The Stability of the Lagrange Triangular Point and a Theorem of Arnold

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1. INTRODUCTION

Deprit and Deprit-Bartholomé [5] computed the normal form for the Hamiltonian of the planar restricted three body problem at the Lagrange equilateral triangle equilibrium point up to terms of order 4. This normalization was carried out for all mass ratios μ smaller than Routh's critical mass ratio $\mu_1 = \frac{1}{2}(1 - \sqrt{69}/9)$ except for two values μ_2 and μ_3 . At μ_2 and μ_3 the ratio of the linearized frequencies is 1:2 and 1:3, respectively, and so low order resonance terms appear. This normalization was carried out in order to apply the KAM theory to prove the stability of this equilibrium point. From the coefficients of the normal form they computed a quantity D_4 (defined below) which when non-zero establishes the stability of the equilibrium point. They found that $D_4 \neq 0$ for $0 < \mu < \mu_1, \ \mu \neq \mu_2$, μ_3 and μ_c . The value μ_c does not correspond to resonance but is simply a point where the quantity D_4 changes sign. In this paper we shall establish the full stability of the Lagrange equilibrium point in the planar restricted three body problem even in the case when $\mu = \mu_c$. We do this by computing by machine the normal form up to terms of order 6 and then applying a theorem of Arnold [3] which establishes the stability of the equilibrium even in degenerate cases.

Russman [10] announced a theorem which implies that the equilibrium is isoenergetic stable when $\mu = \mu_c$. By "isoenergetic stable" he means that the system is stable only on the energy surface defined by the energy at the equilibrium itself. Unfortunately there is not even an indication of the proof.

Arnold announced his theorem in a Doklady note with only a sketch of a proof. The sketch indicates that his proof requires completely redoing all

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the estimates ordinarily needed to prove the theorem from scratch. Because of this we present an argument in Section 2 which reduces Arnold's generalization to a simple application of the invariant curve theorem.

In Section 3 we present the full normal form up to terms of order 6 and show that a second quantity $D_6 \neq 0$ at μ_c . This computation was done by a computer package developed by the second author. First the value of μ_c was computed where the coefficients were carried as floating point numbers as is quite common in the celestial mechanics literature. It is more precise and pleasing to know D_6 as a function of μ and then to evaluate it at the critical value μ_c in order to establish the stability criteria. Therefore the computation was carried out using rational numbers as coefficients. Since this introduces a multitude of new terms and far greater complexity a full discussion of the method is given in Section 4.

2. Arnold's Theorem

Consider a Hamiltonian H in the canonical coordinates x_1, x_2, y_1, y_2 of the form

$$H = H_2 + H_4 + \dots + H_{2n} + H^* \tag{1}$$

where

(i) H is real analytic in a neighborhood of the origin in \mathbb{R}^4 ;

(ii) H_{2k} , $1 \le k \le n$, is a homogeneous polynomial of degree k in I_1 , I_2 , where

$$I_i = \frac{1}{2}(x_i^2 + y_i^2);$$

(iii) H^* has a series expansion which starts with terms at least of order 2n + 1;

- (iv) $H_2 = \omega_1 I_1 \omega_2 I_2$; ω_i positive constants;
- (v) $H_4 = \frac{1}{2}(AI_1^2 2BI_1I_2 + CI_2^2); A, B, C$ constants.

There are several implicit assumptions in stating that H is of the above form. Since H is at least quadratic in $x_1,..., y_2$ the origin is assumed to be the equilibrium point in question. Since $H_2 = \omega_1 I_1 - \omega_2 I_2$ is the Hamiltonian of two harmonic oscillators with frequencies ω_1 and ω_2 , the linearization at the origin of the system of equations whose Hamiltonian is H is two harmonic oscillators. Since H_2 is not sign definite a simple appeal to Liapunov's stability theorem cannot be made. Since $H_2,..., H_{2n}$ depend only on I_1 and I_2 the Hamiltonian is assumed to be in Birkhoff's normal form up to terms of degree 2n. This usually requires some non-resonance assumptions on the frequencies ω_1 and ω_2 , but for our purposes it suffices to assume that H is in this form. Arnold's theorem [3] is:

THEOREM. The origin is stable for the system whose Hamiltonian is (1) provided for some k, $2 \le k \le n$, $D_{2k} = H_{2k}(\omega_2, \omega_1) \ne 0$ or equivalently provided H_2 does not divide H_{2k} .

