Stability of Skew Dynamical Systems*

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We study discrete skew product systems over an almost periodic system. We generalize the concepts of hyperbolic structure, shadowing, basic set, etc., in order to prove the natural generalization of Smale's Ω-stability theorem for skew product systems. Some of the lemmas are extensions of classical results on almost periodic differential equations.   © 1996 Academic Press, Inc.

1. INTRODUCTION

The theory of differentiable dynamical systems was motivated by the geometric theory of autonomous ordinary differential equations. In a similar manner, our motivation is the study of the geometric theory of almost periodic ordinary differential equations. Given an almost periodic system of a differential equation, the Miller–Sell construction [11, 15] defines a skew product flow with an almost periodic base. Since an almost periodic flow admits a cross section, the skew product flow admits a cross section also [10]. Thus we are led to the study of a discrete skew product system. In particular, we study the following system.

Let M be a compact n-dimensional differential manifold and T a compact metric space. Suppose

$$f\colon M \times T \to M \times T$$

is a homeomorphism, where

$$f(x, t) = (\Psi(x, t), V(t))$$

and $V: T \to T$ is a homeomorphism also. For fixed t let $\Psi(\cdot, t): M \to M$ be a diffeomophism and let its partial derivatives vary continuously in t. Let $V: T \to T$ be almost periodic. Here "almost periodic" is in the sense of Bohr [16]. Since V is almost periodic for any $t \in T$ and $k \in Z^+$, the orbit of V^k through t is dense in T. Such a system is called a *smooth skew dynamical system over the base* $V: T \to T$ or simply a *skew system*.

Our main result is a generalization of Smale's Ω-stability theory, which asserts that an ordinary dynamical system is Ω-stable if it satisfies axiom A and has the no-cycle property. Once the correct definitions are given the proofs are the same, mutatis mutandis, and therefore several proofs are skipped or only outlined. More complete proofs can be found in [21]. Several of the lemmas shed new light on the stability of the local structures of almost periodic systems under perturbations.

Since the system is defined on $M \times T$ where T is only a metric space, we cannot impose a hyperbolicity condition in the T direction. Therefore, our perturbations are only in the M direction and the map on T is held fixed. When thinking about almost periodic differential equations, this is like allowing the size of the harmonics of the external force to vary, while requiring the frequencies of the external force to be fixed. One of the main tools is the stable manifold theorem. The proof of the stable manifold theorem in [7] can be naturally extended here, but we will follow the proof in [14]. We like the geometry of this proof.

Actually, a further generalization is possible. Almost all that follows can be generalized to a fiber-preserving map of fiber bundle; however, the incremental generality does not warrant the additional notational complexity.

2. DEFINITIONS, STABLE MANIFOLDS, AND SHADOWING

In this section, we will extend the definitions of hyperbolic structure, shadowing orbit, pseudo-orbit, etc. in order to discuss the Stable Manifold Theorem and the Shadowing Lemma. This version of the Shadowing Lemma for skew systems is stronger than the version found in [10].

The *orbit* through (x, t) is the set $O(x, t) = \{f^k(x, t), k \in Z\}$. A sequence

$$\{(x_n, t_n), n = 0, \pm 1, \pm 2, ...\}$$

in $M \times T$ is called an α-*pseudo-orbit* if $t_{n+1} = V(t_n)$, and

$$d[f(x_n, t_n), (x_{n+1}, t_{n+1})] < \alpha \qquad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

A pseudo-orbit $\{(x_n, t_n), n = 0, \pm 1, \pm 2...\}$ is *β-shadowed* by an orbit $\{f^k(x, t), k \in Z\}$ if $t = t_0$ and $d[f^k(x, t), (x_k, t_k)] < \beta$ for $k = 0, \pm 1, \pm 2...$. The *limit set L*, *nonwandering set Ω*, and *invariant set* are defined in the usual way [17]. A closed invariant set Λ is *isolated* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{-\infty}^{\infty} f^n(\bar{U})$, where $\bar{U}$ is the closure of U.

We make the set of all functions $f, g: M \times T \to M \times T$, $f(x, t) = (\Phi(x, t), V(t))$, satisfying above assumptions into a metric space by using the $C^{1, 0}$ topology on this space, that is, two functions are close if and only if their partial derivatives with respect to x and their values are close.

Suppose f is a skew system on $M \times T$ and $(x, t) \in M \times T$. Since M is a differential manifold, M has a tangent space $T_x M$ at x. We call $T_x M \times \{t\}$ with the natural linear structure the *tangent space* of $M \times T$ at (x, t) and denote it by $T_{(x, t)} M \times T$. Because $f(x, t)$ has a continuous partial derivative with respect to x and $f(x, t)$ maps $M \times \{t\}$ to $M \times \{V(t)\}$ for fixed t, $Df_x(x, t)$ maps $T_{(x, t)} M \times T$ to $T_{f(x, t)} M \times T$. For convenience, we will use Df, instead of Df_x, to denote the partial derivative of $f(x, t)$ with respect to x. Because the tangent spaces $T_x M$ and $T_y M$ are isomorphic and essentially $T_{(x, t)} M \times T$ and $T_{(y, \tau)} M \times T$ are just $T_x M$ and $T_y M$, we will denote the tangent space $T_{(x, t)} M \times T$ by E for all (x, t) in $M \times T$.

Let Λ be an invariant set of f. Λ is a *hyperbolic* invariant set or f has a *hyperbolic structure on Λ* if there exist constants C, μ, λ, $0 < \mu < 1 < \lambda$, and a continuous map $P: \Lambda \to L(E, E)$, such that $P(x, t)$ is a linear projection operator that satisfies the following:

(1) $P(f(x, t)) Df(x, t) = Df(x, t) P(x, t)$,

(2) $\|Df^k(x, t) P(x, t)\| \leqslant C\mu^k$, where $(x, t) \in \Lambda, k \geqslant 0$,

(3) $\|Df^k(x, t)[I - P(x, t)]\| \geqslant C\lambda^k$, where $(x, t) \in \Lambda, k \geqslant 0$.

Let

$$E^s_{(x, t)} = P(x, t)(E), \qquad E^u_{(x, t)} = [I - P(x, t)](E).$$

We call $E^s_{(x, t)}$ and $E^u_{(x, t)}$ the *stable and unstable spaces at* (x, t) respectively. Thus, for any $(x, t) \in \Lambda$, the tangent space $T_{(x, t)} M \times T$ can be split into the direct sum of $E^s_{(x, t)}$ and $E^u_{(x, t)}$. Because $P(x, t)$ is continuous, $E^s_{(x, t)}$ and $E^u_{(x, t)}$ depend on (x, t) continuously. (1) implies $Df_{(x, t)}(E^s_{(x, t)}) = E^s_{f(x, t)}$ and $Df_{(x, t)}(E^u_{(x, t)}) = E^u_{f(x, t)}$. We will assume that the constant C in the definition of the hyperbolic structure is 1 by using the adapted norm [10]. In the following, when studying the map $f(y, t)$ near (x, t) for fixed t, we will use the splitting $E^s_{(x, t)} \times E^u_{(x, t)}$ of $T_{(x, t)} M \times T$. We will identify the ball of radius r and center (x, t),

$$B_{(x, t)}(r) = B^s_{(x, t)}(r) \times B^u_{(x, t)}(r),$$

in the tangent space $T_{(x, t)}M \times T$ with a neighborhood of (x, t) in $M \times \{t\}$, where $B^s_{(x, t)}(r)$ and $B^u_{(x, t)}(r)$ are the balls of radius r and center (x, t) in $E^s_{(x, t)}$ and $E^u_{(x, t)}$ respectively. In the following, all balls will be to be closed.

The set

$$W^s_{(x, t)} = \{(y, t) \in M \times \{t\}, \lim_{n \to \infty} d[f^n(x, t), f^n(y, t)] = 0\}$$

is called the *stable manifold at* (x, t). Since the proof of the local stable manifold theorems follows one of the standard proofs [14], we shall only sketch the proof here. Also see [21].

