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Hill's Lunar Equations and the Three-Body Problem*,†

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This paper uses the method of symplectic scaling to derive Hill's lunar equations from the equations of the three-body problem. This derivation gives a precise asymptotic statement about the relation between Hill's equations and the three-body problem. It is shown that any non-degenerate periodic solution of Hill's equation whose period is not a multiple of 2π can be continued into the full three-body problem.

1. INTRODUCTION

One of Hill's major contributions to celestial mechanics was his reformulation of the main problem of lunar theory; that is, he gave a new definition for the equations of the first approximation for the motion of the moon [2]. Since his equations of the first approximation contained more terms, the perturbations were smaller and hence he was able to obtain series representations for the position of the moon which converge more rapidly than the previously obtained series. Indeed for many years lunar ephemerides were computed from the series developed by Brown who used the main problem as defined by Hill. Even today most of the searchers for more accurate series solutions for the motion of the moon use Hill's definition of the main problem.

Previous to Hill, the main problem consisted of two Kepler problems—one describing the motion of the earth and moon about their center of mass and the other describing the motion of the sun and the center of mass of the earth–moon system. The coupling terms between the two Kepler problems are neglected at the first approximation. Delaunay used this

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definition of the main problem for his solution of the lunar problem, but after 20 years of computation was unable to meet the observational accuracy of his time.

In Hill's definition of the main problem the sun and the center of mass of the earth–moon system still satisfy a Kepler equation but the motion of the moon is described by a different system of equations known as Hill's lunar equations. Using heuristic arguments about the relative size of various physical constants, he concluded that certain other terms were sufficiently large that they should be incorporated into the main problem. This heuristic grouping of terms does not lead to a precise description of the relationship between the equations of the first approximation and the full problem. Even crude error estimates are difficult to obtain.

In a popular description of Hill's lunar equations one is asked to consider the motion of an infinitesimal body (the moon) which is attracted to a body (the earth) fixed at the origin. The infinitesimal body moves in a rotating coordinate system which rotates so that the positive x-axis points to an infinite body (the sun) which is infinitely far away. The ratio of the two infinite quantities is taken so that the gravitational attraction of the sun on the moon is finite. Although picturesque, this definition does not obviate the connection between Hill's lunar equations and the full three-body problem.

In this paper we shall use the method of symplectic scaling of the Hamiltonian in order to give a precise derivation of the main problem of lunar theory. Under one set of assumptions we shall derive the main problem as used by Delaynay and under another, the main problem as given by Hill. The derivations are precise asymptotic statements about the limiting behavior of the three-body problem and so can be used to give precise estimates on the deviation of the solutions of the first approximation and the full solutions. (The estimates are not sharp in the practical sense.) These derivations give a mathematically sound justification for the choice of Hill's definition of the main problem. We would like to suggest that the method of symplectic scaling is the proper method for defining the main problem for any mechanical problem. This method was used in [3] to define the main problem for three other problems in celestial mechanics.

As an illustration of how this precise asymptotic formula can be used we prove a theorem about the continuation of periodic solutions from Hill's lunar equations to the full three-body problem. We prove that any non-degenerate periodic solutions of Hill's lunar equations whose period is not a multiple of 2π can be continued into the full three-body problem. A similar theorem holds for symmetric periodic solutions.

2. Defining the Main Problem

In this section we shall show how to introduce scaled symplectic coordinates into the problem of three bodies in such a way that Hill's equations are the equations of the first approximation. We shall explore other scaled variables and see why they lead to poor approximations.

Consider a frame which rotates with constant angular frequency equal to one with reference to a fixed Newtonian frame and let $x_0, x_1, x_2; y_0, y_1, y_2$ be the position and momentum vectors relative to the rotating frame of three particles of masses m_0, m_1, m_2. If the particles are attracted to one another by Newton's law of gravity then the Hamiltonian defining the equations of motion of the three particles is

$$H = \sum_{i=0}^{2} \left\{ \frac{\|y_i\|^2}{2m_i} - x_i^T J y_i \right\} - \sum_{0 \leq i < j \leq 2} \frac{m_i m_j}{\|x_i - x_j\|}, \tag{1}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In our informal discussions we shall refer to the particles of mass m_0, m_1 and m_2 as the earth, moon and sun, respectively. Since we wish to eliminate the motion of the center of mass and also scale the distance between the earth and moon we choose to represent the equations in Jacobi coordinates. That is, we perform the following symplectic change of coordinates on (1),