Proof. Assume now that $D_4 = D_6 = \cdots = D_{2n-2} = 0$ but $D_{2n} \neq 0$. Under this assumption there exist homogeneous polynomials F_{2k} , k = 2,..., n-2, of degree k such that $H_{2k} = H_2 F_{2k-2}$. So the Hamiltonian (1.1) becomes

$$H = H_2(1 + F_2 + \dots + F_{2n-4}) + H_{2n} + H^*.$$
⁽¹⁾

Introduce canonical action angle variables $(I_1, I_2, \phi_1, \phi_2)$ by $I_i = \frac{1}{2}(x_i^2 + y_i^2)$, $\phi_i = \arctan(y_i/x_i)$. Since we wish to consider a small neighborhood of the origin scale the variables by $I_i = \varepsilon^2 J_i$, where ε is a small positive parameter. This is a canonical change of variables with multiplier ε^{-2} . Let $F = 1 + \varepsilon^2 F_2 + \cdots + \varepsilon^{2n-4} F_{2n-4}$ so that

$$H = H_2 F + \varepsilon^{2n-2} H_{2n} + O(\varepsilon^{2n-1}).$$
⁽²⁾

Fix a bounded neighborhood of the origin, say $|J_i| \leq 4$, and call it N so that the remainder term is uniformly $O(\varepsilon^{2n-1})$. We shall restrict our attention to this neighborhood henceforth without explicit mention. Let h be a new parameter which is to lie in the bounded interval [-1, 1]. Since $F = 1 + O(\varepsilon^2)$ we can rewrite (2) as

$$H - \varepsilon^{2n-1}h = KF \tag{3}$$

where

$$K = H_2 + e^{2n-2}H_{2n} + O(e^{2n-1}).$$
(4)

Since $F = 1 + O(\varepsilon^2)$, for sufficiently small ε the function F is positive (on N) so the level set L where $H = \varepsilon^{2n-1}h$ is the same as the level set where K = 0. Let $z = (J_1, J_2, \phi_1, \phi_2)$, ∇ be the gradient operator with respect to z and J be the usual 4×4 skew symmetric matrix of Hamiltonian mechanics. In this notation the equations of motion are

$$\dot{z} = J\nabla H = (J\nabla K)F + K(J\nabla F).$$
⁽⁵⁾

On the level set L where K=0 this equation becomes

$$\dot{z} = (J\nabla K) F. \tag{6}$$

For sufficiently small ε the function F is positive so reparameterize by $d\tau = F dt$ and the equation (6) becomes

$$z' = J\nabla K, \qquad ' = \frac{d}{d\tau}.$$
 (7)

Thus, we have shown that in the neighborhood N, for small ε , the flow defined by H on the level set where $H = \varepsilon^{2n-1}h$ is a reparameterization of the flow defined by K on the level set K = 0. Thus it suffices to consider the flow defined by K on the set K = 0. To that end, the equations of motion defined by K are

$$J'_{i} = O(\varepsilon^{2n-1})$$

$$\phi'_{1} = -\omega_{1} - \varepsilon^{2n-2} \frac{\partial H_{2n}}{\partial J_{1}} + O(\varepsilon^{2n-1})$$

$$\phi'_{2} = \omega_{2} - \varepsilon^{2n-2} \frac{\partial H_{2n}}{\partial J_{2}} + O(\varepsilon^{2n-1}).$$
(8)

Now following the computations in [8] we compute the Poincaré map for the section defined by $\phi_2 \equiv 0 \mod 2\pi$ in the level set K = 0.