THEOREM 2.1 (Local Stable Manifold Theorem). *Assume* $f: M \times T \to M \times T$ *is a skew system and* Λ *is a closed hyperbolic invariant set with constants* $\mu, \lambda,$ *and* C. *Then there is an* $r > 0$ *such that for any* (x, t) *in* Λ *there is a* C^1 *embedded disk*

$$W^s_{(x, t)}(r) = \{(y, t) \in B_{(x, t)}(r): f^n(y, t) \in B_{f^n(x, t)}(r) \text{ for } n = 0, 1, 2...\}$$

which is tangent to $E^s_{(x, t)}$ *at* (x, t). *In fact,* $W^s_{(x, t)}(r)$ *is a graph of a* C^1 *function*

$$\varphi_{(x, t)}: B^s_{(x, t)}(r) \to B^u_{(x, t)}(r)$$

with $\varphi_{(x, t)}(0) = 0$ *and* $D\varphi_{(x, t)}(0) = 0$, *i.e.,*

$$W^s_{(x, t)}(r) = \{(u, \varphi_{(x, t)}(u)): u \in B^s_{(x, t)}(r)\}.$$

Moreover there is an $\alpha, 0 < \alpha < 1$, *such that*

$$\begin{aligned}
W^s_{(x, t)}(r) &= \bigcap_{n=0}^{\infty} f^{-n}(B_{f^n(x, t)}(r)) \\
&= \{(y, t) \in B_{(x, t)}(r): f^n(y, t) \in B_{f^n(x, t)}(r) \text{ and} \\
&\quad |f^n(y, t) - f^n(x, t)| \leqslant \alpha^n |x - y| \text{ for } n = 0, 1, 2...\}.
\end{aligned}$$

Also, $W^s_{(x, t)}(r)$ *continuously depends on* (x, t). *Furthermore,* $\varphi_{(x, t)}(z)$ *and* $D\varphi_{(x, t)}(z)$ *continuously depend on* (x, t).

Even though the theorem looks like the Stable Manifold Theorem in regular dynamical systems and the proof is similar, the geometric picture is different. Here f has a partial derivative with respect to x only and we talk about the hyperbolic structure only in the x direction: so we can expect a local stable manifold around (x, t) in $M \times \{t\}$.

Proof. For each point (x, t) in Λ, we can identify the ball $B_{(x, t)}(r)$ of radius $r > 0$ in the tangent space $T_{(x, t)}M \times T$ with a neighborhood of (x, t)

in $M \times \{t\}$. Because M is compact, we can choose $r > 0$ small enough so that we can identify the ball $B_{(x, t)}(r)$ of radius $r > 0$ in the tangent space $T_{(x, t)}M \times T$ with a neighborhood of (x, t) in $M \times \{t\}$ for all (x, t) in $M \times T$. Thus, the map from a neighborhood of (x, t) in $M \times \{t\}$ to a neighborhood of $f(x, t)$ in $M \times \{V(t)\}$ induces a C^1 map $f: B_{(x, t)}(r) \to B_{f(x, t)}(C_0 r)$, where $C_0 > \lambda$ is a constant. C_0 can be taken uniformly for all (x, t) in Λ since Λ is compact. We will choose r small enough so that for all $(x, t) \in \Lambda$ we can identify $B_{(x, t)}(C_0 r)$ in $T_{(x, t)}M \times T$ with a neighborhood of (x, t) in $M \times \{t\}$.

Since f is hyperbolic on Λ, there are μ and λ, $0 < \mu < 1 < \lambda$, such that

$$\|Df_{(x, t)} \,|\, B^s_{(x, t)}(r)\| < u \qquad \text{and} \qquad m(Df_{(x, t)} \,|\, B^u_{(x, t)}(r)) > \lambda$$

for all (x, t) in Λ, where $m(A)$ is the minimum norm of A,

$$m(A) = \inf_{u \neq 0} \frac{\|Au\|}{\|u\|}.$$

Take $\varepsilon > 0$ small enough so that $\mu + 2\varepsilon < 1$ and $\lambda - 6\varepsilon > 1$. Since Df maps $E^s_{(x, t)}$ and $E^s_{(x, t)}$ onto $E^s_{f(x, t)}$ and $E^u_{f(x, t)}$ respectively,

$$Df_{(x, t)} = \begin{pmatrix} A^{ss}(x, t) & 0 \\ 0 & A^{uu}(x, t) \end{pmatrix}.$$

Given such an $\varepsilon > 0$, there is an $r > 0$ small enough so that for $(x, t) \in \Lambda$ and $(y, t) \in B_{(x, t)}(r)$,

$$Df_{(y, t)} = \begin{pmatrix} A^{ss}(y, t) & A^{su}(y, t) \\ A^{us}(y, t) & A^{uu}(y, t) \end{pmatrix}$$

satisfies the following: $\|A^{ss}(y, t)\| < \mu$, $\|A^{su}(y, t)\| < \varepsilon$, $m(A^{uu}(y, t)) > \lambda$, $\|A^{us}(y, t)\| < \varepsilon$, $\|A^{ss}(y, t) - A^{ss}(x, t)\| < \varepsilon$, and $\|A^{uu}(y, t) - A^{uu}(x, t)\| < \varepsilon$. Here we used the fact that Df is continuous and Λ is compact.

Let

$$W^s_{(x, t)}(r) = \{(y, t) \in B_{(x, t)}(r) : f^n(y, t) \in B_{f^n(x, t)}(r) \qquad \text{for} \quad n = 0, 1, 2...\},$$

where we think of the local stable manifold as being represented in local coordinates on $T_{(x, t)}M \times T$. We want to prove that it is a graph of a C^1 function

$$\varphi: B^s_{(x, t)}(r) \to B^u_{(x, t)}(r)$$

satisfying $\varphi(0) = 0$ and $D\varphi(0) = 0$. The stable cone $C^s_{(x, t)}$ and unstable cone $C^u_{(x, t)}$ at (x, t) are defined as follows:

$$C^s_{(x,\,t)} = \{(v^s_{(x,\,t)}, v^u_{(x,\,t)}) \in E^s_{(x,\,t)} \times E^u_{(x,\,t)} : |v^u_{(x,\,t)}| \leqslant |v^s_{(x,\,t)}|\}$$
$$= \{v_{(x,\,t)} \in T_{(x,\,t)} M \times T : |v^u_{(x,\,t)}| \leqslant |v^s_{(x,\,t)}|\},$$
$$C^u_{(x,\,t)} = \{v_{(x,\,t)} \in T_{(x,\,t)} M \times T : |v^u_{(x,\,t)}| \geqslant |v^s_{(x,\,t)}|\}.$$

We are going to divide the proof into four steps. First we will prove that the local stable manifold at (x, t) is a graph of some function

$$\varphi : B^s_{(x,\,t)}(r) \to B^u_{(x,\,t)}(r).$$

Second we are going to prove that φ is 1-Lipschitz. Then we will prove that φ is C^1, $\varphi(0) = 0$, and $D\varphi(0) = 0$. And at last we will prove that

$$W^s_{(x,\,t)}(r) = \bigcap_{n=0}^{\infty} f^{-n}\{B_{f^n(x,\,t)}(r)\}$$
$$= \{(y, t) \in B_{(x,\,t)}(r) : f^n(y, t) \in B_{f^n(x,\,t)}(r) \quad \text{and}$$
$$|f^n(y, t) - f^n(x, t)| \leqslant \alpha^n |y - x| \quad \text{for} \quad n = 0, 1, 2...\}$$

and the local stable manifold $W^s_{(x,\,t)}(r)$ continuously depends on (x, t).

The following lemmas outline the proof of the Stable Manifold Theorem, which follows the four steps. However, because the similarity of the proofs of these lemmas to the proofs of those lemmas in regular dynamical systems, the proofs to these lemmas will not be provided. Readers can compare these lemmas with the corresponding lemmas in Robinson [14] or see [21].