$$\begin{aligned} u_0 &= (m_0 + m_1 + m_2)^{-1}\{m_0 x_0 + m_1 x_1 + m_2 x_2\}, \\ u_1 &= x_1 - x_0, \\ u_2 &= x_2 - (m_0 + m_1)^{-1}\{m_0 x_0 + m_1 x_1\}, \end{aligned} \tag{2}$$

$$\begin{aligned} v_0 &= y_0 + y_1 + y_2, \\ v_1 &= (m_0 + m_1)^{-1}\{m_0 y_1 - m_1 y_0\}, \\ v_2 &= (m_0 + m_1 + m_2)^{-1}\{(m_0 + m_1) y_2 - m_2(y_0 + y_1)\}, \end{aligned} \tag{3}$$

to obtain

$$H = \sum_{i=0}^{2} \left\{ \frac{\|v_i\|^2}{2M_i'} - u_i^T J v_i \right\} - \frac{m_0 m_1}{\|u_1\|} - \frac{m_1 m_2}{\|u_2 - v_0' u_1\|} - \frac{m_0 m_2}{\|u_2 + v_1' u_1\|}, \tag{4}$$

where

$$\begin{gathered} M_0' = m_0 + m_1 + m_2, \qquad M_1' = (m_0 + m_1)^{-1} m_0 m_1, \\ M_2' = (m_0 + m_1 + m_2)^{-1}(m_0 + m_1) m_2, \\ v_0' = (m_0 + m_1)^{-1} m_0, \qquad v_1' = (m_0 + m_1)^{-1} m_1. \end{gathered} \tag{5}$$

Since H is independent of u_0 (the center of mass), its conjugate variable v_0 (total linear momentum) is an integral. Thus there is no loss in generality in taking $u_0 = v_0 = 0$. Thus we shall proceed with the Hamiltonian defined in (4) with the summation extending from $i = 1$ to 2.

With the Hamiltonian in (4) as our starting point we shall proceed to make various assumptions on the size of various quantities until we are led to a definition of the equation of the first approximation for lunar theory. Each of these assumptions lead to a natural scaling of the variables. The first assumption is that the earth and moon have approximately the same mass but their masses are small relative to the mass of the sun. To that effect we let

$$m_0 = \varepsilon^{2\gamma}\mu_0, \qquad m_1 = \varepsilon^{2\gamma}\mu_1, \qquad m_2 = \mu_2, \tag{6}$$

where ε is a small positive parameter and γ is a positive integer to be chosen later. Since the masses of m_0 and m_1 are of order $\varepsilon^{2\gamma}$, so will be their momenta provided their velocities are of order 1. Although it is not altogether necessary, it will make the discussion clearer if we scale the momenta first in order to take this observation into account. In order to limit the proliferation of symbols the arrow notation common to scaling problems will be used; however, these actually represent changes of coorldinates. Thus we make the substitutions $v_1 \to \varepsilon^{2\gamma}v_1$, $v_2 \to \varepsilon^{2\gamma}v_2$ in (4). With this symplectic change of variables with multiplier $\varepsilon^{2\gamma}$ the Hamiltonian becomes

$$\begin{aligned} H &= H_1 + H_2 + O(\varepsilon^{2\gamma}), \\ H_1 &= \frac{\|v_1\|^2}{2M_1} - u_1^T J v_1 - \frac{\varepsilon^{2\gamma}\mu_0\mu_1}{\|u_1\|}, \\ H_2 &= \frac{\|v_2\|^2}{2M_2} - u_2^T J v_2 - \frac{\mu_1\mu_2}{\|u_2 - \nu_0 u_1\|} - \frac{\mu_0\mu_2}{\|u_0 + \nu_1 u_1\|}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} M_1 &= (\mu_0 + \mu_1)^{-1}\mu_0\mu_1, \qquad M_2 = \mu_0 + \mu_1, \\ \nu_0 &= (\mu_0 + \mu_1)^{-1}\mu_0, \qquad \nu_1 = (\mu_0 + \mu_1)^{-1}\mu_1. \end{aligned} \tag{8}$$

Note that the $O(\varepsilon^{2\gamma})$ depends only on $\|v_1\|$ and $\|v_2\|$.

The next assumption is that the distance between the earth and moon ($\|u_1\| = \|x_1 - x_0\|$) is small relative to the distance between the sun and the center of mass of the earth–moon system ($\|u_2\|$). We effect this assumption by making the change of variables $u_1 \to \varepsilon^{2\alpha}u_1$, where α is a positive integer to be chosen later. This is not a symplectic change of variables, but this will be corrected with further changes of variables given below. This change of variables makes H_2 in (7) independent of u_1 to the lowest order. Specifically

$$H_2 = H_3 + O(\varepsilon^{4\alpha}),$$
$$H_3 = \frac{\|v_2\|^2}{2M_2} - u_2^T J v_2 - \frac{\mu_2(\mu_0 + \mu_1)}{\|u_2\|}. \tag{9}$$

Note that the term of order $\varepsilon^{2\alpha}$ is zero due to the particular form of the constants v_0 and v_1. H_3 is the Hamiltonian of the Kepler problem, where a fixed body of mass μ_2 is located at the origin and another body of mass $\mu_0 + \mu_1$ moves in a rotating frame and is attracted to the fixed body of Newton's law of gravity. One can think of the fixed body as the sun and the other body as the union of the earth and moon.