From the last equation in (8) the time T required for ϕ_2 to increase by 2π is given by

$$T = \frac{2\pi}{\omega_2} \left(1 + \frac{\varepsilon^{2n-2}}{\omega_2} \frac{\partial H_{2n}}{\partial J_2} \right) + O(\varepsilon^{2n-1}).$$
(9)

Now integrate the second to last equation in (8) from time $\tau = 0$ to time $\tau = T$. Let ϕ_1 at $\tau = 0$ be ϕ_0 and ϕ_1 at time $\tau = T$ be ϕ so

$$\widetilde{\phi} = \phi_0 + \left(-\omega_1 - \varepsilon^{2n-2} \frac{\partial H_{2n}}{\partial J_1}\right) T + O(\varepsilon^{2n-1})$$

$$\widetilde{\phi} = \phi_0 - 2\pi \left(\frac{\omega_1}{\omega_2}\right) - \varepsilon^{2n-2} \left(\frac{2\pi}{\omega_2^2}\right) \left(\omega_2 \frac{\partial H_{2n}}{\partial J_1} + \omega_1 \frac{\partial H_{2n}}{\partial J_2}\right) + O(\varepsilon^{2n-1}).$$
(10)

In the above, the partials of H_{2n} are evaluated at (J_1, J_2) . From the relation K=0 one calculates that $J_2 = (\omega_1/\omega_2) J_1 + O(\varepsilon^2)$. Substitute this into (10) to eliminate J_2 and use Euler's theorem on homogeneous polynomials to get

$$\tilde{\phi} = \phi_0 + \alpha + \varepsilon^{2n-2} \beta J_1^{n-1} + O(\varepsilon^{2n-1}) \tag{11}$$

where $\alpha = -2\pi(\omega_1/\omega_2)$ and $\beta = -2\pi(n/\omega_2^{n+1}) H_{2n}(\omega_2, \omega_1)$. By assumption $D_{2n} = H_{2n}(\omega_2, \omega_1) \neq 0$ and so $\beta \neq 0$. With the equation $J_1 \rightarrow J_1 + O(\varepsilon^{2n-1})$.

equation (11) defines an area preserving map of an annular region, say $\frac{1}{2} \leq J_1 \leq 3$ for ε sufficiently small. There is an invariant curve for this map given by $J_1 = p(\phi)$, where $1 \leq p(\phi) \leq 2$ for all ϕ [9]. Thus there is an $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$ all solutions of (8) which start on K = 0 with initial condition J_1 less than 1 must have J_1 less than 2 for all τ . This is true for all $h \in [-1, 1]$ due to the uniformity in the error estimate. Since on K = 0 we know that $J_2 = (\omega_1/\omega_2) J_1 + O(\varepsilon^2)$, a bound on J_1 implies a bound on J_2 . Thus there are constants c_1 and c_2 such that if $J_1(\tau)$, $J_2(\tau)$ satisfy equations (8), start on K = 0, and $|J_i(0)| < c_i$ then $|J_i(\tau)| < c_2$ for all τ and all $h \in [-1, 1]$.

Going back to the original variables $(I_1, I_2, \phi_1, \phi_2)$ and the original H this means that for all ε sufficiently small, all solutions of the equations defined by the Hamiltonian (1) which start on $H = \varepsilon^{2n-1}h$ and satisfy $|I_i(0)| \le \varepsilon^2 c_1$ must satisfy $|I_i(t)| \le \varepsilon^2 c_2$ for all t. Since h is arbitrary in [-1, 1] the sets where $|I_i| \le \varepsilon^2 c_1$ (or $\varepsilon^2 c_2$) and $H = \varepsilon^{2n-1}h$ represent full neighborhoods of the origin and clearly they can be made arbitrary small by taking ε small. Thus the origin is stable and the theorem has been proved.

3. Application to the Restricted Three Body Problem

The preceding theorem will now be applied to the planar restricted problem of three bodies. Deprit and Deprit-Bartholomé [5] have used Arnold's theorem to show that the Lagrangian equilibrium point L_4 is stable for mass ratios μ smaller than Routh's critical mass ratio $\mu_1 = \frac{1}{2}(1 - \sqrt{69/9})$ except for three values μ_2 , μ_3 and μ_c . They have computed D_4 to be

$$D_4 = -\frac{36 - 541\omega_1^2 \omega_2^2 + 644\omega_1^4 \omega_2^4}{8(1 - 4\omega_1^2 \omega_2^2)(4 - 25\omega_1^2 \omega_2^2)}.$$
 (12)

At the mass ratios μ_2 and μ_3 the ratio of the frequencies ω_1 to ω_2 is 1:2 and 1:3, respectively. In this case the normal form of the Hamiltonian contains more lower order terms and is different from the form required for Arnold's theorem. Markeev [7] and also Alfriend [1, 2] have shown that L_4 is unstable when the mass ratio is equal to μ_2 or μ_3 .