LEMMA 2.1. *Suppose* $f(x, t) = (\Psi(x, t), V(t))$ *is a continuous map from* $U \times T$ *to* $E \times T$ *where* U *is an open set in* E, *and* f *has a continuous partial derivative with respect to the first variable. Assume* $(x_0, t) \in U \times T$ *and let* $L = Df(x_0, t)$ *be an invertible linear map. We denote* $f(x_0, t)$ *by* $(y_0, V(t))$. *Let* β *be any number with* $0 < \beta < 1$. *Let* $r > 0$ *satisfy*

(1) $B_{(x_0,\,t)}(r)$ *is contained in* $U \times \{t\}$ *and*

(2) $\|L - Df(x, t)\| \leqslant m(L)(1 - \beta)$ *for all* $(x, t) \in B_{(x_0,\,t)}(r)$.

Then $f(B_{(x_0,\,t)}(r)) \supset L(B_{(x_0,\,t)}(\beta r)) \supset B_{(y_0,\,V(t))}(m(L)\,\beta r)$. *Moreover, every point* $(y, V(t))$ *in* $L(B_{(x_0,\,t)}(\beta r))$ *has exactly one preimage* $(x, t) \in B_{(x_0,\,t)}(r)$, *either* $f(x, t) = (y, V(t))$ *or* $\Psi(x, t) = y$. *So, the inverse function* f^{-1} *exists with domain* $B_{(y_0,\,V(t))}(m(L)\,\beta r)$ *and range* $B_{(x_0,\,t)}(r)$ *and has a continuous partial derivative with respect to the first variable and is from*

DEFINITION. Assume σ is a C^1 function from $B^u_{(x,\,t)}(r)$ into $B^s_{(x,\,t)}(r)$. Its graph is called an *unstable disk* at (x, t) if $Lip(\sigma) \leqslant 1$. Similarly we can define a *stable disk*.

LEMMA 2.2. *If $D_0(x, t)$ is an unstable disk at (x, t), then*

(1) $D_1(f(x, t)) = f(D_0(x, t)) \cap B_{f(x, t)}(r)$ *is an unstable disk at* $f(x, t)$ *and* $diam\{\pi^u_{(x, t)}[f^{-1}(D_1(f(x, t))) \cap D_0(x, t)]\} \leqslant 2r(\lambda - \varepsilon)^{-1}$

2. $D_n(f^n(x, t)) = f(D_{n-1}(f^{n-1}(x, t))) \cap B_{f^n(x, t)}(r)$ *is an unstable disk at* $f^n(x, t)$ *and* $diam\{\pi^u_{(x, t)}[\bigcap_{j=0}^n f^{-j}(D_j(f^j(x, t)))]\} \leqslant 2r(\lambda - \varepsilon)^{-n}$.

LEMMA 2.3. *The local stable manifold*

$$W^s_{(x, t)}(r) = \{(y, t) : f^n(y, t) \in B_{f^n(x, t)}(r) \qquad for \ all \quad n \geqslant 0\}$$

is the graph of some function

$$\varphi : B^s_{(x, t)}(r) \to B^u_{(x, t)}(r)$$

satisfying $\varphi(0) = 0$.

LEMMA 2.4. *Assume* (y, t), $(z, t) \in B_{(x, t)}(r)$ *and* $(z, t) \in \{(y, t)\} + C^u_{(x, t)}$. *Then*

(1) $|\pi^s_{f(x, t)}[f(z, t) - f(y, t)]| \leqslant (\mu + \varepsilon) |\pi^u_{(x, t)}(z - y)|$

(2) $|\pi^u_{f(x, t)}[f(z, t) - f(y, t)]| \geqslant (\lambda - \varepsilon) |\pi^u_{(x, t)}(z - y)|$

(3) $f(z, t) \in \{f(y, t)\} + C^u_{f(x, t)}$.

Similarly, if

$$f^{n-1}(y, t), f^{n-1}(z, t) \in B_{f^{n-1}(x, t)}(r) \ and \ f^{n-1}(z, t) \in \{f^{n-1}(y, t)\} + C^u_{f^{n-1}(x, t)},$$

then

$$|\pi^u_{f^n(x, t)}[f^n(z, t) - f^n(y, t)| \geqslant (\lambda - \varepsilon)^n |\pi^u_{(x, t)}(z - y)|$$
$$and \ f^n(z, t) \in \{f^n(y, t)\} + C^u_{f^n(x, t)}.$$

LEMMA 2.5. *The local stable manifold,* $W^s_{(x, t)}(r)$, *is a graph of an* 1-*Lipschitz function*

$$\varphi : B^s_{(x, t)}(r) \to B^u_{(x, t)}(r).$$

Define

$$C^s_{(y, t)}(n) = (Df^n(y, t))^{-1} \{f^n(y, t) + C^s_{f^n(x, t)}\}$$
$$C^s_{(y, t)}(n, \eta) = C^s_{(y, t)}(n) \cap \{(y, t) + (\pi^s_{(x, t)})^{-1} B^s_{(x, t)}(\eta)\}.$$

LEMMA 2.6.   *If* $(y, t) \in B_{(x, t)}(r)$, *then*

$$Df(f^{n-1}(y, t))\{f^{n-1}(y, t) + C^u_{f^{n-1}(x, t)}\}$$

$$\subset \left\{(v^s, v^u) \in E_{f^n(y, t)}, |v^s| \leqslant \frac{\mu + \varepsilon}{\lambda - \varepsilon} |v^u|\right\}$$

and $C^s_{(y, t)}(n) \subset C^s_{(y, t)}(n-1)$ *for* $n = 1, 2, 3....$ *Furthermore,* $\bigcap_{n=0}^{\infty} C^s_{(y, t)}(n)$ *is a graph of some linear map from* $E^s_{(x, t)}$ *to* $E^u_{(x, t)}$.

Suppose the local stable manifold is the graph of

$$\varphi: B^s_{(x, t)}(r) \to B^u_{(x, t)}(r).$$

We anticipate that, this linear map is the derivative of φ. But to prove this, we need the following lemma.

LEMMA 2.7.   *Assume* $(y, t) \in W^s_{(x, t)}(r)$ *and* $n > 0$. *Then there exists an* $\eta(n) > 0$ *such that the following are true*:

(1)   *If* $|z - y| < \eta$ *and* $f^n(z, t) \in \{f^n(y, t) + C^s_{f^n(x, t)}\}$, *then* $(z, t) \in C^s_{(y, t)}(n-1, \eta)$ *and*

2.   *If* $(z, t) \in W^s_{(x, t)}(r)$ *and* $|z - y| < \eta$, *then* $(z, t) \in C^s_{(y, t)}(n-1, \eta)$.

LEMMA 2.8.   *The local stable manifold is* C^1 *and is tangent to* $E^s_{(x, t)}$ *at* (x, t).

LEMMA 2.9.

$$W^s_{(x, t)}(r) = \bigcap_{n=0}^{\infty} f^{-n}(B_{f^n(x, t)}(r))$$

$$= \{(y, t) \in B_{(x, t)}(r): f^n(y, t) \in B_{f^n(p)}(r) \text{ and }$$

$$|f^n(y, t) - f^n(x, t)| \leqslant \alpha^n |y - x| \text{ for } n = 0, 1, 2...\}$$

and the local stable manifold $W^s_{(x, t)}(r)$ *continuously depends on* (x, t).

LEMMA 2.10. $\varphi_{(x, t)}(u)$ *and* $D\varphi_{(x, t)}(u)$ *continuously depend on* (x, t).

This finishes the outline of the proof of the Local Stable Manifold Theorem.   ∎

The Local Unstable Manifold Theorem can be stated and proved in a similar way. It will not be repeated here.