The third and final assumption that we shall make is that the center of mass of the earth–moon system moves on a nearly circular orbit about the sun. Thus we need to prepare H_3 before effecting this assumption by a change of coordinates. Since H_3 is the Hamiltonian of a Kepler problem in rotating coordinates, one of the circular orbits becomes a circle of critical points for H_3. Specifically, H_3 has a critical point $u_2 = a$, $v_2 = -M_2 Ja$ for any constant vector a satisfying $\|a\|^3 = \mu_2$. Introduce coordinates,

$$Z = \begin{pmatrix} u_2 \\ u_2 \end{pmatrix}$$

and a constant vector

$$Z_0 = \begin{pmatrix} a \\ -M_2 Ja \end{pmatrix}$$

so that H_3 is a function of Z and $\nabla H_3(Z_0) = 0$. By Taylor's theorem

$$H_3(Z) = H_3(Z_0) + \tfrac{1}{2}(Z - Z_0)^T S(Z - Z_0) + O(\|Z - Z_0\|^3), \tag{10}$$

where S is the Hessian of H_3 evaluated at Z_0. Since constants are lost in the formation of the equations of motion we shall ignore the constant $H_3(Z_0)$ in our further discussions. Thus since we seek solutions which are nearly circular, we seek solutions where Z is close to Z_0. Thus we make the change of variables $Z - Z_0 \to \varepsilon^\beta U$, where β is again a positive integer to be chosen.

So far, starting with (7) we have proposed the following changes of variables $u_1 \to \varepsilon^{2\alpha} u_1$ and $Z - Z_0 \to \varepsilon^\beta U$. In order to have a symplectic change of variables (of multiplier $\varepsilon^{2\beta}$) we must make the further change $v_1 \to \varepsilon^{2(\beta-\alpha)} v_1$. Thus we propose the following symplectic change of variables in (7):

$$\begin{aligned} u_1 &\to \varepsilon^{2\alpha} u_1, \\ v_1 &\to \varepsilon^{2(\beta-\alpha)} v_1, \\ Z - Z_0 &\to \varepsilon^{\beta} U. \end{aligned} \tag{11}$$

Moreover we have introduced three positive integers α, β and γ as measures of the order of magnitude of three physical quantities. One of the variables α, β or γ could be fixed, but since we seek integer solutions it is best not to choose one of them too early.

First consider the main problem as defined by Delaunay. In this case the earth–moon system is a Kepler problem and so we must choose the scaling so that the kinetic energy and potential energy in H_1 are of the same order of magnitude. This leads to the restriction that $2\beta = \alpha + \gamma$. Also the difference between H_2 and H_3 which is of order $\varepsilon^{4\alpha}$ must be of higher order than either of the energy terms in H_1. This leads to the inequality $2\alpha > \beta$.

Since the equality $2\beta = \alpha + \gamma$ and the inequality $2\alpha > \beta$ do not lead to a unique solution we choose a small solution in integers, say, $\alpha = 2$, $\beta = 3$, $\gamma = 4$. With this choice the Hamiltonian becomes

$$H = \varepsilon^{-2} \left\{ \frac{\|v_1\|^2}{2M_1} - \frac{\mu_0 \mu_1}{\|u_1\|} \right\} + \left\{ \frac{1}{2} U^T S U - u_1^T S v_1 \right\} + O(\varepsilon^2). \tag{12}$$

Other choices of α, β and γ consistent with the two contraints lead to qualitatively similar scaled Hamiltonians. That is, the terms $U^T S U$ and $u_1^T J v_1$ are always of order zero and the terms $\|v_1\|^2$ and $1/\|u_1\|$ are of order $\varepsilon^{2\beta - 4\alpha}$, which has a negative exponent. In order to better understand this transformed Hamiltonian let us make one further change of variables. Define a new time by $\tau = \varepsilon^{-2} t$ and thus a new Hamiltonian by $K = \varepsilon^2 H$ so that the problem defined in the new time is defined by

$$K = \frac{\|v_1\|^2}{2M_1} - \frac{\mu_0 \mu_1}{\|u_1\|} + \varepsilon^2 \left\{ \frac{1}{2} U^T S U - u_1^T J v_1 \right\} + O(\varepsilon^4). \tag{13}$$