In the interval $(0, \mu_1)$, μ_c is the one value where the discriminant D_4 of Arnold's theorem is zero. Our theorem applies to this case. We will show that $D_6 \neq 0$ and thus L_4 is also stable for this value of μ .

The planar restricted problem of three bodies is defined by the Hamiltonian function

$$H = \frac{1}{2} \left(P_1^2 + P_2^2 \right) + Q_2 P_1 - Q_1 P_2 - \frac{1 - \mu}{\rho_1} - \frac{\mu}{\rho_2}$$

where ρ_1 is the distance of the particle at (Q_1, Q_2) to the primary body of mass $1 - \mu$ at $(-\mu, 0)$ and ρ_2 is its distance to the other body of mass μ at $(1 - \mu, 0)$.

The substitution

$$Q_{1} \rightarrow Q_{1} + \frac{1}{2}(1 - 2\mu)$$

$$Q_{2} \rightarrow Q_{2} + \frac{1}{2}\sqrt{3}$$

$$P_{1} \rightarrow P_{1} - \frac{1}{2}\sqrt{3}$$

$$P_{2} \rightarrow P_{2} + \frac{1}{2}(1 - 2\mu)$$

translates the origin of the coordinate system to the equilibrium point L_4 and the transformed Hamiltonian reads

$$H = \frac{1}{2} \left(P_1^2 + P_2^2 \right) + Q_2 P_1 - Q_1 P_2 - \frac{\gamma}{2} Q_1 - \frac{\sqrt{3}}{2} Q_2 - \frac{1+\gamma}{2\rho_1} - \frac{1-\gamma}{2\rho_2}.$$
 (13)

For convenience we have set

$$\gamma = 1 - 2\mu$$

as it simplifies the expansion of the last two terms in (13). Since

$$\rho_1^2 = 1 + Q_1 + \sqrt{3}Q_2 + Q_1^2 + Q_2^2$$

and

$$\rho_2^2 = 1 - Q_1 + \sqrt{3} Q_2 + Q_1^2 + Q_2^2$$

the expansion of ρ_2^{-1} can be obtained from the one for ρ_1^{-1} by setting $Q_1 \rightarrow -Q_1$. Moreover the expansion of the last two terms in (13) as series in Q_1 and Q_2 can be obtained from the one for ρ_1^{-1} by multiplying with γ those terms which have an odd exponent for Q_1 .

The terms of second order in the expansion of (13) are

$$H_2 = \frac{1}{2} \left(P_1^2 + P_2^2 \right) + Q_2 P_1 - Q_1 P_2 + \frac{1}{2} Q_1^2 - \frac{3\sqrt{3}\gamma}{4} Q_1 Q_2 - \frac{5}{8} Q_2^2.$$
(14)

The first step requires a linear transformation which brings these second order terms into their normal form. From the corresponding system of linear differential equations one obtains the characteristic equation

$$\lambda^4 + \lambda^2 + \frac{27}{16}(1 - \gamma^2) = 0.$$

Its roots are distinct and purely imaginary for $\mu < \mu_1$. The four eigenvalues

are $\pm i\omega_1$, $\pm i\omega_2$. The real numbers ω_1 and ω_2 are determined unambigously by

$$0 < \omega_2 < \frac{1}{2}\sqrt{2} < \omega_1 < 1$$

$$\omega_1^2 + \omega_2^2 = 1$$

$$\omega_1^2 \omega_2^2 = \frac{27}{16}(1 - \gamma^2).$$

The resulting symplectic transformation to new coordinates q_1 , q_2 and their conjugate momenta p_1 , p_2 was given by Breakwell and Pringle [4]. It can also be found in Deprit and Deprit-Bartholomé [5]. We will repeat it here in the form in which we have implemented it by machine. The transformation is given by

$$\begin{pmatrix} Q_1\\ Q_2\\ P_1\\ P_2 \end{pmatrix} = A \begin{pmatrix} q_1 \omega_1 / k l_1\\ q_2 \omega_2 / k l_2\\ p_1 / \omega_1 k l_1\\ p_2 / \omega_2 k l_2 \end{pmatrix}$$