LEMMA 2.11 (Shadowing Lemma). *Let Λ be a closed hyperbolic invariant set of $f: M \times T \to M \times T$. Then there are a neighborhood U of Λ and a neighborhood W of f having the following properties*:

(1) *For any $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudo-orbit $\{(x_i, t_i)\}$ of g has a β-shadowing orbit $\{g^i(x_0', t_0)\}$, where $\{(x_i, t_i)\} \subset U$ and $g \in W$.*

(2) *There is a $\beta_0 > 0$ such that if $0 < \beta < \beta_0$, then the β-shadowing orbit $\{g^i(x_0', t_0)\}$ is uniquely determined by the α-pseudo-orbit $\{(x_i, t_i)\}$.*

(3) *If Λ is an isolated invariant set of f, then for any $g \in W$, g has a closed isolated hyperbolic invariant set Δ in U and the shadowing orbit $\{g^i(x_0', t_0)\}$ is in Δ.*

Proof. First we can continuously extend the splitting $E^s_{(x,t)} \times E^u_{(x,t)}$ from Λ to a neighborhood G of Λ. Suppose U is a neighborhood of Λ such that $\bar{U} \subset G$, where $\bar{U}$ is the closure of U. For $\beta > 0$, we choose $r, \varepsilon > 0$ small enough such that $r < \beta$, $\lambda - 6\varepsilon > 1$, $\mu + 5\varepsilon < 1$, and $B_{(x,t)}(r) \subset G$ for any $(x, t) \in \bar{U}$. Notice that for fixed $(x, t) \in \Lambda$, $Df(x, t)$ is of the form

$$\begin{pmatrix} A^{ss}(x, t) & 0 \\ 0 & A^{uu}(x, t) \end{pmatrix}. \tag{1}$$

We can choose $r > 0$ small enough so that for all $(y, t) \in B_{(x,t)}(r)$,

$$Df(y, t) = \begin{pmatrix} A^{ss}(y, t) & A^{su}(y, t) \\ A^{us}(y, t) & A^{uu}(y, t) \end{pmatrix} \tag{2}$$

satisfying

$$\|A^{ss}(y, t)\| < \mu, \qquad m(\|A^{uu}(y, t)\|) > \lambda,$$
$$\|A^{su}(y, t)\| < \varepsilon, \qquad \|A^{us}(y, t)\| < \varepsilon, \tag{3}$$
$$\|A^{ss}(y, t) - A^{ss}(x, t)\| < \varepsilon, \qquad \|A^{uu}(y, t) - A^{uu}(x, t)\| < \varepsilon.$$

Because $Df(x, t)$ continuously depends on (x, t) and we have extended the splitting continuously to $G \supset \bar{U}$, the above property still holds for $(x, t) \in \bar{U}$ if U is small enough. Notice that $\bar{U}$ is compact, we can choose an $r > 0$ small enough so that for all $(x, t) \in \bar{U}$ and all $(y, t) \in B_{(x,t)}(r)$, $Df(y, t)$ is of the form (2) with (3) holding.

Because we are using $C^{1,0}$ topology, if g is close to f, then Dg is close to Df. So we can choose a neighborhood W of f such that for all $(x, t) \in \bar{U}$, $g \in W$ and for all $(y, t) \in B_{(x,t)}(r)$, $Dg(y, t)$ is of the form (2) with (3) holding.

Because $\bar{U}$ is compact, we can choose W and $\alpha > 0$ small enough such that $g(B_{(x, t)}(r)) \subset B_p(C_0 r)$, whenever $d(g(x, t), p) < \alpha$, $(x, t) \in U$, $p \in M \times \{V(t)\}$, and $g \in W$, where C_0 is the constant in the proof of the Stable Manifold Theorem. We will also choose $r > 0$ small enough so that we can identify $B_{(x, t)}(C_0 r)$ as a neighborhood of (x, t) in $M \times \{t\}$. Notice that $\lambda - 6\varepsilon > 1$ and $\mu + \varepsilon < 1$. For any $(x, t) \in \bar{U}$, we can choose $\alpha > 0$ small enough so that if $\sigma: B^u_{(x, t)}((\lambda - 6\varepsilon)r) \to B^s_{(x, t)}((\mu + \varepsilon)r)$ satisfies $\|D\sigma\| < (\mu + \varepsilon)(\lambda - \varepsilon)^{-1}$, then $B_{(y, t)}(r) \cap graph(\sigma)$ is an unstable disk at (y, t) whenever $d(x, y) < \alpha$. Since $P(x, t)$ in the definition of hyperbolic set continuously depends on (x, t), this property can be extended to a neighborhood of (x, t). That is, for any $(x, t) \in \bar{U}$, there is a neighborhood N of (x, t) and an $\alpha > 0$ such that if $\sigma: B^u_{(y, \tau)}((\lambda - 6\varepsilon)r) \to B^s_{(y, \tau)}((\mu + \varepsilon)r)$ satisfies $\|D\sigma\| < (\mu + \varepsilon)(\lambda - \varepsilon)^{-1}$, then $B_{(z, \tau)}(r) \cap graph(\sigma)$ is an unstable disk at (z, τ), whenever $d(y, z) < \alpha$ and $(y, \tau) \in N$. Since $\bar{U}$ is compact, there is an $\alpha > 0$ such that if $\sigma: B^u_{(x, t)}((\lambda - 6\varepsilon)r) \to B^s_{(x, t)}((\mu + \varepsilon)r)$ satisfies $\|D\sigma\| < (\mu + \varepsilon)(\lambda - \varepsilon)^{-1}$, then $B_{(y, t)}(r) \cap graph(\sigma)$ is an unstable disk at (y, t), whenever $d(x, y) < \alpha$ and $x \in \bar{U}$. In the same way, for $\delta > 0$ with $(1 + \delta) < \lambda - \varepsilon$, we can choose $\alpha > 0$ small enough so that if $(x, t) \in \bar{U}$ and $d(x, y) < \alpha$, then $diam([I - P(y, t)](graph(\chi))) < (1 + \delta)2r$, where $\chi: B^u_{(x, t)}(r) \to B^s_{(x, t)}(r)$ is an unstable disk at (x, t). We choose α, δ, and W small enough so all conditions above are satisfied. Now we need the following lemmas.

LEMMA 2.12. *Suppose* $\{(x_i, t_i)\}$ *is an α-pseudo-orbit in* $\bar{U}$ *of g and* $D_0(x_0, t_0)$ *is an unstable disk of $g \in W$ at (x_0, t_0). Then the following are true*:

(1) $D_1(x_1, t_1) = g(D_0(x_0, t_0)) \cap B_{(x_1, t_1)}(r)$ *is an unstable disk at (x_1, t_1) and* $diam\{[I - P(x_0, t_0)](g^{-1}(D_1(x_1, t_1)) \cap D_0(x_0, t_0))\} \leqslant 2r(1 + \delta)(\lambda - \varepsilon)^{-1}$

(2) $D_n(x_n, t_n) = g(D_{n-1}(x_{n-1}, t_{n-1})) \cap B_{(x_n, t_n)}(r)$ *is an unstable disk at* (x_n, t_n) *and* $diam\{[I - P(x_0, t_0)](\bigcap_{i=0}^{n} g^{-i}(D_i(x_i, t_i))\} \leqslant 2r(1 + \delta)^n (\lambda - \varepsilon)^{-n}$ *for $n = 0, 1, 2...$.*

LEMMA 2.13. $\bigcap_{n=0}^{\infty} g^{-n}(B_{(x_n, t_n)}(r))$ *is a stable disk at (x_0, t_0) and* $\bigcap_{n=-\infty}^{0} g^{-n}(B_{(x_n, t_n)}(r))$ *is an unstable disk at (x_0, t_0).*

The proofs of these two lemmas are similar to the proofs of Lemma 2.2 and 2.5 in the proof of Stable Manifold Theorem, so they will be omitted here.

Now let's continue the proof of the Shadowing Lemma. By Lemma 2.13, $\bigcap_{n=-\infty}^{\infty} g(B_{(x_n, t_n)}(r))$ contains one and only one point because the stable disk and unstable disk intersect at one and only one point. We denote it by (x_0', t_0). Since $(x_0', t_0) \in g^{-n}(B_{(x_n, t_n)}(r))$, $g^n(x_0', t_0)) \in B_{(x_n, t_n)}(r)$ for $n = 0, \pm 1, \pm 2...$. Notice that $r < \beta$. Hence $\{g^n(x_0', t_0), n = 0, \pm 1, \pm 2...\}$ is

an orbit that β-shadows the α-pseudo-orbit $\{(x_n, t_n), \ n = 0, \pm 1, \pm 2 \ldots\}$. This finishes the proof of part (1).

