac{1}{2}(x_2^2 + y_2^2), \quad I_3 = x_3 y_3.$$

Now we compute

$$\Delta = A\omega_2^2 - 2B\omega_2\omega_1 + C\omega_1^2 \approx 0.621151 \neq 0,$$

so Arnold's Theorem applies and there are invariant two-dimensional tori in the center manifold of Hill's problem at $\mathscr{L}_2$ and also in the center manifold of the restricted problem at $\mathscr{L}_2$ for μ near 0.

In order to apply Theorem 2.3 to prove the existence of bridges connecting the generalized Sitnikov family with the Lyapunov family for μ small, we need to compute the derivative of the eigenvalues linearized equation at the equilibrium point $\mathscr{L}_2(\mu)$ with respect to $\delta = \mu^{1/3}$.

The first correction term in (13) is

$$H = L + \frac{\delta}{2}\{2q_1^3 - 3q_1q_2^2 - 3q_1q_3^2\} + O(\delta^2). \tag{23}$$

With this we compute that the equilibrium point $\mathscr{L}_2$ is at

$$\begin{aligned} &q_1 = -3^{-1/3} + 3^{-5/3}\delta, && q_2 = 0, && q_3 = 0, \\ &p_1 = 0, && p_2 = -3^{-1/3} + 3^{-5/3}\delta, && p_3 = 0. \end{aligned}$$

We can then compute that

$$\begin{aligned} \omega_1(\delta) &= \sqrt{2\sqrt{7} - 1} + \frac{(35 - 2\sqrt{7})\, 3^{2/3}\sqrt{2\sqrt{7} - 1}}{126}\delta \\ &\approx 2.07159 + 1.0160\delta + \cdots, \\ \omega_2(\delta) &= 2 + \frac{3^{2/3}}{2}\delta \approx 2.0000 + 1.04004\delta + \cdots, \\ \lambda(\delta) &= \sqrt{2\sqrt{7} + 1} + \frac{(35 + 2\sqrt{7})\, 3^{2/3}\sqrt{2\sqrt{7} + 1}}{126}\delta \\ &\approx 2.50829 + 1.66840\delta + \cdots. \end{aligned}$$

Thus

$$\omega_1'(0) = \frac{(35 - 2\sqrt{7})\, 3^{2/3}\sqrt{2\sqrt{7} - 1}}{126} \approx 1.0160, \qquad \omega_2'(0) = \frac{3^{2/3}}{2} \approx 1.04004,$$

and so

$$\begin{aligned} M_1 &= \frac{\omega_1(0)\,\omega_2'(0) - \omega_2(0)\,\omega_1'(0)}{A\omega_2(0) - B\omega_1(0)} \approx 0.8133658, \\ M_2 &= \frac{\omega_1(0)\,\omega_2'(0) - \omega_2(0)\,\omega_1'(0)}{B\omega_2(0) - C\omega_1(0)} \approx -0.7937111. \end{aligned} \tag{24}$$

Since the M_1 and M_2 have opposite signs Theorem 2.3 applies and so there are bridges between the planar Lyapunov family and the generalized Sitnikov family for μ near 0.

