JOURNAL OF DIFFERENTIAL EQUATIONS **39**, 2–38 (1981)

Periodic Solutions of the N-Body Problem

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Received July 29, 1978; revised August 30, 1979

This paper proves the existence of six new classes of periodic solutions to the N-body problem by small parameter methods. Three different methods of introducing a small parameter are considered and an appropriate method of scaling the Hamiltonian is given for each method. The small parameter is either one of the masses, the distance between a pair of particles or the reciprocal of the distances between one particle and the center of mass of the remaining particles. For each case symmetric and non-symmetric periodic solutions are established. For every relative equilibrium solution of the $(N-1)$-body problem each of the six results gives periodic solutions of the N-body problem. Under additional mild non-resonance conditions the results are roughly as follows. Any non-degenerate periodic solutions of the restricted N-body problem can be continued into the full N-body problem. There exist periodic solutions of the N-body problem, where $N-2$ particles and the center of mass of the remaining pair move approximately on a solution of relative equilibrium and the pair move approximately on a small circular orbit of the two-body problems around their center of mass. There exist periodic solutions of the N-body problem, where one small particle and the center of mass of the remaining $N-1$ particles move approximately on a large circular orbit of the two body problems and the remaining $N-1$ bodies move approximately on a solution of relative equilibrium about their center of mass. There are three similar results on the existence of symmetric periodic solutions.

I. INTRODUCTION

The special properties and intrinsic complexities of the N-body problem have captured the attention of many mathematicians over the centuries. Since no general solution is known, investigators have sought and found many special classes of solutions such as periodic and escape solutions. This paper is devoted to codifying and extending a large body of the results on the existence of periodic solutions of the N-body problem by small parameter methods.

The simplest periodic solutions of the N-body problem are those where the particles move uniformly along concentric circular orbits in a fixed plane. In

* This research was partially supported by NSF Grant MCS 780 1425.

0022-0396/81/010002–37$02.00/0

an appropriately chosen rotating coordinate system these solutions appear at rest and so they are called relative equilibria solutions. The geometric placement of these particles in this coordinate system is called a central configuration. Even though this is the simplest class of solutions it is manifold and seemingly unclassifiable. Usually central configurations are classified up to similarity transformations and this conversion will be used in this general discussion. For the two-body problem there is only one central configuration and for the three-body problem there are the equilateral triangle configurations of Lagrange and the collinear configurations of Euler (see Siegel and Moser [29] or Wintner [33] for a thorough discussion). For $N > 3$ only special configurations are known and a complete classification seems very difficult. Moulton [22] and Smale [30] have shown that there are precisely $N!/2$ collinear central configurations of the N-body problem and many other special cases are known. Palmore [25] has obtained a sharp lower estimate for the number of central configurations as a function of N.

The next simplest class of periodic solutions and the class we address are obtained by small parameter methods. This class consists of periodic solutions of the $(N+1)$-body problem, where N particles (or centers of masses of clusters of particles) move approximately on a relative equilibrium solution and the remaining particle moves approximately on a solution of Kepler's problem or a restricted problem. The results obtained here contain the central results of Arenstorf [2, 3, 5], Barrar [6], Conley [9], Crandall [10], Moulton [20, 21], Perron [26] and Siegel [28]. Specific references to these earlier works will be given at the appropriate points in the text.

The unifying theme exploited here is a scaling technique to introduce a small parameter and an appropriate definition of non-degenerate relative equilibrium. The small parameter may be a mass, a distance between two particles or the reciprocal of the distance from one particle to the center of mass of the remaining particles. The central configuration may or may not be symmetric and so six cases are considered. However, in each case a small parameter is introduced and the Hamiltonian scaled so that to a certain order the Hamiltonian of the $(N+1)$-body problem decouples into the sum of two terms each of which is the Hamiltonian of a simpler problem. In each case one of the terms is the Hamiltonian of the linearization of the N-body problem about a relative equilibrium. The nature of the other term depends on the manner in which the small parameter is introduced—if the small parameter is one of the masses then the other term is the Hamiltonian of the restricted $(N+1)$-body problem and if the small parameter is a distance or a reciprocal of a distance then the other term is the Hamiltonian of the Kepler problem in rotating coordinates. An approximate periodic solution is obtained by taking the relative equilibrium solution and a periodic solution of either the Kepler or the restricted problem. We then show that under mild non-resonance conditions these periodic solutions persist when the higher-order coupling terms are reintroduced. The proof relies on an appeal to either the standard implicit function theorem or the implicit function theorem of Arenstorf [2].

This paper is divided into four chapters with this introduction being the first. The contents of the remaining chapters and their sections are as follows:

II. Background

A. *Equations of Motion and Relative Equilibria.* The Hamiltonian of the N-body problem in fixed and rotating coordinates are given. Relative equilibrium solution and characteristic exponent of a relative equilibrium are defined. Some properties of the relative equilibria of the three-body problem are summarized.

B. *Reduction of Dimension.* The dimension of the system is reduced by fixing the center of mass, linear and angular momentum and identifying configurations which differ by a rotation. This defines the reduced space. The relation between the characteristic exponents on the full and reduced space is derived. Non-degenerate relative equilibrium is defined.

C. *Jacobi Coordinates.* An inductive definition of Jacobi coordinates is given and the Hamiltonian of the N-body problem and angular momentum are given in these coordinates. The calculation of the characteristic exponents of the relative equilibrium of the two-body problem is present in this section.

D. *Symmetry Conditions.* Some basic properties of systems which admit a discrete symmetry are given. A criteria for the existence of symmetric periodic solution is given in intrinsic form.

E. *Non-Degenerate Relative Equilibria.* This paper introduces a new definition of non-degenerate relative equilibria and this section proves that the new and traditional definitions are equivalent.

III. Non-symmetric Periodic Orbits

A. *Small Mass.* In this section it is shown that under mild non-resonance assumptions there are periodic solutions of the $(N+1)$-body problem where one particle of small mass moves approximately on a periodic solution of the restricted problem and the other N particles move approximately on a relative equilibrium solution.

B. *Bifurcation of a Primary.* In this section it is shown that under mild non-resonance conditions there are solutions of the $(N+1)$-body problem where $N-1$ particles and the center of mass of the other pair move approximately on a relative equilibrium solution and the pair move approximately on a small circular solution of the two-body problem about their center of mass.

C. *Orbits at Infinity.* In this section it is shown that under mild non-resonance conditions there are periodic solutions of the $(N+1)$-body problem where one particle of small mass and the center of mass of the remaining particles move approximately on a large circular solution of the two-body problem and the other N particles move approximately on a relative equilibrium solution.

IV. Symmetric Periodic Orbits

If the relative equilibria admits a line of symmetry additional symmetric periodic solutions can be shown to exist. The three sections of this chapter have the same title as those of the previous chapter and treat the same cases with additional symmetry assumptions.

The numbering of formulas and equations will begin anew in each section and a reference to an unqualified number will refer to the current section. However, "Eq. II.A.3" will refer to Eq. (3) in Section A of Chapter II. Vectors will be considered as column vectors but written as row vectors in the text.

The first major result of this paper is in Section III.A and the reader may skip Sections II.C,D and E to obtain this result. Sections II.A and B are needed for all subsequent sections, Section II.C for III.B,C, and IV.B,C; Section III.A (resp. III.B,C) for IV.A (resp. IV.B,C).

II. Background

A. *Equations of Motion and Relative Equilibria*

In this paper only the planar N-body problem will be treated. Let $q_1,\ldots, q_N \in \mathbb{R}^2$ be the position vectors of N particles in a Newtonian frame of reference. Let the particles have masses $m_1,\ldots, m_N$ and moments $p_1,\ldots,p_N$, respectively. The Hamiltonian of the N-body problem is

$$H = H_N = K + V, \tag{1}$$

where

$$K = \sum_{i=1}^{N} (\|p_i\|^2/2m_i) \tag{2}$$

is the total kinetic energy and

$$V = -\sum_{1 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|q_i - q_j\|} \tag{3}$$

is the potential energy of the system. The equations of motion are

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = p_i/m_i, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = \sum_{j=1}^{N}{}' \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} \end{aligned} \tag{4}$$

or

$$\ddot{q}_i = \sum_{j=1}^{N}{}' \frac{m_j (q_j - q_i)}{\|q_j - q_i\|^3}, \tag{5}$$

where $i = 1, \ldots, N$. Here and below the prime on the summation sign indicates that the term where $i = j$ is excluded.