To prove part (2), we choose β_0 as r. It is obvious that for any β, $0 < \beta < \beta_0$, the orbit which β-shadows the α-pseudo-orbit is uniquely determined by the α-pseudo-orbit because the stable and the unstable disks intersect at one and only one point.

Now we prove part (3). Since Λ is an isolated invariant set of f, there is an isolated neighborhood G' of Λ such that $\bigcap_{n=-\infty}^{\infty} f^n(\overline{G'}) = \Lambda$. Without loss of generality, we assume that the above U and r are small enough that $B_{(x,\,t)}(r) \subset G'$ for all $(x, t) \in U$. Since $\bigcap_{n=-\infty}^{\infty} f^n(\overline{G'}) = \Lambda$, there is an integer m such that $\bigcap_{n=-m}^{m} f^n(\overline{G'}) \subset U$. Therefore if W is small, $\bigcap_{n=-m}^{m} g^n(\overline{G'}) \subset U$ for any $g \in W$. If we denote $\bigcap_{n=-\infty}^{\infty} g^n(\overline{G'}) = \Delta$, then $\Delta \subset U$ and G' is an isolated neighborhood of Δ. Because Δ is the largest invariant set in G' and the orbit which β-shadows the α-pseudo-orbit is in G', it is in Δ. If W is small, g has a hyperbolic structure on Δ for all $g \in W$. This finishes the proof of the Shadowing Lemma. ∎

This is a stronger version of the Shadowing Lemma than the one found in [10], but some say the one given above was known to Lamont Cranston and Margo Lane.

COROLLARY 2.1. *Suppose Λ is a closed hyperbolic invariant set of f. There are a neighborhood U of Λ, a neighborhood W of f, and an $\alpha > 0$ such that if $g \in W$ and $\{g^n(x, t)\}$, $\{g^n(y, t)\} \subset U$ are α-shadowed by each other, then $x = y$.*

3. LOCAL STABILITY OF ALMOST PERIODIC SETS

Suppose A is a closed invariant set of f. A is called a *base-like set* if the projection map, $p: A \to T$, $p(x, t) = t$, is a homeomorphism. A point (x, t) is called a *base-like point* if the closure of its orbit is a base-like set. A point (x, t) is called a *periodic-like point* if (x, t) is a base-like point of f^k for some k, that is, if the closure of the orbit of f^k through (x, t) is a base-like set. The smallest such $k > 0$ is called the period of the periodic-like point. The closure of the orbit through a periodic-like point is called a *periodic-like set*. Periodic-like sets will play the same role as periodic points did in the classical theory.

Suppose f and g: $M \times T \to M \times T$ are two skew dynamical systems. We say the two systems are topologically skew-conjugate if there is a homeomorphism

$$H(x, t): M \times T \to M \times T, \qquad H(x, t) = (h(x, t), t),$$

such that $H \circ f = g \circ H$. In this case we also say H skew-conjugates f and g. Under skew-conjugacy, the base-like point is invariant. In the following, we will call skew-conjugacy just conjugacy. We have the following obvious result.

LEMMA 3.1. *Suppose f and g are toplogically conjugate and H conjugates f and g. If (x, t) is a base-like (periodic-like) point of f, then $H(x, t)$ is a base-like (periodic-like) point of g.*

If (x, t) is a base-like point of f, then the closure of the orbit of f through (x, t), denoted by A, is homeomorphic to T. The restriction of f to A is in fact topologically conjugate to V on T, where V is the map in the definition of f. Furthermore the projection map p conjugates $f \mid A$ and V.

In skew dynamical systems, the smallest closed invariant sets seem to be the base-like sets since $\{ V^k(t), k \in Z \}$ is dense in T. A natural question is whether a hyperbolic base-like set is stable. The following two theorems answer the question positively.

THEOREM 3.1. *Suppose A is a hyperbolic base-like set of f. Then there are a neighborhood U of A and a neighborhood W of f such that for every $g \in W$, there is one and only one hyperbolic base-like set S of g in U and $f \mid A$ is topologically conjugate to $g \mid S$.*

Proof. Since A is a closed hyperbolic invariant set, there is an $r > 0$ such that

$$\bigcap_{n=0}^{\infty} f^{-n}(B_{f^n(x, t)}(r))$$

is the local stable manifold of f at (x, t) for any $(x, t) \in A$ by the Stable Manifold Theorem. Similarly,

$$\bigcap_{n=-\infty}^{0} f^{-n}(B_{f^n(x, t)}(r))$$

is the local unstable manifold of f at (x, t) for any $(x, t) \in A$. Therefore,

$$\bigcap_{n=-\infty}^{\infty} f^{-n}(B_{f^n(x, t)}(r))$$

contains one and only one point. Here $B_{f^n(x, t)}(r)$ is the ball neighborhood of $f^n(x, t)$ with radius r in the tangent space $T_{f^n(x, t)} M \times T$, which we identify with a neighborhood of $f^n(x, t)$ in $M \times \{ V^n(t) \}$, and the point is the intersection of the local stable and unstable manifolds of f at (x, t), which

is (x, t) itself. Let $D = \bigcup_{(x, t) \in A} B_{(x, t)}(r)$. D is a closed neighborhood of A. For any fixed t,

$$\left\{ \bigcap_{n = -\infty}^{\infty} f^n(\bar{D}) \right\} \bigcap \{M \times \{t\}\} = \bigcap_{n = -\infty}^{\infty} f^{-n}(B_{f^n(x, t)}(r)) = \{(x, t)\},$$

where (x, t) is the only point in A whose second coordinate is t. Thus $\bigcap_{n = -\infty}^{\infty} f^n(\bar{D}) = A$: so D is an isolated neighborhood of A. By the Shadowing Lemma, there are a neighborhood W of f and a neighborhood U of A such that for any $g \in W$, g has a closed hyperbolic invariant S contained in U. Suppose β_0 is the positive number in the Shadowing Lemma. That is for any $0 < \beta < \beta_0$, there is an $\alpha > 0$ such that any α-pseudo-orbit of g in U can be uniquely β shadowed by an orbit in S, where $g \in W$.

Without loss of generality, we suppose W is contained in the α-neighborhood of f. Then

$$\{g^k(y, t), k \in Z\}$$

is an α-pseudo-orbit of f in U for all (y, t) in S. By the Shadowing Lemma, there is one and only one point (x, t) in A such that $\{f^k(x, t), k \in Z\}$ β-shadows $\{g^k(y, t), k \in Z\}$. On the other hand, for any $(x, t) \in A$, there is one and only one point $(y, t) \in S$ such that $\{g^k(y, t), k \in Z\}$ β-shadows $\{f^k(x, t), k \in Z\}$. For any $t \in T$, there is one and only one point (x, t) in A whose second coordinate is t. Now for this t, there is one and only one point (y, t) in S with the second coordinate t. If there are two points (y, t) and (z, t) in S for this t, then both $\{g^k(y, t), k \in Z\}$ and $\{g^k(z, t), k \in Z\}$ are β-shadowed by $\{f^k(x, t), k \in Z\}$. This implies that $\{g^k(y, t), k \in Z\}$ and $\{g^k(z, t), k \in Z\}$ β-shadow $\{f^k(x, t), k \in Z\}$, contradicting the uniqueness of shadowing. Thus the map

$$H: S \to A, \qquad H(y, t) = (x, t)$$

is one-to-one and onto, and it is continuous by the Shadowing Lemma. Therefore, H is a homeomorphism from S to A. Notice that for any $(y, t) \in S$, $g(y, t)$ and $f \circ H(y, t)$ have the same second coordinates: so

$$H \circ g(y, t) = f \circ H(y, t),$$

and $H \circ g = f \circ H$. Hence $f \mid A$ and $g \mid S$ are topologically conjugate. ∎

The theorem above guarantees that under small perturbations the hyperbolic base-like set is stable. In fact, under small perturbations the local structure near a hyperbolic base-like set is stable. The following is a Hartman–Grobman type theorem.