where the 4×4 matrix A is

$$A = \frac{1}{2} \begin{pmatrix} 0 & 0 & l_1^2 & l_2^2 \\ -8 & -8 & -3\sqrt{3}\gamma & 3\sqrt{3}\gamma \\ -1 - 4\omega_1^2 & -1 - 4\omega_2^2 & 3\sqrt{3}\gamma & 3\sqrt{3}\gamma \\ 3\sqrt{3}\gamma & 3\sqrt{3}\gamma & 9 - 4\omega_1^2 & -9 + 4\omega_2^2 \end{pmatrix}$$

and

$$l_{1}^{2} = 9 + 4\omega_{1}^{2}$$
$$l_{2}^{2} = 9 + 4\omega_{2}^{2}$$
$$k^{2} = \omega_{1}^{2} - \omega_{2}^{2}.$$

The normal form for (14) is then

$$H_2 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2).$$

The normalization of higher order terms is performed more easily with complex coordinates. We therefore introduce another symplectic transformation to complex position coordinates x_1 , x_2 and their conjugate momenta y_1 , y_2 by

$$q_{j} = \frac{kl_{j}}{\omega_{j}} x_{j} + \frac{i}{2kl_{j}} y_{j},$$

$$p_{j} = ikl_{j} x_{j} + \frac{\omega_{j}}{2kl_{j}} y_{j},$$

$$j = 1, 2$$

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This transformation differs from the one usually employed at this stage. Normally factors of $\sqrt{2}$ and $\sqrt{\omega_j}$ are used to give the transformation a more symmetric appearance. For us this is unimportant and would actually cause more work if we had to keep track of fractional exponents. The factor kl_j is included in the transformation to simplify the computer programming. The combined transformation is then

$$\begin{pmatrix} Q_{1} \\ Q_{2} \\ P_{1} \\ P_{2} \end{pmatrix} = (A + iAB) \begin{pmatrix} x_{1} \\ x_{2} \\ y_{1}/2k^{2}l_{1}^{2} \\ y_{2}/2k^{2}l_{2}^{2} \end{pmatrix}$$
(15)

with

$$B = \begin{pmatrix} 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & \omega_2 \\ \omega_1^{-1} & 0 & 0 & 0 \\ 0 & \omega_2^{-1} & 0 & 0 \end{pmatrix}$$

The second order terms (14) read then

$$H_2 = i\omega_1 x_1 y_1 - i\omega_2 x_2 y_2. \tag{16}$$

Since $I_j = ix_j y_j$, j = 1, 2 establishes the relationship to the action variables we have the proper form for the second order terms.

After the Hamiltonian function H of (13) has been developed as a sum of homogeneous polynomials in the new set of variables (x_1, x_2, y_1, y_2) it will be normalized by Deprit's [6] method of Lie transformation. For this we need a notation which is slightly different from the one used above. We will write the Hamiltonian now as

$$H = \sum_{m=0}^{\infty} \frac{1}{m!} H_{m,0}$$
(17)

where the $H_{m,0}$ are the homogeneous polynomials of degree m+2. In particular $H_{0,0}$ is given by (16). The Hamiltonian function (17) will be transformed into

$$H^* = \sum_{n=0}^{\infty} \frac{1}{n!} H_{0,n} \tag{18}$$

with the help of a generating function

$$W = \sum_{j=0}^{\infty} \frac{1}{j!} W_{j+1}.$$
 (19)

The transformation is accomplished with the help of a doubly indexed array $H_{m,n}$ and is defined by

$$H_{m,n} = H_{m+1,n-1} + \sum_{k=0}^{m} \binom{m}{k} L_{W_{k+1}} H_{m+k,n-1}$$
(20)

The Lie derivative $L_W H$ turns out to be the Poisson bracket, that is,

$$L_{W}H = \sum_{j=1}^{2} \frac{\partial H}{\partial x_{j}} \frac{\partial W}{\partial y_{j}} - \frac{\partial H}{\partial y_{j}} \frac{\partial W}{\partial x_{j}}.$$
 (21)

For n=0 the functions $H_{m,n}$ are those of (17) and for m=0 they are those of (18).

The functions W_j which generate the transformation near the origin are selected one by one such that $H_{0,j}$ is in normal form. One finds that

$$H_{0,j} = K + L_{W_i} H_{0,0}, \qquad j = 1, 2, \dots, j = 1, 2,$$

where K contains only known terms, that is it depends only on W_s with s < j.