Since the general solution of these equations is unknown for $N > 2$, investigators have sought special solutions. One special solution class has the N particles moving on concentric circles with uniform velocity. If the center of the circles (and hence the center of mass of this system) is at the origin these solutions must be of the form

$$\begin{aligned} q_i^* &= \exp(-\omega J t)\, a_i, \\ p_i^* &= -m_i \omega J \exp(-\omega J t)\, a_i, \qquad i = 1, \ldots, N, \end{aligned} \tag{6}$$

where $a_1, \ldots, a_N$ are constant vectors, ω is a positive number (the frequency), $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\exp(-\omega J t) = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$. In order for q_i^* and p_i^* of (6) to satisfy Eqs. (4) or (5) the vectors a_i and the number ω must satisfy the system of non-linear algebraic equations

$$\omega^2 a_i + \sum_{j=1}^{N}{}' \frac{m_j (a_j - a_i)}{\|e_j - a_i\|^3} = 0, \qquad i = 1, \ldots, N. \tag{7}$$

If $a_1, \ldots, a_N$ satisfy (7) then the geometric configuration of N particles of masses $m_1, \ldots, m_N$ at positions $a_1, \ldots, a_N$, respectively, is called a central configuration. If a similarity transformation which fixes the origin is applied to a solution set $a_1, \ldots, a_N$ of (7) then one obtains another solution $a'_1, \ldots, a'_N$ with possibly a different ω. Thus central configurations are usually classified up to a similarity transformation. There is clearly only one central configuration if $N = 2$ (take $a_1 = (m_2, 0)$ and $a_2 = (m_1, 0)$). For the three-body problem there are only the equilateral triangle solutions of Lagrange and the collinear configurations of Euler (see Siegel and Moser [29]). Moulton [22] and Smale [30] have shown that there are precisely $N!/2$ collinear central configurations of the N-body problem but a complete classification is still unknown.

If coordinates which rotate with the constant frequency ω are used these solutions will appear at rest. To accomplish this let $q_i = \exp(-\omega Jt)\, x_i$ and $p_i = \exp(-\omega Jt)\, y_i$ so that the Hamiltonian becomes

$$H = \sum_{i=1}^{N} \{\|y_i\|^2/2m_i - \omega x_i^T J y_i\} - \sum_{1 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|x_i - x_j\|} \tag{8}$$

and the equations of motion are

$$\begin{aligned} \dot{x}_i &= \omega J x_i + y_i/m_i, \\ \dot{y}_i &= \omega J y_i + \sum_{j=1}^{N}{}' \frac{m_i m_j (x_j - x_i)}{\|x_j - x_i\|^3}, \qquad i = 1,..., N \end{aligned} \tag{9}$$

(see Siegel and Moser [29]). The condition under which $x_i^* = a_i$ and $y_i^* = m_i \omega J a_i$ be constant solutions of (9) is again that ω and $a_1,..., a_N$ satisfy (7). Henceforth we shall always assume that $\omega = 1$ since this can be accomplished by a change in the time scale.

Let

$$J_k = \begin{pmatrix} 0_k & I_k \\ -I_k & 0_k \end{pmatrix},$$

where 0_k is the $k \times k$ zero matrix and I_k is the $k \times k$ identity matrix (so $J = J_1$). Let $Z = (x_1,..., x_N, y_1,..., y_N)$ and $Z^* = (a_1,..., a_N, -m_1 J a_1,..., -m_N J a_N)$ and so Eqs. (9) become

$$\dot{Z} = J_{2N} \nabla H(Z) \tag{10}$$

and since Z^* is a relative equilibrium $\nabla H(Z^*) = 0$. By Taylor's theorem $H(Z) = H(Z^*) + \frac{1}{2}(Z - Z^*)^T S(Z - Z^*) + O(\|Z - Z^*\|^3)$, where $S = (\partial^2 H/\partial Z^2)(Z^*)$ is the Hessian of H at Z^*. The linearization of Eqs. (9) or (10) about Z^* is

$$\dot{Z} = J_{2N} S Z \tag{11}$$

which is a Hamiltonian system with Hamiltonian $\frac{1}{2}Z^T S Z$. The characteristic values of $J_{2N} S$ will be called the characteristic exponents of the relative equilibrium and its characteristic polynomial will be called the characteristic polynomial of the relative equilibrium. Since the characteristic polynomial of a Hamiltonian matrix is even the characteristic exponents occur in pairs

$\lambda, -\lambda$ of the same multiplicity and the characteristic exponent 0 is of even multiplicity.

In general, it is quite difficult to calculate these characteristic exponents, however for $N = 2$ or 3 they are known. For $N = 2$ there is only one central configuration and it is easy to calculate that the characteristic polynomial is

$$\lambda^2(\lambda^2 + 1)^3 \tag{12}$$

(see Section II.C for this calculation). For $N = 3$ the computation of the characteristic polynomial is carried out in Siegel and Moser [29] and Wintner [33]. The characteristic polynomial for the Lagrange equilateral triangle configuration is

$$\lambda^2(\lambda^2 + 1)^3(\lambda^4 + \lambda^2 + \gamma), \tag{13}$$

where

$$\gamma = \frac{27}{4} \frac{(m_1 m_2 + m_2 m_3 + m_3 m_1)}{(m_1 + m_2 + m_3)^2}. \tag{14}$$

Again 0, $\pm i$ are characteristic exponents. The other four exponents are pure imaginary when $\gamma \leqslant \frac{1}{4}$ and complex when $\gamma > \frac{1}{4}$. A similar polynomial is known for the collinear configuration of Euler and besides 0, $\pm i$ there are one pair of pure imaginary and one pair of real characteristic exponents.

B. *Reduction of Dimension*

The Hamiltonian of the N-body problem (II.A.1) is invariant under the symplectic extension of the group of Euclidean motions of the plane and this fact introduces certain degeneracies which must be accounted for in a perturbation analysis. A Euclidean motion of the plane is of the form $q \to Aq + b$, where A is a rotation matrix and b is a constant vector and the symplectic extension of this transformation requires that $p \to Ap$. It is easy to check that H_N is invariant under the action $(q_1, ..., q_N, p_1, ..., p_N) \to (Aq_1 + b, ..., Aq_N + b, Ap_1, ..., Ap_N)$. This transformation carries a periodic orbit into a periodic orbit and so it follows that the periodic orbits of the N-body problem are not isolated even in an energy level. A theorem of the author's [18] says that in this problem the algebraic multiplicity of the characteristic multiplier $+1$ of a periodic solution must be at least 8. Unless these degeneracies are eliminated the standard methods of perturbation theory cannot be applied.

By a classical theorem this symmetry implies that the equations of motion II.A.4 admit

$$L = p_1 + \cdots + p_N, \tag{1}$$

linear momentum, and

$$I = q_1^T J p_1 + \cdots + q_N^T J p_N, \tag{2}$$

angular momentum, as integrals. Thus part, but not all, of the degeneracies can be eliminated by holding these integrals fixed; that is, consider the equation as defined on the invariant submanifold $B \subset \mathbb{R}^{2N} \times \mathbb{R}^{2N}$, where L and I are some fixed constants. Since B is of the odd dimension, $4N-3$, it cannot be a symplectic submanifold. A periodic solution of the N-body problem restricted to B would have the characteristic multiplier $+1$ of multiplicity at least 5. For simplicity assume that B is the submanifold of $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$, where $L = 0$ and $I = 1$. In this case B is carried into itself by the symplectic extension of the Euclidean group given above and so the periodic orbits are not isolated in B. However, if one identifies a point in B with all its images under this action one obtains a quotient space $D = B/\sim$ which is, in general, of dimension $4N-6$. The action is free and proper and so D is a manifold and by a theorem of [18] it is a symplectic manifold! In fact D inherits its symplectic structure in a natural way from the symplectic structure of $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ and the Hamiltonian H of II.A.1 and the flow defined by H are all well defined on D. In general, a periodic solution of the N-body problem would have the characteristic multiplier $+1$ of multiplicity 2 on D. Thus D is the natural space to study these equations. A relative equilibrium solution would reduce to a point on D. Thus a natural space to study the N-body is on D which is the space obtained from $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ by setting $L = 0$, $J = 1$ and identifying points which differ by a Euclidean motion. The details and generalization of these results are found in the author's paper [18]. This paper also shows that some periodic solutions of the three-body problem have the characteristic multiplier $+1$ of precisely multiplicity 2.