THEOREM 3.2. *Suppose*

$$f: R^n \times T \to R^n \times T, f(x, t) = (\Psi(x, t), V(t))$$

is a homeomorphism with a continuous partial derivative with respect to x, and let $f(0, t) = (0, V(t))$, and $P = \{0\} \times T$ be a hyperbolic base-like set. Then there are neighborhoods U and D of P such that $f \mid U$ and $(Df, V) \mid D$ are topologically conjugate.

We omit the proof since the proof in Palis and Melo [13] carries over with little change.

Suppose (x, t) is a point in $M \times T$. For fixed t, $f(x, t)$ is a diffeomorphism from $M \times \{t\}$ onto $M \times \{V(t)\}$. In particular, $Df(x, t)$ maps $T_{(x, t)}M \times T$ onto $T_{f(x, t)}M \times T$ and both $T_{(x, t)}M \times T$ and $T_{f(x, t)}M \times T$ are isomorphic to R^n. We have commented already that there is an $r > 0$ such that we can identify a neighborhood of any point (y, t) in $M \times \{t\}$ with $B_{(y, t)}(r)$, a ball with radius r and center (y, t) in the tangent space of $M \times T$ at (y, t); so for fixed t, locally we can think that both f and Df map a small neighborhood in $B_{(x, t)}(r)$ into $B_{f(y, t)}(r)$. If A is a hyperbolic base-like set of f, then A has a closed neighborhood $\bigcup_{(x, t) \in A} B_{(x, t)}(r)$ in $M \times T$. Thus both f and Df map one neighborhood of A in $\bigcup_{(x, t) \in A} B_{(x, t)}(r)$ to another neighborhood of A in $\bigcup_{(x, t) \in A} B_{(x, t)}(r)$. Because we are going to consider only the local property, we can think both f and Df map $T_{(x, t)}M \times T$ to $T_{f(x, t)}M \times T$ for fixed t, or as maps from $\bigcup_{(x, t) \in A} T_{(x, t)}M \times T$ to $\bigcup_{(x, t) \in A} T_{(x, t)}M \times T$. However, the maps f and Df on $\bigcup_{(x, t) \in A} T_{(x, t)}M \times T$ behave exactly like the f and Df on $\bigcup_{(0, t) \in A} R^n \times \{(0, t)\}$ in the preceding theorem, so we have the following corollary.

COROLLARY 3.1. *Suppose A is a hyperbolic base-like set of f. Then there are two neighborhoods U and D of A such that the dynamical systems $f \mid U$ and $(df, V) \mid D$ are topologically conjugate.*

Combining this corollary with the stability theorem for base-like set, we have the following corollary.

COROLLARY 3.2. *Suppose A is a hyperbolic base-like set of f. Then there are a neighborhood U of A and a neighborhood W of f such that for any $g \in W$, g has a hyperbolic base-like set S in U. Furthermore there is a neighborhood U' of S such that (f, U) and (g, U') are topologically conjugate.*

4. Ω-STABILITY OF SKEW SYSTEMS

We have shown that the local structure near a base-like set is stable. Now we are going to study the global stability of the nonwandering set Ω.

THEOREM 4.1. *Suppose A is a closed isolated hyperbolic invariant set of f and U is an isolated neighborhood of A. Then there is a neighborhood W of f such that for every $g \in W$, g has a closed isolated hyperbolic invariant set S in U and $f \mid A$ is topologically conjugate to $g \mid S$.*

Proof. The proof is straightforward. The existence of W and S is a direct consequence of the Shadowing Lemma. Using the Shadowing Lemma, we construct the homeomorphism H between A and S so that, $H(x, t) = (y, t)$, where $(x, t) \in A$ and $(y, t) \in S$. Observe that two corresponding points have the same second coordinate. Using this fact and the uniqueness of shadowing, we can prove that H conjugates $f \mid A$ and $g \mid S$. ∎

THEOREM 4.2. *An Anosov skew system is structurally stable.*

Meyer [8] has a complete proof, Frank's proof [4] can be generalized to this case, and the above theorem can also be used to give a proof.

LEMMA 4.1 (Lambda Lemma). *Suppose that A and S are two hyperbolic periodic base-like sets and let $(x, t) \in A$, $(y, t) \in S$. Assume the dimension of the local unstable manifold of (x, t) is larger than zero and less than the dimension of the manifold M. If the stable and unstable manifolds $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection, then for any $\varepsilon > 0$, there is an N and a cell D in $W^u_{f^n(y, t)}$ which is $\varepsilon - C^1$ close to $W^s_{f^n(x, t)}(r)$ if $n \geqslant N$.*

Again the proof of this lemma will be omitted–see [21].

Assume A and S are two hyperbolic periodic-like sets. We say that A and S are *equivalent*, denoted $A \sim S$, if there are $(x, t) \in A$ and $(y, t) \in S$ such that $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection and $W^u_{(x, t)}$ and $W^s_{(y, t)}$ have a transverse intersection. The set of all hyperbolic periodic-like points is denoted by Θ. Suppose $(x, t), (y, \tau) \in \Theta$. Then there are periodic-like sets A and S such that $(x, t) \in A$ and $(y, \tau) \in S$, and we say $(x, t) \sim (y, \tau)$ if $A \sim S$.

The following lemma gives a very nice picture of how equivalent hyperbolic periodic-like sets are related.

LEMMA 4.2. *Suppose A and S are two hyperbolic base-like sets with $A \sim S$. Then for any $t \in T$, $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection and $W^u_{(x, t)}$ and $W^s_{(y, t)}$ have a transverse intersection whenever $(x, t) \in A$ and $(y, t) \in S$. Similarly if A and S are two hyperbolic periodic-like sets and $A \sim S$, then for any $(x, t) \in A$, there is a point $(y, t) \in S$ such that $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection and $W^u_{(x, t)}$ and $W^s_{(y, t)}$ have a transverse intersection.*

Proof. Since A and S are base-like sets, there are continuous functions $a(t)$, $s(t): T \to M$ such that $A = \{(a(t), t), t \in T\}$ and $S = \{(s(t), t), t \in T\}$. To prove $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection for all $(x, t) \in A$ and $(y, t) \in S$, we need only to prove $W^s_{(a(t), t)}$ and $W^u_{(s(t), t)}$ have a transverse intersection for all $t \in T$. Because $A \sim S$, there is a $\tau \in T$ such that $W^s_{(a(\tau), \tau)}$ and $W^u_{(s(\tau), \tau)}$ have a transverse intersection (z, τ), where $W^s_{(a(\tau), \tau)}$ and $W^u_{(s(\tau), \tau)}$ are stable and unstable manifolds at $(a(\tau), \tau) \in A$ and $(s(\tau), \tau) \in S$ respectively. Suppose $W^s_{(x, t)}(r)$ is the local stable manifold at (x, t). Then

$$W^s_{(x, t)}(r) = \{(u, \varphi_{(x, t)}(u), u \in B^s_{(x, t)}(r)\}$$

and $\varphi_{(x, t)}(u)$ and $D\varphi_{(x, t)}(u)$ continuously depend on (x, t) by the Local Stable Manifold Theorem. Let $W^s_{(x, t)}$ be the stable manifold at (x, t). Then

$$W^s_{(x, t)} = \bigcup_{n=0}^{\infty} f^{-n}(W^s_{f^n(x, t)}(r)),$$

and

$$W^s_{(a(\tau), \tau)} = \bigcup_{n=0}^{\infty} f^{-n}(W^s_{f^n(a(\tau), \tau)}(r)).$$

Because $W^s_{(a(\tau), \tau)}$ and $W^u_{(s(\tau), \tau)}$ have a transverse intersection (z, τ), there is a positive integer N such that

$$(z, \tau) \in \bigcup_{n=0}^{N} f^{-n}(W^s_{f^n(a(\tau), \tau)}(r)) = f^{-N}(W^s_{f^N(a(\tau), \tau)}(r)).$$