A typical term in K is $Ax_1^{\alpha_1}x_2^{\alpha_2}y_1^{\beta_1}y_2^{\beta_2}$. If the exponents are such that

$$|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| \neq 0$$

then the term can be eliminated from $H_{0,j}$ by selecting a term of the same form in W_j whose coefficient C is determined by

$$C = \frac{-iA}{(\alpha_1 - \beta_1) \,\omega_1 - (\alpha_2 - \beta_2) \,\omega_2}.$$

If $|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| = 0$ then the term in K belongs to $H_{0,i}$.

Figure 1 gives an outline of a computer program that implements the above procedure. It assumes that a package of computer programs for the formal manipulation of power series in several variables with complex coefficients is available.

A subroutine that is required is ADDTERM. It adds a monomial whose coefficient and exponents are given by the second and third parameter to the series given by the first parameter. In our package this routine is the most basic one, as others can be built up from it. For example the procedure SUB is constructed such that it takes the individual terms of the second series and subtracts them one by one from the series designated by the first parameter. The procedure LIE-DERIVATIVE works in a similar way with the following modification. Before the individual products generated by the Poisson bracket are added they are multiplied by a scalar. In Fig. 1 this scalar is BINOM.

```
LIE-TRIANGLE: Procedure( H,MAXROW ) ;
/* on entry: original Hamiltonian function in array H(0:MAXROW)
   on exit: normalized Hamiltonian through order MAXROW in H(*) */
  IND = 0;
  Do ROW = 1 To MAXROW ;
    IND = IND + 1;
    TEMP = H(ROW)
    Call COPY( F(IND), TEMP ) ;
    Do I = 1 To ROW ;
      M1, BINOM = 1 ;
      MO = ROW - I;
      J = IND;
      Do K = ROW To Max(2.1) By -1:
        J = J - K;
        L = L + 1
        Call LIE-DERIVATIVE( TEMP, BINOM, W(L), F(J) );
        BINOM = ( BINOM * MO ) / M1 ;
        M1 = M1 + 1;
        MO = MO - 1;
        End
      IND = IND + 1;
      If I < ROW Then Call COPY( F(IND), TEMP ) ;
      End :
    F(IND),W(ROW) = Null;
    for all terms in TEMP Do ;
      N1 = exponent of x1 - exponent of y1 ;
      N2 = exponent of x2 - exponent of y2;
      If N1 = 0 & N2 = 0 Then Do
        Call ADDTERM( F(IND), coefficient, exponents ) ;
        coefficient = 0 ;
        End ;
      Else Do ;
        temporary_coeff = (0-1I) * coefficient / ( W1*N1 - W2*N2 ) ;
        Call ADDTERM( W(ROW), temporary-coeff, exponents ) ;
        End ;
      End ;
    Do I = IND - ROW + 1 To IND By -1;
      Call SUB( F(I), TEMP ) ;
      End ;
    Call PURGE( TEMP )
    Call COPY( H(ROW), F(IND) ) ;
    End ;
  End ;
```

FIG. 1. Outline of a program for normalizing a Hamiltonian function.

How to reference the individual coefficients and exponents of a series has not been made precise in Fig. 1. To correct it we would have to describe in some details how series can be represented inside the computer, but this would lead us too far afield.

The variables W1 and W2 correspond to the frequencies ω_1 and ω_2 . We did not declare them on purpose. For now let us say that W_1 and W_2 receive their numerical values in the main program and that the statement

that uses them is an arithmetic expression. This statement which corresponds to (22) has to be replaced when the computations are done with ω_1 and ω_2 as formal variables because then we will divide a rational number by a linear polynomial in ω_1 and ω_2 .

As a final remark to our outline in Fig. 1 note that the index j of the one dimensional array F_j is related to the two indices of the $H_{m,n}$ from our earlier discussion by j = (m+n)*(m+n+1)/2 + n. The functions F_j are built up row by row as seen from the diagram

The functions in the first column F_0 , F_1 , F_3 ,..., are those of the given Hamiltonian. The functions on the diagonal F_0 , F_2 , F_5 ,..., are those of the transformed Hamiltonian.

POLYPAK is our package of computer programs for the manipulation of power series by machine. It is written and maintained for the latest release of the optimizing PL/I compiler of IBM.