In order to ease the computations we shall proceed on a slightly modified reduction which is more closely related to the classical reduction as is given, for example, in Siegel and Moser [29]. First we start with the equations in the rotating coordinates of the previous section. Now the equations are no longer invariant under translations, but the center of mass of the system,

$$C = (m_1 x_1 + \cdots + m_N x_N)/M, \qquad M = m_1 + \cdots + m_N, \tag{3}$$

and the total linear momentum,

$$L = y_1 + \cdots + y_N, \tag{4}$$

satisfy the linear equation

$$\begin{aligned} \dot{C} &= JC + L/M, \\ \dot{L} &= JL. \end{aligned} \tag{5}$$

Thus $L = C = 0$ defines an invariant hyperplane $B_1 \subset \mathbb{R}^{4N}$ of dimension $4N - 4$. The restriction of Eqs. II.A.9 to B_1 will be called the first reduced system. A relative equilibrium of the type discussed in the previous section must lie in B_1. System (5) is linear and its characteristic polynomial is $(\lambda^2 + 1)^2$. Thus the characteristic polynomial of a relative equilibrium of the first reduced system is obtained from the characteristic polynomial of the full system by dividing out the factor $(\lambda^2 + 1)^2$.

Now we claim that B_1 is a symplectic subspace of $\mathbb{R}^{4N}$ and so the first reduced system is Hamiltonian. Consider the vectors

$$\alpha_1 = (0,..., 0; m_1, 0,..., m_N, 0)^T,$$
$$\alpha_2 = (0,..., 0; 0, m_1,..., 0, m_N)^T,$$
$$\beta_1 = (-1/M)(1, 0,..., 1, 0; 0,..., 0)^T,$$
$$\beta_2 = (-1/M)(0, 1,..., 0, 1; 0,..., 0)^T,$$

where $M = \sum m_i$ and the semicolon separates the first n components from the last n components. Then

$$B_1 = \{z \in \mathbb{R}^{4N}: \alpha_1^T JZ = \alpha_2^T JZ = \beta_1^T JZ = \beta_2^T JZ = 0\}$$

and

$$\alpha_1^T J\alpha_2 = \alpha_1^T J\beta_2 = \alpha_2^T J\beta_1 = \beta_1^T J\beta_2 = 0,$$
$$\alpha_1^T J\beta_1 = \alpha_2^T J\beta_2 = 1.$$

The span of $\alpha_1, \alpha_2, \beta_1, \beta_2$ is a symplectic subspace of $\mathbb{R}^{4N}$ and B_1 is its J-complement; thus B_1 is a symplectic subspace.

The Hamiltonian (II.A.9) is invariant under rotations, i.e., the symplectic mapping $x_i \to Ax_i$, $y_i \to Ay_i$ leaves it fixed for all rotations A. Clearly B_1 is invaariant under the same action and so the Hamiltonian of the first reduced system is invariant under this action. Thus the results of the author [18] apply to this system. In particular the first reduced system admits total angular momentum

$$I = x_1^T Jy_1 + \cdots + x_N^T Jy_N \tag{6}$$

as an integral. The gradient of I restricted to B_1 is zero only at the origin and so $B_2' = I^{-1}(c) \subset B_1$ is a regularly embedded submanifold for all $c \neq 0$. The action leaves I and B_2' invariant and is free and proper. Thus the quotient space D obtained from B_2' by identifying the orbits of this action to a point is a smooth manifold. By [18], D is a symplectic manifold and the first reduced system naturally projects to D. This system will be called the (full) reduced system and is the system which will be discussed below.

The local version of this second reduction is discussed in Whittaker [34, pp. 313–14]. Let Z_1^* be a relative equilibrium of the first reduced system, then by the classical theorem given by Whittaker there exist symplectic coordinates $w_1,..., w_{4N-4}$ such that $I = w_1$ and the Hamiltonian H is independent of w_{2N-1}. Thus the equations of motion are

$$\begin{aligned} \dot{w}_1 &= 0, & \dot{w}_{2N-1} &= \frac{-\partial H}{\partial w_1}, \\ \dot{w}_i &= \frac{\partial H}{\partial w_{2N-2+i}}, & \dot{w}_{2N-2+i} &= \frac{-\partial H}{\partial w_i}, \end{aligned} \tag{7}$$

where the partials of H are evaluated at $(w_1,..., w_{2N-2}, \cdot, w_{2N},..., w_{4N-4})$.

Locally the equations for the full reduced system are obtained by setting $w_1 = I_1(Z_2^*)$ in (7) and ignoring the equations for w_1 and w_{2N-1}. It is easy to see from (7) that this has the effect of removing a factor of λ^2 from the characteristic polynomial of the first reduced system.

In summary: *The reduced system is a Hamiltonian system of* $2N-3$ *degrees of freedom on a symplectic manifold D. The manifold D is obtained from the original phase space* $\mathbb{R}^{4N}$ *by setting the center of mass and linear momentum equal to zero, fixing angular momentum* $\neq 0$ *and identifying configurations which differ by a rotation. To a relative equilibrium* Z_0^* *of* Section II.A *there is a corresponding relative equilibrium* Z_2^* *of the reduced system. If the characteristic polynomial of the original relative equilibrium is* $p(\lambda)$ *then the characteristic polynomial of the relative equilibrium of the reduced system is* $p(\lambda)/\lambda^2(\lambda^2+1)^2$.

Similarly let us consider a periodic solution $\phi(t)$ of period τ of the full N-body problem which lies in the first reduced space. By considering (5) and (7) as has been done for equilibrium, this periodic solution has characteristic multipliers $e^{i\tau}, e^{i\tau}, 1, \gamma_4,..., \gamma_N, e^{-i\tau}, e^{-i\tau}, 1, \gamma_4^{-1},..., \gamma_N^{-1}$ and the characteristic multipliers of the projection of this periodic solution on the reduced space are $\gamma_4,..., \gamma_N, \gamma_4^{-1},..., \gamma_N^{-1}$.

The reduced space is the natural space to study relative equilibria and periodic solutions, in general, since all the symmetry has been eliminated. Thus we say that a relative equilibrium is *non-degenerate* if zero is not a characteristic exponent of the relative equilibrium on the reduced space. Thus the relative equilibrium is non-degenerate if zero is a characteristic exponent of multiplicity precisely 2. We shall say that a periodic solution of the N-body problem is non-degenerate if $+1$ is a characteristic exponent of multiplicity precisely 2 on the reduced space.

C. *Jacobi Coordinates*

Jacobi coordinates are ideal for several of the problems considered in this paper for several reasons. First one coordinate locates the center of mass of the system; thus it can be set to zero and ignored in subsequent considerations. Second, another coordinate is the vector from one particle to another and so can be easily scaled in the problem where two of the particles are close. Third, another coordinate is the vector to one particle from the center of mass of the other particles and so can be easily scaled in the problem where one particle is far from the others. Last, the Hamiltonian and angular momentum are relatively simple in these coordinates.

Because of the nature of the problems considered in this paper it is necessary to discuss the N- and $(N+1)$-body problems simultaneously. For later applications it is convenient to consider the $(N+1)$-body problem here and index the masses, position vectors and momentum vectors from 0 to N. Let $x_0, x_1, ..., x_N, y_0, ..., y_N$ be the rotating coordinates used in the previous sections. We follow a suggestion of André Deprit and use an inductive definition of Jacobi coordinates.

Set $g_0 = x_0$ and $\mu_0 = m_0$. Define a sequence of point transformations by

$$T_k: \quad \begin{aligned} u_k &= x_k - g_{k-1}, \\ g_k &= (1/\mu_k)(m_k x_k + \mu_{k-1} g_{k-1}), \\ \mu_k &= m_k + \mu_{k-1} \end{aligned} \tag{1}$$

for $k = 1, ..., N$. Thus μ_k is the total mass and g_k is the center of mass of the particles with index 0, 1,..., k. The vector u_k is the position of the kth particle relative to the center of mass of the previous particles. Consider T_k as a change of coordinates from $g_{k-1}, u_1, ..., u_{k-1}, x_k, ..., x_N$ to $g_k, u_1, ..., u_k, x_{k+1}, ..., x_N$ or simply from g_{k-1}, x_k to g_k, u_k. The inverse of T_k is

$$T_k^{-1}: \quad \begin{aligned} x_k &= (\mu_{k-1}/\mu_k)\, u_k + g_k, \\ g_{k-1} &= (-m_k/\mu_k)\, u_k + g_k. \end{aligned} \tag{2}$$

In order to make the linear symplectic extension of T_k (the Mathieu transformation) define $G_0 = y_0$ and

$$Q_k: \quad \begin{aligned} v_k &= (\mu_{k-1}/\mu_k)\, y_k - (m_k/\mu_k)\, G_{k-1}, \\ G_k &= y_k + G_{k-1} \end{aligned} \tag{3}$$

and

$$Q_k^{-1}: \quad \begin{aligned} y_k &= v_k + (m_k/\mu_k)\, G_k, \\ G_{k-1} &= -v_k + (\mu_{k-1}/\mu_k)\, G_k. \end{aligned} \tag{4}$$

If we denote the coefficient matrix in (1) by A then the coefficient matrices in (2), (3) and (4) are A^{-1}, A^{T-1} and A^T, respectively, and so the pair T_k, Q_k is a symplectic change of coordinates.