5. APPENDIX A: ADDITIONAL FORMULAS

The Hamiltonian H_4 after the change of variables (9) is

$$\begin{aligned}
H_4 = & -\frac{3}{2}x_1^4 + \frac{21\omega_1}{2\beta^2}x_1^2x_2^2 + \frac{42}{\alpha\beta}(x_1^2x_2x_3 + x_1^2x_2y_3) - \frac{201+72\omega_1}{\beta^2\omega_1}x_1^2y_2^2 \\
& - \frac{3\lambda\omega_2}{2\alpha\beta}(x_1^2y_2x_3 - x_1^2y_2y_3) + \frac{285-72\omega_1}{2\alpha^2\omega_1}(x_1^2x_3^2 + x_1^2y_3^2) \\
& + \frac{72\omega_1 - 117}{\alpha^2\omega_1}x_1^2x_3y_3 - \frac{49}{2\beta^4}x_2^4 - \frac{49\omega_1}{2\alpha\beta^3}(x_2^3x_3 + x_2^3y_3) \\
& + \frac{1407+504\omega_1}{2\beta^4}x_2^2y_2^2 + \frac{42\lambda\omega_2}{\alpha\beta^3\omega_1}(x_2^2y_2x_3 - x_2^2y_2y_3) \\
& + \frac{504\omega_1 - 1701}{4\alpha^2\beta^2}(x_2^2x_3^2 + x_2^2y_3^2) + \frac{1113-504\omega_1}{2\alpha^2\beta^2}x_2^2x_3y_3 \\
& + \frac{2814+1008\omega_1}{\alpha\beta^3\omega_1}(x_2y_2^2x_3 + x_2y_2^2y_3) + \frac{21\lambda\omega_2}{\alpha^2\beta^2}(x_2y_2x_3^2 - x_2y_2y_3^2) \\
& - \frac{1505-504\omega_1}{\alpha^3\beta\omega_1}(x_2x_3^3 + x_2y_3^3) + \frac{1113-504\omega_1}{\alpha^3\beta\omega_1}(x_2x_3^2y_3 + x_2x_3y_3^2) \\
& - \frac{27291+9648\omega_1}{16\beta^4}y_2^4 - \frac{(201+72\omega_1)\lambda\omega_2}{2\alpha\beta^3\omega_1}(y_2^3x_3 - y_2^3y_3) \\
& + \frac{4557+2016\omega_1}{16\alpha^2\beta^2}(y_2^2x_3^2 + y_2^2y_3^2) + \frac{6699+2016\omega_1}{8\alpha^2\beta^2}y_2^2x_3y_3 \\
& + \frac{(285-72\omega_1)\lambda\omega_2}{4\alpha^3\beta\omega_1}(y_2x_3^3 - y_2y_3^3) \\
& - \frac{(519-216\omega_1)\lambda\omega_2}{4\alpha^3\beta\omega_1}(y_2x_3^2y_3 - y_2x_3y_3^2) \\
& + \frac{109440-38939\omega_1}{64\alpha^4\omega_1}(x_3^4 + y_3^4) + \frac{26899-9648\omega_1}{16\alpha^4}(x_3^3y_3 + x_3y_3^3) \\
& - \frac{71793-24912\omega_1}{32\alpha^4}x_3^2y_3^2,
\end{aligned} \tag{25}$$

where $\alpha = \sqrt{(8-5\sqrt{2})\lambda}$ and $\beta = \sqrt{(8+5\sqrt{2})\omega_1}$.

The terms in the expansion of the Hamiltonian up to order 4 in the q_i, p_i and v up to order 3 are

$$\begin{aligned}
H^0 &= \tfrac{1}{2}(p_1^2+p_2^2+p_3^2) - p_2 q_1 + p_1 q_2 + 4(-2q_1^2+q_2^2+q_3^2) \\
&\quad -4(8q_1^4+3q_2^4+3q_3^4) + 24(4q_1^2q_2^2 - q_2^2q_3^2 + 4q_3^2q_1^2), \\
H^1 &= 24q_1 + 80q_1(2q_1^2-3q_2^2-3q_3^2), \\
H^2 &= -16 + 144(-2q_1^2+q_2^2+q_3^2) + 2560q_1^2(-q_1^2+3q_2^2+3q_3^2) \\
&\quad -960(q_2^2+q_3^2)^2, \\
H^3 &= 224q_1 + 1920q_1(2q_1^2-3q_2^2-3q_3^2), \\
H^4 &= -64 + 1600(-2q_1^2+q_2^2+q_3^2) \\
&\quad +6720(-8q_1^4+24q_1^2q_2^2-3q_2^4+24q_1^2q_3^2-6q_2^2q_3^2-3q_3^4).
\end{aligned} \tag{26}$$

The expansion of the equations of motion in the q_i, p_i to order 3 and in v up to order 3 are

$$\begin{aligned}
\dot q_1 &= p_1 + q_2, \\
\dot q_2 &= p_2 - q_1, \\
\dot q_3 &= p_3, \\
\dot p_1 &= p_2 + 16q_1 + 128q_1^3 - 192q_1(q_2^2+q_3^2) + 24(-1-20q_1^2+10q_2^2+10q_3^2)\, v \\
&\quad +64q_1(9+160q_1^2-240q_2^2-240q_3^2)v^2 \\
&\quad +32(-7-360q_1^2+180q_2^2+180q_3^2)\, v^3 \\
&\quad +1280q_1(5+168q_1^2-252q_2^2-252q_3^2)\, v^4, \\
\dot p_2 &= -p_1 - 8q_2 + 48q_2(-4q_1^2+q_2^2+q_3^2) + 480q_1q_2 v \\
&\quad +96q_2(-3-160q_1^2+40q_2^2+40q_3^2)v^2 + 11520q_1q_2v^3 \\
&\quad +640q_2(-5-504q_1^2+126q_2^2+126q_3^2)\, v^4, \\
\dot p_3 &= -8q_3 + 48q_3(-4q_1^2+q_2^2+q_3^2) + 480q_1q_3 v \\
&\quad +96q_3(-3-160q_1^2+40q_2^2+40q_3^2)\, v^2 \\
&\quad +11520q_1q_3v^3 + 640q_3(-5-504q_1^2+126q_2^2+126q_3^2)\, v^4.
\end{aligned} \tag{27}$$

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