When the computations are performed with complex floating point arithmetic it suffices to have four exponents for the set of variables (x_1, x_2, y_1, y_2) . From (12) we first determine the critical value $\mu_c \simeq 0.010913667677$. With it we find the other numerical quantities required for the linear transformation (15). After the transformation is available we construct the series for ρ_1^2 . A standard subroutine of POLYPAK provides ρ_1^{-1} . From there we get the expansion of (13) in the form of (16). We then call the subroutine LIE-TRIANGLE to get the normal form for the Hamiltonian function. After the substitution $ix_1 y_1 = \omega_2$ and $ix_2 y_2 = \omega_1$ we find

$$D_4 = 2.3E - 13 + i * 9.8E - 15$$

 $D_6 = -66.6 + i * 1.7E - 11.$

Due to roundoff errors D_4 is not exactly zero. For the same reason both quantities have a small imaginary part. Within the accuracy of the machine such errors have to be allowed for. Since the real part of D_6 is so much larger then the terms in error we can say that for all practical purposes we have demonstrated that $D_6 \neq 0$ and that L_4 is stable at the critical value μ_c .

4. COMPUTATIONS IN RATIONAL ARITHMETIC

Although the computations in floating point arithmetic have determined the stability of L_4 at μ_c it would have been more satisfying if we had obtained an expression for D_6 in a form similar to (12). By substituting the special values for ω_1 and ω_2 into it we also could decide the question of stability.

For this reason we have repeated the normalization with ω_1 and ω_2 kept as formal variables. In addition we need a variable for γ and another one to represent $\sqrt{3}$ as this time the computations will be carried out in rational arithmetic. Furthermore k, l_1 and l_2 appear in denominators of the linear transformation (15). As they can not be expressed as polynomials in ω_1 and ω_2 we introduce additional variables for them.

• More variables are needed still. Additional denominators arise from (22). Since we will stop after finding $H_{0,4}$ and do not compute W_4 we will encounter only denominators in (22) with $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 < 6$. They are

$$\omega_1 \pm \omega_2, 2\omega_1 \pm \omega_2, \omega_1 \pm 2\omega_2, 3\omega_1 \pm \omega_2, \omega_1 \pm 3\omega_2, 4\omega_1 \pm \omega_2, 3\omega_1 \pm 2\omega_1, 2\omega_1 \pm 3\omega_2, \omega_1 \pm 4\omega_2.$$
(23)

A closer inspection reveals that the last four sets are not needed. Although terms with these denominators occur in W_3 they contribute nothing to the normal form at the next order. As an illustration consider the denominator $\omega_1 + 4\omega_2$ in W_3 . It can only occur in combination with $x_1 y_2^4$ or $y_1 x_2^4$. Terms in W_3 contribute to those at order 6 only via $L_{W_3}H_{1,0}$. Considering the form of the Lie derivative (21) it becomes clear that neither $x_1 y_2^4$ nor $y_1 x_2^4$ in W_3 can generate a normalized term $x_1^{\alpha_1} x_2^{\alpha_2} y_1^{\beta_1} y_2^{\beta_2}$ at order 6 whose exponents satisfy $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, and $\alpha_1 + \alpha_2 = 3$. The same argument applies to the other denominators.

It is easiest to suppress the computation of these terms completely by considering and keeping only those terms whose exponents satisfy

$$|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| < 5.$$

This is permissible because the other terms which are affected by this condition belong to $H_{3-i,j}$, j = 0, 1 or 2. But they too can not influence a normalized term at order 6 as their contribution would come from $L_{W_1}H_{3-j,j}$ for j = 0, 1 or 2.

Since $k^2 = \omega_1^2 - \omega_2^2$ serves already as a common denominator for the first pair in (23) we introduce four additional variables to represent the next four pairs of denominators. We thus have a total of 15 variables. We list them in the order in which we used them in our program:

$$x_1, x_2, y_1, y_2, \sqrt{3}, \gamma, \omega_2, \omega_1, l_1, l_2, k,$$

$$(4\omega_1^2 - \omega_2^2)^{1/2}, (\omega_1^2 - 4\omega_2^2)^{1/2}, (9\omega_1^2 - \omega_2^2)^{1/2}, (\omega_1^2 - 9\omega_2^2)^{1/2}.$$

It would have been possible to use one common denominator for all terms at a given order. In this case the numerators would be polynomials in the first eight variables. When we tried this approach the rational coefficients of the individual terms in the numerator became too large and caused an overflow.