An easy calculation yields

$$g_{k-1}^T J G_{k-1} + x_k^T J y_k = g_k^T J G_k + u_k^T J v_k . \tag{5}$$

and

$$\| G_{k-1} \|^2 / 2\mu_{k-1} + \| y_k \|^2 / 2m_k = \| G_k \|^2 / 2\mu_k + \| v_k \|^2 / 2M_k, \tag{6}$$

where $M_k = m_k \mu_{k-1}/\mu_k$.

Since each transformation T_k, Q_k is symplectic for $k = 1,..., N$ the composition is symplectic and so the change of variables from $x_0,..., x_N$, $y_0,..., y_N$ to $g_N, u_1,..., u_N$, $G_N, v_1,..., v_N$ is symplectic. A simple induction on (5) and (6) shows that kinetic energy is

$$K = \sum_{i=0}^{N} \| y_i \|^2 / 2m_i = \| G_N \|^2 / 2\mu_N + \sum_{i=1}^{N} \| v_i \| / 2M_i \tag{7}$$

and total angular momentum I is

$$I = \sum_{i=0}^{N} x_i^T J y_i = g_N^T J G_N + \sum_{i=1}^{N} u_i^T J v_i . \tag{8}$$

Also g_N is the center of mass of the system and G_N is total linear momentum.

This induction definition does not lend itself to simple formulas for the u's and v's in terms of the x's and y's but we require only a few special properties of this representation. First note from (1) that

$$u_1 = x_1 - x_0 . \tag{9}$$

We claim that

$$x_0 = g_k - \sum_{l=1}^{k} (m_l/\mu_l)\, u_l \qquad \text{for} \quad k = 1,..., N. \tag{10}$$

Equation (10) is true when $k = 1$ since (2) gives $g_0 = (-m_1/\mu_1)\, u_1 + g_1$ and $g_0 = x_0$. Assume (10) for $k - 1$. So $x_0 = g_{k-1} - \sum_{l=1}^{k-1} (m_l/\mu_l)\, u_l$, but by (2) again $g_{k-1} = (-m_k/\mu_k)\, u_k + g_k$ and these two formulas yield (10).

Lastly, we claim that

$$x_j - x_i = u_j + \sum_{l=1}^{j-1} \alpha_{jil} u_l \qquad \text{for} \quad 0 \leqslant i < j \leqslant N, \tag{11}$$

where α_{jil} are constants. We prove (11) by induction on N. For $N = 1$ this is just (9). Now assume (11) for $N - 1$. We need only consider $j = N$ and so

$$x_N - x_i = (x_N - x_0) - (x_i - x_0). \tag{12}$$

By (10), $x_0 = g_{N-1} - \sum_{l=1}^{N-1}(m_l/\mu_l)\, u_l$ and by (1), $x_N = u_N + g_{N-1}$. Since $i < 1$ the induction hypothesis yields $x_i - x_0 = u_i + \sum_{l=1}^{i-1} \alpha_{i0l} u_l$. Substituting these last three relations into (12) yields (13).

Let $d_{ji} = x_j - x_i = u_j + \sum_{l=1}^{j-1} \alpha_{jil} u_l$. The Hamiltonian (II.A.8) becomes

$$\begin{aligned} H = \|G_N\|^2/2\mu_N &+ \sum_{i=1}^{N} \|v_i\|/2M_i \\ &- g_N^T J G_N - \sum_{i=1}^{N} u_i^T J v_i \\ &- \sum_{0 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|d_{ij}\|}. \end{aligned} \tag{13}$$

By (11), the last term in (13) (the potential energy) is independent of g_N and so the equations for g_N and G_N are

$$\begin{aligned} \dot{g}_N &= J g_N + G_N/\mu_N, \\ \dot{G}_N &= J G_N. \end{aligned} \tag{14}$$

These are the same equations as II.B.4 since $L = G_N$ and $C = G_N$. Thus the Hamiltonian of the N-body problem on the first reduced space is obtained from (13) by setting $g_N = G_N = 0$. That is,

$$H = \sum_{i=1}^{N} (\|v_i\|/2M_i - u_i^T J v_i) - \sum_{0 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|d_{ji}\|}. \tag{15}$$

Let us calculate the characteristic equation for the relative equilibrium for the two-body problem so $N = 1$ in (15). In this case

$$H = \|v_1\|/2M_1 - u_1^T J v_1 - \sum \frac{m_0 m_1}{\|d_{10}\|}, \tag{16}$$

where $M_1 = m_1 m_0/(m_1 + m_0)$ and $d_{10} = u_1$. Introduce the canonical polar coordinates by

$$\begin{aligned} u_1 &= r\cos\theta, & v_1 &= R\cos\theta - (\Theta/r)\sin\theta, \\ u_2 &= r\sin\theta, & v_2 &= R\sin\theta - (\Theta/r)\cos\theta \end{aligned} \tag{17}$$

so that (16) becomes

$$H = \{R^2 + \Theta^2/r^2\}/(2M_1) - \Theta - m_0 m_1/r. \tag{18}$$

where $M_1 = m_0 m_1/(m_0 + m_1)$. H is independent of θ and so Θ, angular momentum, is an integral. The equations of motion are

$$\dot{\theta} = \frac{\Theta}{Mr^2} - 1, \qquad \dot{\Theta} = 0,$$
$$\dot{r} = \frac{R}{M}, \qquad \dot{R} = \frac{\Theta^2}{Mr^3} - \frac{m_0 m_1}{r^2}. \tag{19}$$

These equations have an equilibrium point when θ is arbitrary and $R = 0$, $\Theta = m_0 m_1 (m_0 + m_1)^{-1/3}$, $r = (m_0 + m_1)^{1/3}$. As indicated in the previous section the equations on the reduced space are obtained by holding Θ fixed and ignoring θ so the equations become

$$\dot{r} = \frac{R}{M}, \qquad \dot{R} = Mr - \frac{m_0 m_1}{r^2}. \tag{20}$$

The linearization of these equations about the equilibrium point $R = 0$, $r = (m_0 + m_1)^{1/3}$ is

$$\dot{r} = R/M, \qquad \dot{R} = -MR. \tag{21}$$

The characteristic polynomial for the equations on the reduced space is therefore $\lambda^2 + 1$ and $\lambda^2(\lambda^2 + 1)^3$ on the full space.

D. *Discrete Symmetries*

Some central configurations admit a discrete symmetry which can be exploited in a perturbation analysis to establish the existence of additional periodic solutions. For example the collinear configuration is symmetric in the line of masses and the equilateral triangle configuration of the three-body problem with equal masses is symmetric in the three medians of the triangle. This section is not needed until Chapter IV.