Here we also require that (z, τ) belongs to the interior of $f^{-N}(W^s_{f^N(a(\tau), \tau)}(r))$. By choosing a larger N if necessary, we also require that (z, τ) belongs to the interior of the set $f^N(W^u_{f^{-N}(s(\tau), \tau)}(r))$. Since $\varphi_{(x, t)}(u)$ and $D\varphi_{(x, t)}(u)$ continuously depend on (x, t) and $f^N(x, t)$ continuously depends (x, t), $W^s_{f^N(x, t)}(r)$ and its tangent spaces continuously depend on (x, t). Notice that f^{-N} is a diffeomorphism from $M \times \{V^N(t)\}$ onto $M \times \{t\}$ and f^{-N} and Df^{-N} continuously depend on (x, t). $f^{-N}(W^s_{f^N(x, t)}(r))$ and its tangent spaces continuously depend on (x, t). In a similar manner, $f^N(W^u_{f^{-N}(y, t)}(r))$ and its tangent spaces continuously depend on (y, t). Because $a(t)$ and $s(t)$ are continuous, there is a neighborhood U of τ in T such that $f^{-N}(W^s_{f^N(a(t), t)}(r))$ and $f^N(W^u_{f^{-N}(s(t), t)}(r))$ have a transverse intersection for all $t \in U$. Thus $W^s_{(a(t), t)}$ and $W^u_{(s(t), t)}$ have a transverse intersection for all $t \in U$. Because any orbit of V is dense in T,

$$\{V^n(U), n = 0, \pm 1, \pm 2, ...\}$$

is a cover of T. On the other hand, for any fixed t, $f(x, t)$ is a diffeomorphism from $M \times \{t\}$ onto $M \times \{V(t)\}$, thus if $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection, $W^s_{f(x, t)}$ and $W^u_{f(y, t)}$ also have a transverse intersection. Similarly if $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection, then $W^s_{f^{-1}(x, t)}$ and $W^u_{f^{-1}(y, t)}$ have a transverse intersection too, and by repeating this procedure, we can see that if $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection, $W^s_{f^n(x, t)}$ and $W^u_{f^n(y, t)}$ have a transverse intersection for any integer n. Notice that for any $t \in U$, $W^s_{(a(t), t)}$ and $W^u_{(s(t), t)}$ have a transverse intersection and

$$\{ V^n(U), n = 0, \pm 1, \pm 2, ... \}$$

is a cover of T. $W^s_{(a(t), t)}$ and $W^u_{(s(t), t)}$ have a transverse intersection for all $t \in T$. So $W^s_{(x, t)}$ and $W^u_{(y, t)}$ have a transverse intersection whenever $(x, t) \in A$ and $(y, t) \in S$. In the same way we can prove $W^u_{(x, t)}$ and $W^s_{(y, t)}$ have a transverse intersection whenever $(x, t) \in A$ and $(y, t) \in S$.

If A and S are two periodic-like sets with period m and n respectively, then A contains m branches $A_1, ..., A_m$ and S contains n branches $S_1, ..., S_n$. Each of the branches is a base-like set of f^{mn}. If $A \sim S$, then there are $(x, \tau) \in A_i$ and $(y, \tau) \in S_j$ such that $W^s_{(x, \tau)}$ and $W^u_{(y, \tau)}$ have a transverse intersection and $W^u_{(x, \tau)}$ and $W^s_{(y, \tau)}$ have a transverse intersection. Using the lemma above, we know that for any $t \in T$, $W^s_{(z, t)}$ and $W^u_{(w, t)}$ have a transverse intersection and $W^u_{(z, t)}$ and $W^s_{(w, t)}$ have a transverse intersection where (z, t) is the only one point in A_i with the second coordinate t and (w, t) is the only one point in S_j with the second coordinate t. Suppose $(x, t) \in A$. Then $(x, t) \in A_l$ for some $1 \leqslant l \leqslant m$, and there is a k, $0 \leqslant k < n$ such that $f^k(x, t) \in A_l$. Therefore there is a point $(y, \tau) \in S$, $V^k(t) = \tau$, such that $W^s_{f^k(x, t)}$ and $W^u_{(y, \tau)}$ have a transverse intersection and $W^u_{f^k(x, t)}$ and $W^s_{(y, \tau)}$ have a transverse intersection. Thus $W^s_{(x, t)}$ and $W^u_{f^{-k}(y, \tau)}$ have a transverse intersection and $W^u_{(x, t)}$ and $W^s_{f^{-k}(y, \tau)}$ have a transverse intersection. However, $f^{-k}(y, \tau) \in S$. This finishes the proof of the lemma. ∎

THEOREM 4.3.   *The relation $\sim$ is an equivalence relation on Θ.*

Proof.   Suppose (x_1, t_1), (x_2, t_2), $(x_3, t_3) \in \Theta$ and $(x_1, t_1) \sim (x_2, t_2)$, $(x_2, t_2) \sim (x_3, t_3)$. Then there are periodic-like sets A, S, and L such that $(x_1, t_1) \in A$, $(x_2, t_2) \in S$ and $(x_3, t_3) \in L$. To prove the relation $\sim$ is an equivalence relation on Θ, we only need to prove $A \sim L$ under the assumption that $A \sim S$ and $S \sim C$. Because A, S, and L is a periodic-like set, there is K such that each branch of A, S and L are base-like sets of f^K. Without loss of generality, we suppose that A, S, and L are all base-like sets of f.

Assume $(x, \tau) \in A$, $(y, \tau) \in S$ and $(z, \tau) \in L$. By the lemma above, $W^u_{(x, \tau)}$ and $W^s_{(y, \tau)}$ have a transverse intersection (v, τ) and $W^u_{(y, \tau)}$ and $W^s_{(z, \tau)}$ have a transverse intersection (w, τ). Suppose r and ε are two small positive

numbers. Using the Lambda Lemma, there is an N such that if $n \geqslant N$, there exists a cell neighborhood D_n of (v, τ) in $W^u_{(x, \tau)}$ such that $f^n(D_n)$ is $\varepsilon/3 - C^1$ close to $W^u_{f^n(y, \tau)}(r)$, where $W^u_{f^n(y, \tau)}(r)$ is the local stable manifold at $f^n(y, \tau)$. In particular, $f^N(D_N)$ is $\varepsilon/3 - C^1$ close to $W^u_{f^N(y, \tau)}(r)$, where $W^u_{f^N(y, \tau)}(r)$ is the local stable manifold at $f^N(y, \tau)$. We know that

$$W^u_{f^N(x, \tau)} = \bigcup_{n=0}^{\infty} f^n(W^u_{f^{-n}(f^N(x, \tau))}(r))$$

and it is not difficult to see that $f^N(D_N) \subset W^u_{f^N(x, \tau)}$. Therefore, there is a positive integer m such that $f^N(D_N)$ is contained in the interior of $\bigcup_{n=0}^{m} f^n(W^u_{f^{-n}(f^N(x, \tau))}(r))$. However,

$$\bigcup_{n=0}^{m} f^n(W^u_{f^{-n}(f^N(x, \tau))}(r)) = f^m(W^u_{f^{-m}(f^N(x, \tau))}(r)).$$

Since A, S, and L are base-like sets, there are continuous functions $a(t)$, $s(t)$ and $l(t)$ such that

$$A = \{(a(t), t), t \in T\}, S = \{(S(t), t), t \in T\}, \text{ and } L = \{(l(t), t), t \in T\}.$$

Notice that

$$f^{-m}(f^N(x, \tau)) = (a(V^{-m}(V^N(\tau))), V^{-m}(V^N(\tau))).$$

In the proof of last lemma, we proved that for fixed m, $f^m(W^u_{f^{-m}(x', t)}(r))$ and its tangent spaces continuously depend on (x', t). Using this conclusion at $(a(V^N(\tau)), V^N(\tau))$, we see that $a(t)$ is continuous. There is a neighborhood U_1 of $V^N(\tau)$ in T such that if $t \in U_1$, $f^m(W^u_{f^{-m}(a(t), t)}(r))$ is $(\varepsilon/3) - C^1$ close to $f^m(W^u_{f^{-m}(f^N(x, \tau))}(r))$. Likewise, $s(t)$ is continuous and $\varphi_{(x', t)}(u)$ and $D\varphi_{(x', t)}(u)$ continuously depend on (x', t), where

$$\{(\varphi_{(x', t)}(u), u), u \in B^u_{(x', t)}(r)\} = W^u_{(x', t)}(r).$$

There is a neighborhood U_2 of $V^N(\tau)$ in T such that if $t \in U_2$, $W^u_{(s(t), t)}(r)$ is $(\varepsilon/3) - C^1$ close to

$$W^u_{(b(V^N(\tau)), V^N(\tau))}(r).$$

We denote $U_1 \cap U_2$ by U. Therefore, there is a cell in $f^m(W^u_{f^{-m}(a(t), t)}(r))$ which is $\varepsilon - C^1$ close to $W^u_{(s(t), t)}(r)$ for all $t \in U$.