The other extreme is to work only with partial fractions. In this approach the multitude of different terms becomes a problem, as any series in POLYPAK can have at most 32767 individual terms.

Our approach tries to strike a balance between these two extremes. By selecting seven variables that stand for certain factors in the denominators we will get more terms than necessary but their rational coefficients will be small enough for our program to handle them.

The variables which we have introduced are not independent of each other. We will use this to simplify an intermediate result after all of its terms have been found. The following are straight forward substitutions:

$$\sqrt{3}^2 = 3, \gamma^2 = 1 - \frac{16}{27}\omega_2^2 + \frac{16}{27}\omega_2^4, \omega_1^2 = 1 - \omega_2^2.$$

They are used so that the exponents of $\sqrt{3}$, γ and ω_1 take only one of two values, i.e., 0 or 1 for $\sqrt{3}$ and γ . Unfortunately the symmetry between ω_1 and ω_2 is destroyed by these substitutions. Our strategy is to find a common denominator for similar terms and to express the resulting numerator as a polynomial in ω_2 , so that we can cancel any common factors. In this method we do not work with one denominator for all terms and therefore we were able to keep the size of the coefficients within the capabilities of our program.

Figure 2 lists the normalized terms of order 6, as they have been computed by our program. It will be noted that we print real coefficients only. Actually the entire series has to be multiplied by $\sqrt{-1}$. It turns out that in rational arithmetic all coefficients were either purely real or imaginary. It was therefore more efficient to do the computations in real arithmetic and to remember during the programming which series had to be multiplied with the imaginary unit.

From the terms of $H_{0,4}$ we compute D_6 as required by our theorem. After some simplifications we find

$$\begin{split} D_6 &= (-3105/4 + 1338449/48\sigma - 489918305/1728\sigma^2 \\ &+ 7787081027/6912\sigma^3 - 2052731645/1296\sigma^4 - 1629138643/324\sigma^5 \\ &+ 1879982900/81\sigma^6 + 368284375/81\sigma^7)/(\omega_1 * \omega_2 * (\omega_1^2 - \omega_2^2)^5 \\ &* (4 - 25\sigma)^3 * (9 - 100\sigma)) \end{split}$$

where we have set $\sigma = \omega_1^2 \omega_2^2$.

FIG. 2. Coefficients for normalized Hamiltonian at order 6. $d_1 = (4\omega_1^2 - \omega_2^2)^{1/2}$ $d_2 = (\omega_1^2 - 4\omega_2^2)^{1/2}, d_3 = (9\omega_1^2 - \omega_2^2)^{1/2}, d_4 = (\omega_1^2 - 9\omega_2^2)^{1/2}.$

When we substitute in (24) the special values for ω_1 and ω_2 we obtain the same answer as the one given by the computations in floating point arithmetic, that is $D_6 \approx -66.6$. This provides the necessary check on the correctness of our formula for D_6 .

Another check on the correctness of our program is that at order 4 it reproduces the formula (12) for D_4 . Due to the large amount of calculations that is required to arrive at this formula only a few verifications of (12) have been performed but nothing about it has been published. As our computations by machine can be seen as an independent check for the work of Deprit and Deprit-Bartholomé in [5] we would like to take this opportunity to affirm the correctness of their work.

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5. FINAL REMARKS

All computations were performed on the Amdahl V7 of the University of Cincinnati. The machine was running under the operating system MVS of IBM. The computations in floating point arithmetic took 3 seconds of CPU time whereas the computations in rational arithmetic required 17 minutes and 30 seconds.

The use of rational numbers is only in a small part responsible for the large diference in the computing times. The main reason is that we need 15 variables instead of only 4 in the first case and that the coefficients of the various powers of position and momenta variables have become rational functions in the additional 11 variables and are not simple floating point numbers anymore. A significant amount of time was spent in checking and removing common factors in these rational functions.

It might be of interest to redo these calculations in a general purpose system like MACSYMA or REDUCE. They allow for arbitrary length rational numbers and all factoring could be postponed until the end. On the other hand the large number of terms that have to be computed may overwhelm these systems.

More information on POLYPAK and a copy of the program can be obtained from the second author upon request.

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