Consider a central configuration of the N-body problem with masses $m_1, \ldots, m_N$ and position vectors (in rotating coordinates) $a_1, \ldots, a_N$ which admits a line of symmetry. Choose a coordinate system u, v for the corresponding restricted $(N+1)$-body problem (cf. Section III.A or [17]) so that the u_1-axis is the line of symmetry. Then the Hamiltonian

$$H_R = \|v\|^2/2 - u^T J v - \sum_{j=1}^{N} \frac{m_j}{\|a_j - u\|} \tag{1}$$

of this restricted $(N+1)$-body problem is invariant under the substitution

$$u_1 \to u_1, \qquad v_1 \to -v_1,$$
$$u_2 \to -u_2, \qquad v_2 \to v_2, \tag{2}$$

where $u = (u_1, u_2)$, etc. In this case an easy and classical argument establishes that if a solution crosses the line of symmetry orthogonally at times 0 and T ($\neq 0$) then this solution is $2T$-periodic and the orbit of this solution is symmetric with respect to the line of symmetry. Consider the class of solutions which cross the line of symmetry at time 0; specifically let $u = \phi(t, \alpha, \beta)$, $v = \psi(t, \alpha, \beta)$ be the solution of the restricted problem which satisfies

$$\begin{aligned} \phi_1(0, \alpha, \beta) &= \alpha, & \psi_1(0, \alpha, \beta) &= 0, \\ \phi_2(0, \alpha, \beta) &= 0, & \psi_2(0, \alpha, \beta) &= \beta. \end{aligned} \tag{3}$$

Then this solution with $\alpha = \alpha_0$, $\beta = \beta_0$ is $2T_0$-periodic if (T_0, α_0, β_0) satisfy the equations

$$\phi_2(T_0, \alpha_0, \beta_0) = 0, \qquad \psi_1(T_0, \alpha_0, \beta_0) = 0. \tag{4}$$

Such a symmetric periodic solution will be called non-degenerate if the Jacobian

$$\frac{\partial(\phi_2, \psi_1)}{\partial(t, \alpha, \beta)} (T_0, \alpha_0, \beta_0) \tag{5}$$

has rank 2. The implicit function theorem applied to Eqs. (4) implies that a non-degenerate symmetric periodic solution persists under small symmetric perturbations of the Hamiltonian.

Now consider the N-body problem in rotating coordinates introduced in Section II.A. The Hamiltonian is invariant under the substitution

$$\begin{aligned} x_{j1} &\to x_{j1}, & y_{j1} &\to -y_{j1}, \\ x_{j2} &\to -x_{j2}, & y_{j2} &\to y_{j2}. \end{aligned} \tag{6}$$

In order to treat the full N-body problem on the reduced space the classical results and ideas given above must be generalized slightly. The notation and elementary facts of symplectic geometry used below can be found in [1] and [18]. Let P be a symplectic manifold of dimension $2d$ with symplectic two form Ω, $f: P \to P$ an anti-symplectic involution (i.e., $f \circ f =$ identity and $Df(\Omega) = -\Omega$), and $K: P \to \mathbb{R}$, a Hamiltonian which is invariant under f (i.e., $K \circ f = K$). In our example $P = \mathbb{R}^4$ with the usual symplectic structure, f is given by (2) and $K = H_R$ in (1) for the restricted problem and $P = \mathbb{R}^{4N}$ with the usual symplectic structure, f is given by (6) and $K = H$ in II.A.8 for the N-body problem.

For the restricted problem an initial condition $p = (u_0, v_0)$ is an orthogonal crossing of the line of symmetry if and only if $f(p) = p$. In general, we shall say that $p \in P$ is a *symmetric initial condition* if $p \in Q = \{q \in P: f(q) = q\}$ and we will call Q the *symmetry manifold.*

LEMMA 1. *Q is a Lagrangian submanifold of P. For any $p \in Q$ there exist symplectic coordinates (ζ, ν) at p such that p corresponds to $\zeta = \nu = 0$, locally Q corresponds to $\nu = 0$, and such that the Jacobian matrix of f in these coordinates is* $\operatorname{diag}(I_d, -I_d)$.

Proof. Let $p \in Q$ and choose any symplectic coordinate system at p. Let F be the Jacobian matrix of f at p in these coordinates so that $F^T J_d F = -J$ and $F^2 = I$. Since $F^2 = I$ the only eigenvalues of F are ± 1 and F is diagonalizable. Let $F\alpha = \alpha$ and $F\beta = \beta$ so $\alpha^T J_d \beta = \alpha^T J_d F\beta = (-F\alpha^{-1})^T J_d \beta = -\alpha^T J_d \beta$ or $\alpha^T J_d \beta = 0$. Thus the eigenspace corresponding to the eigenvalue $+1$ (and for -1) is isotropic. Since the tangent space at p is the direct sum of the two eigenspaces these spaces must be of maximal dimension and thus are Lagrangian.

By the preliminary results of [15] there exist symplectic coordinates z at p such that $z(p) = 0$ and in these coordinates

$$F = \begin{pmatrix} I_d & 0_d \\ 0_d & -I_d \end{pmatrix}.$$

Let $f(z) = Fz + \Phi(z)$, where $\Phi(0) = D\Phi(0) = 0$, and define a change of coordinates (not necessarily symplectic) by $w = g(z) = z + \frac{1}{2}F\Phi(z)$. Since $f^2 =$ identity, $F\Phi(z) = -\Phi(Fz + \Phi(z))$ and this implies $g \circ f = \bar{f} \circ g$, where $\bar{f}$ is the linear map $w \to Fw$. Thus in w coordinates f is the linear reflection $w \to Fw$ and so locally the fixed set of f is a d-dimensional manifold. Since the eigenspace corresponding to $+1$ is the tangent space to Q at p, Q is a Lagrangian submanifold. The further statement on the existence of symplectic coordinates (ζ, ν) such that Q is locally $\nu = 0$ follows from the general theorems of [32].

LEMMA 2. *If $\gamma(t)$ is a solution of $dK^\#$ such that $\gamma(0) \in Q$ and $\gamma(T) \in Q$ for $T > 0$ then $\gamma(t)$ is $2T$-periodic and the orbit of γ is invariant under f.*

Proof. Since K is f invariant and f is anti-symplectic $dK(f(x))^\# = -Df(x)\, dK(x)^\#$ for all $x \in P$. Let $\delta(t) = f(\gamma(2T - t))$ so

$$\begin{aligned} \dot{\delta}(t) &= -Df(\gamma(2T - \gamma t))\, \dot{\gamma}(2T - t) \\ &= -Df(\gamma(2T - t))\, dK(\gamma(2T - t))^\# \\ &= dK(f(\gamma(2T - t)))^\# = dK(\delta(t))^\#. \end{aligned}$$

Thus $\delta(t) = f(\gamma(2T - t))$ and $\gamma(t)$ are both solutions of $dK^\#$ and are equal when $t = T$; therefore, by the uniqueness theorem for ordinary differential equations $f(\gamma(2T-t))=\gamma(t)$. Thus $f(\gamma(0))=\gamma(0)=\gamma(2T)$ which implies γ is $2T$-periodic.

LEMMA 3. *Let $p \in Q$ be a critical point of K and (ζ, ν) the symplectic coordinates of Lemma* 1. *Then*

$$K(\zeta,\nu)=K(0,0)+\tfrac{1}{2}\{\zeta^T A\zeta+\nu^T B\nu\}+\sigma(\|\zeta\|^2+\|\nu\|^2),$$

where A and B are $d \times d$ symmetric matrices.

Proof. Let S be the Hessian of K at p. Since K is f invariant $F^TSF=S$, where F is the Jacobian matrix of f at p. From the proof of Lemma 1, F in these coordinates is

$$\begin{pmatrix} I_d & 0_d \\ 0_d & -I_d \end{pmatrix}$$

and so

$$S=\begin{pmatrix} A & 0_d \\ 0_d & B \end{pmatrix}.$$

Let $\Xi(t,p)$ be the solution of $dK^{\#}$ such that $\Xi(0,p)=p$, p_0 such that $\gamma(t)=\Xi(t,p_0)$, where $\gamma(t)$ is the solution of Lemma 2 and $q_0=\gamma(T)$. Thus $\Xi\colon(T,p_0)\to q_0$ and $D\Xi\colon T_T\mathbb{R}\times T_{p_0}P\to T_{q_0}(P)$. The solution $\gamma(t)$ will be called a *non-degenerate* symmetric periodic solution if $T_{q_0}P=(T_{q_0}Q)+(D\Xi(T_T\mathbb{R}\times T_{p_0}Q))$. This condition is the same as the condition for transversal intersection of Q and $\Xi\colon\mathbb{R}\times Q\to P$ at q_0. For the restricted problem the two concepts of non-degeneracy are the same. General transversality theory [16] or a simple application of the implicit function theorem implies that a non-degenerate symmetric periodic solution persists under small symmetric perturbations of the Hamiltonian.

E. *Non-degenerate Relative Equilibrium*

In Section II.B the relative equilibrium was assumed to be non-degenerate in the sense that zero is not a characteristic exponent of the relative equilibrium on the reduced space. Other researchers use a different definition of non-degenerate which will be shown to be equivalent to the definition used here. Therefore the results of Palmore [23, 24] establish that almost all relative equilibria are non-degenerate and in particular the collinear relative equilibria in the N-body problem are non-degenerate. This section can be skipped completely upon first reading.