On another hand, $W^u_{(y, \tau)}$ and $W^s_{(z, \tau)}$ have a transverse intersection (w, τ). Using the Lambda Lemma again, there is an N' such that if $n \geqslant N'$, there is a cell neighborhood D_{-n} of (w, τ) in $W^s_{(z, \tau)}$ such that $f^{-n}(D_{-n})$ is

$C^1\varepsilon$—close to $W^s_{f^{-n}(y,\,\tau)}(r)$, where $W^s_{f^{-n}(y,\,\tau)}(r)$ is the local stable manifold at $f^{-n}(y,\tau)$. Because

$$\{V^{-k}(\tau), k = 0, 1, 2, ...\}$$

is dense in T, there is a $k \geqslant N'$ such that $V^{-k}(\tau) \in U$. Since $k \geqslant N'$, $f^{-k}(D_{-k})$ is $\varepsilon - C^1$ close to $W^s_{f^{-k}(y,\,\tau)}(r)$. If we denote $V^{-k}(\tau)$ by t, then $t \in U$. Thus there is a cell in $W^u_{(a(t),\,t)}(r)$ which is $\varepsilon - C^1$ close to $W^u_{(s(t),\,t)}(r)$, and $W^u_{(a(t),\,t)}(r)$ and $f^{-k}(D_{-k})$ have a transverse intersection. Notice that

$$f^{-k}(D_{-k}) \subset W^s_{f^{-k}(z,\,\tau)},$$

and $W^s_{f^{-k}(z,\,\tau)}$ and $W^u_{(a(t),\,t)}$ have a transverse intersection. By the lemma above, for any $t \in T$, $W^s_{(l(t),\,t)}$ and $W^u_{(a(t),\,t)}$ have a transverse intersection. Similarly we can prove that for any $t \in T$, $W^u_{(l(t),\,t)}$ and $W^s_{(a(t),\,t)}$ have a transverse intersection. Therefore $A \sim L$. ∎

We denote the closure of the set of all the periodic-like points by $\bar\Theta$.

LEMMA 4.3. *If $\bar\Theta$ is hyperbolic, then there are only finitely many equivalence classes $\Lambda_1, ..., \Lambda_n$ such that $\Theta = \bigcup_{i=1}^n \Lambda_i$.*

Proof. Note that the equivalence relation $\sim$ partitions Θ. Fix $\tau \in T$ and consider the set $\Theta \cap \{M \times \{\tau\}\}$. We have an equivalence relation on this set, inherited from the relation above. Because $\bar\Theta$ is hyperbolic, there is an $\varepsilon > 0$ such that if $(x, \tau), (y, \tau) \in \Theta \cap \{M \times \{\tau\}\}$ and $d[(x, \tau), (y, \tau)] < \varepsilon$, then $W^s_{(x,\,\tau)}(r)$ and $W^u_{(y,\,\tau)}(r)$ intersect transversely and $W^u_{(x,\,\tau)}(r)$ and $W^s_{(y,\,\tau)}(r)$ intersect transversely. Thus there are only finitely many equivalent classes in $\Theta \cap \{M \times \{\tau\}\}$. If we denote them by $\Pi_1, ..., \Pi_n$, then

$$\Lambda_i = \{(x, t) \in \Theta, A \cap \Pi_i \neq \phi, \text{ where } A \text{ is the periodic-like}$$

$$\text{set containing } (x, t)\}.$$

Therefore, $\Lambda_1, ..., \Lambda_n$ are all of the equivalence classes of Θ. ∎

From this proof, we see that for any fixed $\tau \in T$,

$$\Theta \bigcap \{M \times \{\tau\}\} = \bigcup_{i=1}^n \Pi_i$$

and $d(\Pi_i, \Pi_j) > \varepsilon$, $i \neq j$.

Definition: Suppose $A_1, A_2, ..., A_m$ are sets. If there is an $\varepsilon > 0$ such that $d(A_i, A_j) \geqslant \varepsilon$ for $i \neq j$, then we say that $A_1, A_2, ..., A_m$ have the *separated property* and if $\Lambda_1, ...\Lambda_n$ have the separated property, then we say Θ has the *separated property*.

If Θ has the separated property, then $\bar{\Theta}$ can be divided into n closed disjoint invariant sets, namely, the closures of $\Lambda_1, ..., \lambda_n$. The closure of Λ_i will still be denoted by Λ_i. Note that $\bar{\Theta} = \bigcup_{i=1}^n \Lambda_i$.

LEMMA 4.4. *If $\bar{\Theta}$ is hyperbolic and Θ has the separated property, then $\bar{\Theta}$ can be divided into finitely many closed disjoint invariant sets $\Lambda_1, ..., \Lambda_n$ such that each Λ_i is topologically transitive.*

We use the usual definition of the limit set $L(f)$ and nonwandering set $\Omega(f)$. They are closed invariant sets. Suppose Θ is the set of periodic-like points. It is not difficult to see that $\Theta \subset L(f) \subset \Omega(f)$.

Suppose Θ is dense in $\Omega(f)$. Then $\bar{\Theta} = \Omega(f)$. According to the lemma above, $\Omega(f)$ can be divided into finitely many topologically transitive sets in which the periodic-like points are dense. If $\Omega(f)$ has the no-cycle property, then we have a filtration—see [21].

Definition: If $\Omega(f)$ is a hyperbolic closed invariant set and Θ is dense in $\Omega(f)$, then we say that $\Omega(f)$ satisfies *axiom A*.

THEOREM 4.4. *Assume Θ has the separated property and $\Omega(f)$ satisfies axiom A. If $\Omega(f)$ has the no-cycle property, then f is Ω-stable.*

Proof. [Sketch] Notice that here, the "hyperbolic" means something different than it does for regular dynamical systems. Since $\Omega(f)$ satisfies axiom A, $\bar{\Theta} = \Omega(f)$, and by the separated property of Θ,

$$\bar{\Theta} = \Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_n.$$

However, $\Omega(f)$ has the no-cycle property. Without loss of generality, we can assume that the ordering by indices is a filtration ordering. Thus there is a filtration

$$\Phi = M_0 \subset M_1 \subset \cdots M_n = M \times T$$

such that

$$\Lambda_i = \bigcap_{j=-\infty}^{\infty} f^j(M_i - M_{i-1}).$$

Thus, each Λ_i is a closed isolated invariant sets. We have proved the stability of hyperbolic isolated closed invariant sets. Thus there is a neighborhood W of f such that each Λ_i topologically conjugates to a closed invariant set Δ_i of g for any $g \in W$. Because periodic-like points are invariant under conjugacy and periodic-like points are dense in each Λ_i, $\bigcup_{i=1}^n \Delta_i \subset \Omega(g)$. If W is small, this filtration is also a filtration adapted to g. Using this we can prove $\bigcup_{i=1}^n \Delta_i \supset \Omega(g)$. This gives the conjugacy of f and g. Therefore, f is Ω-stable. ∎

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