From the general theory of ordinary differential equations neglecting a term of order ε^4 in the worst possible case leads to an error of the form $O(\varepsilon^4) e^{L\tau} = O(\varepsilon^4) e^{L\varepsilon^{-2} t}$, where L is a constant. Thus neglecting the higher order terms is only valid for very short times. Since any choice of α, β and γ consistent with the constraints leads to the same qualitative form for the Hamiltonian tere is no way to overcome this difficulty. Clearly we must drop the inequality $2\alpha > \beta$ and incorporate more terms into the main problem.

Let us proceed to define the main problem as suggested by Hill. Since we still wish to have the two energy terms in H_1 of the same order of magnitude we still impose the restriction $2\beta = \alpha + \gamma$. The essential problem in the previous attempt was the fact that H_3 was not a good enough approximation of H_2. Following Hill, we expand the two troublesome terms in H_2 in a Legendre series as follows,

$$\frac{\mu_0\mu_1}{\|u_2+v_0u_1\|}+\frac{\mu_0\mu_2}{\|u_2-v_1u_1\|}$$
$$=\frac{\mu_2(\mu_0+\mu_1)}{\|u_2\|}+\frac{1}{\|u_2\|}\sum_{k=2}^{\infty} b_k\rho^k P_k(\cos\theta), \tag{14}$$

where $\rho=\|u_1\|/\|u_2\|$, $b_k=\mu_1\mu_2 v_0^k+\mu_0\mu_2(-v_1)^k$, θ is the angle between u_1 and u_2, and P_k is the kth Legendre polynomial. Thus (7) becomes

$$H=H_1+H_2-\frac{1}{\|u_2\|}\sum_{k=2}^{\infty} b_k\rho^k P_k(\cos\theta)+O(\varepsilon^{2\gamma}). \tag{15}$$

Hill said that the first term in the series should be of the same order of magnitude as the terms in H_1 and this leads to the condition $2\alpha=\beta$. The smallest positive integer solution of $2\alpha=\beta$ and $2\beta=\alpha+\gamma$ is $\alpha=1$, $\beta=2$, $\gamma=3$. With this choice of scale factors the Hamiltonian becomes

$$H=\frac{\|v_1\|^2}{2M_1}-u_1^TJv_1-\frac{\mu_0\mu_1}{\|u_1\|}-\frac{\beta_2}{\mu_2}\|u_1\|^2 P_2(\cos\theta)$$
$$+\frac{1}{2}U^TSU+O(\varepsilon^2). \tag{16}$$

Now from the general theory of differential equations, neglecting the $O(\varepsilon^2)$ terms leads to an error of order ε^2 on a bounded time interval. Thus defining the main problem as the Hamiltonian in (16) without the $O(\varepsilon^2)$ terms is a far better choice.

In order to reduce the number of constants in (16) we shall make one further scaling of the variables. We shall introduce new variables ξ and η to eliminate the subscripts and use the fact that $P_2(x)=\frac{1}{2}(1-3x^2)$. Also we choose $a=(\mu_2^{1/3},0)$ so that the abscissa points at the sun. Make the symplectic change of coordinates

$$\begin{aligned} u_1&=(\mu_0+\mu_1)^{1/3}\,\xi,\\ v_1&=(\mu_0+\mu_1)^{1/3}\,M_1\eta,\\ U&=(\mu_0+\mu_1)^{1/3}\,M_1^{1/2}V \end{aligned} \tag{17}$$

so that (16) becomes

$$H = \frac{\|\eta\|^2}{2} - \xi^T J\eta - \frac{1}{\|\xi\|} + (3\xi_1^2 - \|\xi\|^2)$$
$$+ V^T SV + O(\varepsilon^2). \tag{18}$$

Our choice of scaled variables has eliminated all parameters in Hill's equations. Note that we have fixed the time scale by requiring that the period of the sun's motion be 2π.