First investigate the meaning of non-degenerate in the sense of this paper. Let

$$x_i=a_i, \qquad y_i=m_iJa_i \tag{1}$$

be a relative equilibrium, i.e., a constant solution of II.A.8 with $\omega = 1$. Let $X = (x_1, ..., x_N)$, $Y = (y_1, ..., y_N)$, $K = \operatorname{diag}(J, ..., J)$, $A = (a_1, ..., a_N)$ and $S = -(\partial^2 V/\partial X^2)(A)$. Note that K commutes with S and M. The linearization of II.A.8 about the relative equilibrium $X = A$, $Y = MKA$ is

$$\begin{aligned} \dot{X} &= KX + M^{-1}Y, \\ \dot{Y} &= KY + SX. \end{aligned} \tag{2}$$

The relative equilibrium is degenerate in the sense of this paper if and only if there exist vectors $U = (u_1, ..., u_N)$, $V = (v_1, ..., v_N)$ and a real number α such that

$$\begin{aligned} &\text{(i)} && \textstyle\sum m_i u_i = \sum v_i = 0, \\ &\text{(ii)} && U^T MA + A^T KV = 0, \\ &\text{(iii)} && (U, V) \text{ is not parallel to } (KA, MA), \\ &\text{(iv)} && KU + M^{-1}V = \alpha KA, \\ & && SU + KV = \alpha MA. \end{aligned} \tag{3}$$

Condition (3i) asserts that (U, V) lie in the first reduced space. Condition (3ii) asserts that (U, V) are tangent to the angular momentum manifold at the relative equilibrium. Condition (3iii) asserts that (U, V) is not the zero vector in the full reduced space since (KA, MA) is tangent to the orbit of the rotation action at the relative equilibrium. Condition (3iv) asserts that the vector (U, V) is mapped onto the tangent to the rotation action and hence onto the zero vector in the quotient space.

Other researchers [23–25, 30] interpret Eqs. II.A.7 as the necessary and sufficient conditions for A to be a critical point of V restricted to the manifold $X^T MX = 1$, where ω^2 is the Lagrange multiplier. The solution A must have its center of mass at the origin and so $\sum m_i a_i = 0$. The function V is invariant under the rotation action $X \to e^{K\theta} X$ and so it is well defined on the quotient space obtained by identifying X and $e^{K\theta} X$. Thus the other researchers define a manifold $\tilde{M}$ obtained from R^{2N} by restricting X to satisfy $\sum m_i X_i = 0$, $X^T MX = 1$ and identifying X and $e^{K\theta} X$. The function V is naturally projected to a smooth function $\tilde{V}$ on $\tilde{M}$. Also a relative equilibrium A projects to a critical point $\tilde{A}$ of $\tilde{V}$ on $\tilde{M}$. The other definition of a non-degenerate relative equilibrium is that the Hessian of $\tilde{V}$ at $\tilde{A}$ is non-singular. A necessary and sufficient condition for a relative equilibrium to be non-degenerate in this sense is the existence of a vector $W = (w_1, ..., w_N)$ and a real number β such that

$$\begin{aligned} &\text{(i)} \quad \sum m_i w_i = 0, \\ &\text{(ii)} \quad W^T MA = 0, \\ &\text{(iii)} \quad W \text{ is not parallel to } KA, \\ &\text{(iv)} \quad (S + M)W = \beta MA. \end{aligned} \tag{4}$$

The interpretation of these conditions is similar to the previous interpretations.

In order to show these two definitions are equivalent a simple algebraic identity is needed. By II.A.7

$$MA - \frac{\partial V}{\partial X}(A) = 0 \tag{5}$$

and since $\partial V/\partial X$ is homogeneous of degree -2

$$t^2 \frac{\partial V}{\partial X}(tA) = \frac{\partial V}{\partial X}(A) \qquad \text{for all } t. \tag{6}$$

Differentiate (6) with respect to t and then set $t = 0$ to obtain

$$\partial \frac{\partial V}{\partial X}(A) + \frac{\partial^2 V}{\partial X^2}(A)A = 0 \tag{7}$$

(Euler's formula). Combining (5) and (7) yields

$$SA = 2MA. \tag{8}$$

Now assume there exist W and β satisfying (4). Define $U = W + \beta A$ and $V = MKW + \beta MKA$. It is easy to check that U and V satisfy (3i) and (3ii). With the aid of (8) it is easy to check that (U, V) satisfy (3iv) with $\alpha = 2$. Thus (3iii) must be verified. Assume that U is parallel to KA and so

$$U = \delta KA, \tag{9}$$

where $\delta \neq 0$. Thus

$$\begin{aligned} &W + \beta A = -KA, \\ &(S + M)(W + \beta A) = -\delta K(S + M)A, \\ &4\beta MA = -3\delta KMA, \\ &0 = 4\beta A^T KMA = 3\delta A^T MA \end{aligned}$$

and so $\delta = 0$. This contradiction implies U is not parallel to KA.

Conversely assume there exist U, V and α such that (3) holds. Let $W = U - (\alpha/2)A$. Then (4i) follows from (3i). Solving for V from the first equation in (3iv) yields $V = -MKU - \alpha MKA$ which when substituted into (3ii) yields (4ii). The proof that (4iii) implies (3iii) is similar to the proof of the converse given above. From the second equation in (3iv)

$$SU + KV = \alpha MA,$$
$$SU + K(MKU + \alpha MKA) = \alpha MA,$$
$$(S + M)U = 2\alpha MA$$

which is (4iv) with $\beta = 2\alpha$.

Thus the two defintions of non-degenerate relative equilibrium are the same.

III. Non-symmetric Periodic Orbits

A. *Small Mass*

In this section we show that under mild non-resonance assumptions a non-degenerate periodic solution of the restricted problem can be continued into the full $(N + 1)$-body problem. This result follows easily from a standard perturbation lemma after the Hamiltonian of the $(N + 1)$-body problem with one small mass has been scaled. This scaling shows that the restricted problem is the first approximation of the full problem with one small mass.

A non-trivial periodic solution of an autonomous Hamiltonian system has the characteristic multiplier $+1$ with algebraic multiplicity at least equal to 2. Roughly speaking one multiplier is $+1$ because the system is autonomous and one characteristic multiplier is $+1$ because the system admits an integral. A periodic solution of an autonomous Hamiltonian system will be called non-degenerate if the multiplicity of the characteristic multiplier $+1$ is precisely 2. A non-degenerate periodic solution is stable under small Hamiltonian perturbations as is seen from the following lemma.

Let P_ϵ be a smooth one parameter family of symplectic manifolds and $K_\epsilon: P_\epsilon \to \mathbb{R}$ a smooth one parameter family of Hamiltonians for $|\varepsilon| \leqslant \varepsilon_0$. Let $\phi(t)$ be a non-degenerate, T-periodic solution of the system whose Hamiltonian is K_0. Let $k_0 = K_0(\phi(t))$. Then there exists an $\varepsilon_1 > 0$ and smooth functions $T(\varepsilon, k)$, $\Phi(t, \varepsilon, k)$ such that

(i) $T(0, k_0) = T$, $\Phi(t, 0, k_0) = \phi(t)$,

(ii) *$\Phi(t, \varepsilon, k)$ is a $T(\varepsilon, k)$-periodic solution of the system whose Hamiltonian is K_ϵ,*

(iii) $K_\varepsilon(\Phi(t, \varepsilon, k)) = k$

for $|\varepsilon| < \varepsilon_1$ *and* $|k - k_0| < \varepsilon_1$.

This is an elementary and classical result (see Abraham and Marsden [1]). The proof is a simple application of the implicit function theorem to the cross section map restricted to an energy level.

The solution $\Phi(t, \varepsilon, k)$ will be called a continuation of $\phi(t)$.

Consider the $(N+1)$-body problem in rotating coordinates where the particles are indexed from 0 to N and consider the zeroth mass to be small by setting $m_0 = \varepsilon^2$. The Hamiltonian II.A.8 becomes

$$H_{N+1} = \|y_0\|^2/2\varepsilon^2 - x_0^T J y_0 - \sum_{j=1}^{N} \frac{\varepsilon^2 m_j}{\|x_j - x_0\|} + H_N, \tag{1}$$

where H_N is the Hamiltonian of the N-body problem with particles indexed from 1 to N. As in Section II.A let $Z = (x_1, ..., x_N; y_1, ..., y_N)$ and $Z^* = (a_1, ..., a_N; -m_1 J a_1, ..., -m_N J a_N)$ so Z^* is a relative equilibrium. By Taylor's theorem

$$H_N(Z) = H_N(Z^*) + \tfrac{1}{2}(Z - Z^*)^T S(Z - Z^*) + O(\|Z - Z^*\|^3), \tag{2}$$

where S is the Hessian of H_N at Z^*. In (1) make the change of variables

$$x_0 = \xi, \qquad y_0 = \varepsilon^2 \eta, \qquad Z = Z^* - \varepsilon U. \tag{3}$$

So $x_i = a_i + O(\varepsilon)$. This change of variables is symplectic with multiplier ε^2 and thus (1) becomes

$$H_{N+1} = \left\{ \|\eta\|^2/2 - \xi^T J \eta - \sum_{j=1}^{N} \frac{m_j}{\|a_j - \xi\|} \right\} + \frac{1}{2} U^T S U + O(\varepsilon). \tag{4}$$

Thus to lowest order in ε the Hamiltonian of the $(N+1)$-body problem decouples into the Hamiltonian of the restricted $(N+1)$-body problem,

$$H_R = \|\eta\|^2/2 - \xi^T J \eta - \sum_{j=1}^{N} \frac{m_j}{\|a_j - \xi\|} \tag{5}$$

and the Hamiltonian of the linearization of the N-body problem about the relative equilibrium Z^*,

$$H_L = \tfrac{1}{2} U^T S U. \tag{6}$$

Thus, when $\varepsilon = 0$, the equations of motion are

$$\dot{\xi} = J\xi + \eta,$$
$$\dot{\eta} = J\eta - \sum_{1}^{N} \frac{m_j(a_j - \xi)}{\|a_j - \xi\|^3} \tag{7}$$

and

$$\dot{U} = J_{2N} SU. \tag{8}$$

Let $M = \varepsilon^2 + m_1 + \cdots + m_N$ and $U = (u_1, \ldots, u_N, v_1, \ldots, v_N)$ so $x_i = a_i - \varepsilon u_i$ and $y_i = -m_i J a_i - \varepsilon v_i$. Since the center of mass of the relative equilibrium is fixed at the origin $\sum_1^N m_i a_i = 0$. Thus the center of mass of the system is

$$C = \{\varepsilon^2 \xi - \varepsilon(m_1 u_1 + \cdots + m_N u_N)\}/M \tag{9}$$

and linear momentum is

$$L = \varepsilon^2 \eta - \varepsilon(v_1 + \cdots + v_N) \tag{10}$$

and angular momentum is

$$I = \varepsilon^2 \xi^T J\eta - \sum_{1}^{N} (a_i - \varepsilon u_i)^T J(m_i J a_i + \varepsilon v_i). \tag{11}$$

From (9), (10) and (11) we see that the manifold B_ε of the reduced space depends smoothly on ε. Now apply the lemma stated above to the system on the reduced space whose original Hamiltonian is (4) to give:

Let $\phi(t)$ be a periodic solution of the restricted problem (7) *with period τ and characteristic multipliers* $1, 1, \beta, \beta^{-1}$, *where* $\beta \neq 1$. *Let the characteristic exponents of the relative equilibrium be* $0, 0, \pm i, \pm i, \pm \alpha_4, \ldots, \pm \alpha_N$, *where* $\alpha_j \tau \not\equiv 0 \bmod 2\pi i$ *for* $j = 4, \ldots, N$. *Then the τ-periodic solution* $\xi = \phi(t)$, $U \equiv 0$ *of Eqs.* (7) *and* (8) *can be continued into the* $(N+1)$*-body problem on the reduced space.*

By the lemma given above it is enough to show that the periodic solution $\xi = \phi(t)$, $U \equiv 0$ is non-degenerate on the reduced space. By the results of Section II.B passing to the reduced space eliminates $0, 0, \pm i, \pm i$ as characteristic exponents of the relative equilibrium and so the characteristic multipliers of this periodic solution are

$$1, 1, \beta^{\pm 1}, \exp \pm\alpha_4 \tau, \ldots, \exp \pm\alpha_N \tau.$$

Thus the multiplicity of the characteristic multiplier $+1$ is exactly 2 and the lemma applies.

An immediate consequence is: *Any non-degenerate periodic solution of the classical restricted three-body problem whose period is not a multiple of* 2π *can be continued into the three-body problem, since the relative equilibrium of the two-body problem has* $\alpha_4 = i$ *by Section* II.C.

This corollary is the main result of Hadjidemetrious [11].

B. *Bifurcation of a Primary*

Another method of introducing a small parameter into the $(N+1)$-body problem is to assume that the distance between two of the particles is small. In this case we shall show that there are periodic solutions where $N-1$ particles and the center of mass of the other pair move approximately on a relative equilibrium solution and the pair move approximately on a small circular orbit of the two-body problem about their center of mass.

Consider the $(N+1)$-body problem written in Jacobi coordinates as discussed in Section II.C. Assume that the center of mass and linear momentum are fixed at the origin and so the Hamiltonian is (cf. II.C.16)

$$H_{N+1} = \sum_{i=1}^{N} \{\|v_i\|/2M_i - u_i^T J v_i\} - \sum_{0 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|d_{ji}\|} \tag{1}$$

and total angular momentum is

$$I = \sum_{i=1}^{N} u_i^T J v_i. \tag{2}$$

By II.C.10 the vector u_1 is the position vector of the first particle relative to the zeroth particle and we wish to consider the case when these two particles are close, thus we make the change of variables

$$u_1 = \varepsilon^4 \xi, \tag{3}$$

where ε is a small positive parameter. This change of variables is not symplectic, but compensation will be made later. The Hamiltonian becomes

$$H_{N+1} = \|v_1\|^2/2M_1 - \varepsilon^4 \xi^T J v_1 - \frac{m_0 m_1}{\varepsilon^4 \|\xi\|} + H_N + O(\varepsilon^4), \tag{4}$$

where H_N is the Hamiltonian of the N-body problem in a rotating coordinate system. Note that the $O(\varepsilon^4)$ terms do not contain the momentum terms $v_1, \ldots, v_N$. Angular momentum becomes

$$I = \varepsilon^4 \xi^T J v_1 + \sum_{i=2}^{N} u_i^T J v_i. \tag{5}$$

As before define $Z = (u_2, \ldots, u_N, v_2, \ldots, v_N)$ and let Z^* be a relative equilibrium so

$$H_N(Z) = H_N(Z^*) + \tfrac{1}{2}(Z - Z^*)^T S(Z - Z^*) + O(\|Z - Z^*\|^3). \tag{6}$$

Now change variables by

$$\begin{aligned} \eta &= v_1/\varepsilon^2, \\ \varepsilon U &= Z - Z^* \end{aligned} \tag{7}$$

and change time and the Hamiltonian by

$$t = \varepsilon^6 \tau, \qquad H_{N+1} - H_N(Z^*) = \varepsilon^{-6} K. \tag{8}$$

The composition of (3) and (7) is a symplectic change of variables with multiplier ε^2, and so the new Hamiltonian becomes

$$K = \left\{ \|\eta\|^2/2M_1 - \frac{m_0 m_1}{\|\xi\|} \right\} + \varepsilon^6 \left\{ -\xi^T J\eta + \frac{1}{2} U^T S U \right\} + O(\varepsilon^7). \tag{9}$$

Thus to zeroth order in ε the Hamiltonian K is the Hamiltonian of the Kepler problem and at sixth order the rotation term of the Kepler problem and the quadratic terms of the relative equilibrium appear.

The gradient of angular momentum at the relative equilibria Z^* is nonzero and so the angular momentum integral becomes

$$I = I' + \varepsilon I_1 U + O(\varepsilon^2), \tag{10}$$

where I' is $I(Z_0^*)$ and I_1 is the gradient of the angular momentum at Z^* written as a row vector. Holding I fixed is equivalent to holding $\varepsilon^{-1}(I - I') = I_1 V + O(\varepsilon)$ fixed. Thus the reduction to the full reduced space is smooth in ε.