3. Continuation of Periodic Solutions

Hill proposed to construct a lunar theory by first finding a periodic solution of the system defined by the Hamiltonian

$$H' = \frac{1}{2}\|\eta\|^2 - \xi^T J\eta - \frac{1}{\|\xi\|} + (3\xi_1^2 - \|\xi\|^2) \tag{19}$$

and then continue this solution into the full problem. (The equations defined by (19) are known as Hill's lunar equations.) We shall justify this procedure by proving:

THEOREM. *Any non-degenerate periodic solution of Hill's lunar equations whose period is not a multiple of 2π can be continued into the full three body problem.*

More precisely, let $\xi = \phi_0(t)$, $\eta = \psi_0(t)$ be a τ_0 periodic solution of Hill's lunar equations with characteristic multipliers 1, 1, β, β^{-1}. Assume that this solution is non-degenerate; i.e., $\beta \neq 1$, and $\tau_0 \neq n2\pi$ for any integer n. Then there exist smooth functions $\phi(t, \varepsilon) = \phi_0(t) + O(\varepsilon^2)$, $\psi(t, \varepsilon) = \psi(t) + O(\varepsilon^2)$, $\tau(\varepsilon) = \tau_0 + O(\varepsilon^2)$ and $V(\varepsilon) = O(\varepsilon^2)$ defined for all t and small ε such that $\xi = \phi(t, \varepsilon)$, $\eta = \psi(t, \varepsilon)$, $V = V(\varepsilon)$ is a $\tau(\varepsilon)$ periodic solution of the system whose Hamiltonian is (18) (i.e., the three-body problem). Moreover the characteristic multipliers of this periodic solution are 1, 1, 1, 1, $\exp(\pm i\tau_0 + O(\varepsilon^2))$, $\beta + O(\varepsilon^2)$, $\beta^{-1} + O(\varepsilon^2)$.

We have carefully set up the equations so that the proof of this theorem is almost exactly the same as the proof of the analogous theorem for the restricted N-body problem given in [3]. (See in particular Sections II.C and III.A). Thus we shall only outline the proof of the above theorem here.

The system defined by (7) admits the total angular momentum integral

$$J = u_1^T J v_1 + u_2^T J v_2. \tag{21}$$

As before let $Z = (u_2, v_2)$ and let c be the row vector which is the gradient of $u_2^T J v_2$ with respect to Z evaluated at Z_0. Since $Z_0 \neq 0$ it follows that $c \neq 0$. The scaling which reduces (7) to (14) reduces (21) to

$$\begin{aligned} J &= \varepsilon^4 u_1^T J v_1 + \varepsilon^2 c V + O(\varepsilon^4) \\ &= \varepsilon^2 \{cV + O(\varepsilon^2)\}. \end{aligned} \tag{22}$$

The further scaling to obtain (18) only changes the constant vector c. Thus to lowest order in ε the angular momentum vector only depends on the u_2, v_2 or V coordinates. That is most of the angular momentum is in the sun and earth–moon system. Thus to lowest order the elimination of the angular momentum integral and its conjugate variable affects only the u_2, v_2 coordinates. Introduce polar coordinates in the u_2 plane and extend them to obtain a symplectic coordinate system on the u_2, v_2 space. Call these coordinates r, θ, R, Θ. To lowest order in ε, Θ is the total angular momentum and so when we fix angular momentum and ignore its conjugate variable we effectively eliminate Θ and θ. Fixing angular momentum and ignoring its conjugate variable reduces (18) to

$$H = H' + \frac{1}{2}\left\{\frac{R^2}{M} + Mr^2\right\} + O(\varepsilon^2) \tag{23}$$

(see [3, Sect. II.C]). Thus to zeroth order in ε the Hamiltonian of the three-body problem decouples into the sum of the Hamiltonian for Hill's lunar problem and the Hamiltonian of a harmonic oscillator.

When $\varepsilon = 0$ the equations of motion defined by (23) are decoupled and one easily sees that $\xi = \phi_0(t)$, $\eta = \psi_0(t)$, $R = r = 0$ is a τ_0 periodic solution with characteristic multiplier 1, 1, β, β^{-1}, $e^{i\tau_0}$, $e^{-i\tau_0}$. Since we assume that τ_0 is not a multiple of 2π, this periodic solution has precisely two characteristic multipliers equal to one and so is non-degenerate. Thus the standard theorem of perturbation analysis [1] says that this solution can be continued as a periodic solution of the full problem when $\varepsilon \neq 0$. This completes the outline of the proof.

We also note that Hill's lunar equations are symmetric with respect to the ξ_1 axis in the same manner as the restricted three-body problem is symmetric with respect to the line of masses. A symmetric periodic solution is one that crosses the line of symmetry orthogonally at two distinct times. In [3, Sect. II.D] the concept of non-degenerate symmetric periodic solution is defined and it should be noted that a periodic solution might be degenerate ($\beta = 1$) but still be a non-degenerate symmetric periodic solution. Following the arguments in [3, Sect. IV.A] we can also prove:

THEOREM. *Any non-degenerate symmetric periodic solutions of Hill's lunar equations whose period is not a multiple of 2π can be continued into the full three-body problem.*

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