For the moment neglect the $O(\varepsilon^7)$ terms in (9) and consider the approximate equations

$$\begin{aligned} \xi' &= \eta/M_1 + \varepsilon^6 J\xi, \\ \eta' &= -m_0 m_1 \xi/\|\xi\|^3 + \varepsilon^6 J\eta, \\ U' &= J_{2N} S U, \end{aligned} \tag{11}$$

where $' = d/d\tau$. A periodic solution of these equations is

$$\begin{aligned} \xi^* &= \exp(\omega J\tau)a, \\ \eta^* &= M_1 \delta J \exp(\delta J\tau)a, \\ U^* &\equiv 0, \end{aligned} \tag{12}$$

where $\omega = \delta + \varepsilon^6$, $\delta = 1/m_0 + m_1$, and a is a constant vector with $\|a\| = 1$. The period map in an energy level is the identity map up to terms of order $O(\varepsilon^5)$ and so care must be taken in calculating the characteristic multipliers. Change variables by

$$\xi = e^{\omega J \tau}\zeta \tag{13}$$

so that the first two equations in (11) become

$$\zeta'' + 2\delta J\zeta' - \delta^2\zeta = -(\delta^2\zeta/\|\zeta\|^3). \tag{14}$$

The Jacobian of $\zeta/\|\zeta\|^3$ at $a = (1, 0)$ is $R = \left(\begin{smallmatrix} -2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and so the linearization of (14) about a is

$$\zeta'' + 2\delta J\zeta - \delta^2\zeta = -\delta^2 R\zeta \tag{15}$$

from which it is easy to calculate the characteristic polynomial

$$\lambda^2\{\lambda^2 + \delta^2\}. \tag{16}$$

Let the relative equilibrium have characteristic exponents $0, 0, \pm i, \pm i, \pm\alpha_4, \ldots, \pm\alpha_N$, where $\alpha_i \neq 0$ for $i = 4, \ldots, N$. Then the characteristic exponents of the solutions (12) of Eqs. (11) are

$$\begin{aligned} &1, 1, \exp\left(\pm\frac{i2\pi\delta}{\delta + \varepsilon^6}\right) = 1 \pm \varepsilon^6\frac{2\pi i}{\delta} + O(\varepsilon^{12}); \\ &1, 1, \exp(\pm i\varepsilon^6 2\pi/\omega), \exp(\pm i\varepsilon^6 2\pi/\omega), \exp(\pm\varepsilon^6 2\pi\alpha_4/\omega), \ldots, \\ &\exp(\pm\varepsilon^6 2\pi\alpha_N/\omega). \end{aligned} \tag{17}$$

On the reduced space the characteristic multipliers are

$$1, 1, 1 \pm \varepsilon^6 2\pi i/\delta, 1 \pm \varepsilon^6 2\pi i/\delta, 1 \pm \varepsilon^6\alpha_4 2\pi/\omega, \ldots, 1 \pm \varepsilon^6 a_N 2\pi/\omega \tag{18}$$

plus items of order ε^{12} or higher. Thus the characteristic multipliers are of the form $1, 1, 1 \pm \varepsilon^6\beta_4 + O(\varepsilon^{12}), \ldots, 1 \pm \varepsilon^6\beta_N + O(\varepsilon^{12})$, where $\beta_i \neq 0$ for $i = 4, \ldots, N$.

In order to continue this solution into the full $(N + 1)$-body problem an extension of the classical perturbation theorem quoted in Section III.A must be proved. This extension is very similar to the continuation theorem given in Henrard [14].

Let $\phi_0(t, \varepsilon)$ be a $T_0(\varepsilon)$-periodic solution of a Hamiltonian system with smooth Hamiltonian $L_0(u, \varepsilon)$, where $u \in 0$ is an open set $\subset \mathbb{R}^{2m}$, $|\varepsilon| \leqslant \varepsilon_0$ with characteristic multipliers

$$1, 1, 1 \pm \varepsilon^p\gamma_2 + O(\varepsilon^{p+1}), \ldots, 1 \pm \varepsilon^p\gamma_m + O(\varepsilon^{p+1}),$$

where $\gamma_j \neq 0$ for $j = 2, \ldots, m$. Let the period map in an energy level be the identity map up to order ε^{p-1}. Then for any smooth function $\tilde{L}(u, \varepsilon)$ there exists an $\varepsilon_1 > 0$ and smooth functions $T_1(\varepsilon)$, $\phi_1(t, \varepsilon)$ for $|\varepsilon| \leqslant 1$ such that $\phi_1(t, \varepsilon)$ is a $T_1(\varepsilon)$-periodic solution of the system whose Hamiltonian is $L_1(u, \varepsilon) = L_0(u, \varepsilon) + \varepsilon^{p+1}\tilde{L}(u, \varepsilon)$, where $T_1(\varepsilon) = T_0(\varepsilon) + O(\varepsilon^{p+1})$ and $\phi_1(0, \varepsilon) = \phi_0(0, \varepsilon) + O(\varepsilon^{p+1})$.

Proof. At $\phi_0(0, 0) \in 0$ choose a hyperplane transversal to $\dot{\phi}_0(0, 0)$. This hyperplane will be transversal to both flows for ε small enough. Consider the intersections $\sigma_0(\varepsilon)$ and $\sigma_1(\varepsilon)$ of this hyperplane and the level surfaces $L_0(u, \varepsilon) = L_0(\phi_0(0, 0), 0)$ and $L_1(u, \varepsilon) = L_0(\phi_0(0, 0), 0)$. For ε small and near $\phi_0(0, 0)$ both σ_0 and σ_1 are symplectic manifolds of dimension $2m - 2$ and the period maps P_0 and P_1 are defined. Let v be local coordinates in σ_0 and σ_1 with $v = 0$ corresponding to $\phi_0(0, 0)$. The hypothesis gives $P_1 = P_0 + O(\varepsilon^{p+1})$ and $P_0(v, \varepsilon) = v + \varepsilon^p Q(v) + O(\varepsilon^{p+1})$, where $Q(0) = 0$ and the Jacobian matrix of Q at 0 has eigenvalues $\pm\gamma_2, \ldots, \pm\gamma_m$. To find a periodic solution of the system with Hamiltonian L_1 one must solve

$$P_1(v, \varepsilon) = v$$

or

$$v + \varepsilon^p Q(v) + O(\varepsilon^{p+1}) = v$$

or

$$Q(v) + O(\varepsilon) = 0.$$

The implicit function theorem implies that this last equation has a smooth solution $\bar{v}(\varepsilon)$ such that $\bar{v}(0) = 0$. The solution $\phi_1(t, \varepsilon)$ is then the solution of the system with Hamiltonian L_1 with initial condition $\bar{v}(\varepsilon)$ at $t = 0$.

This elementary perturbation lemma proves that the solutions (12) can be continued into the full $(N + 1)$-body problem.

The condition that the relative equilibrium be non-degenerate is very weak. For $N = 2$ or 3 all the relative equilibria are non-degenerate, also Palmore [23] has shown that the collinear relative equilibrium is non-degenerate for all N and all masses (cf. Section II.E). Palmore [24] also has established that almost all central configurations are non-degenerate.

For $N = 2$ the above result gives the so-called Hill solutions of the three-body problem established by Moulton [21], and also discussed by Siegel [28] and Conley [9]. If the relative equilibrium is the triangular configuration given by Lagrange then the above establishes the existence of the periodic solutions of the four-body problem given in Crandall [10]. If the relative equilibrium is the collinear configuration of the N-body problem then the above establishes the existence of the periodic solutions of the $(N + 1)$-body problem given in Perron [26].

C. *Orbits at Infinity*

A small particle which is far from the other N particles would exert very little influence, and so it is natural to assume that there are periodic solutions of the $(N+1)$-body problem where N particles move approximately on a relative equilibrium solution and a small particle moves on a nearly circular orbit at a great distance. It is very easy to prove that there are two families of periodic solutions of the restricted $(N+1)$-body problem which are nearly circles of large radius (see, for example, Moulton [20] for $N=2$ or Meyer [17], in general). These periodic solutions are nearly 2π-periodic and even on the reduced space the linearized equations of the N-body problem about a relative equilibrium have 2π-periodic solutions. Thus this case is very close to resonance. By the calculations in Meyer [17] the families of periodic solutions of large radius of the restricted problem are non-degenerate and these periods differ slightly from 2π. Thus by the result of Section III.A these solutions can be continued into the full $(N+1)$-body problem for small mass. Since these periodic solutions are obtained in a two-step proof there is no obvious relation between the orders of magnitude. All one knows is that the mass must be made small after a large radius is chosen. In this section, we present a method of scaling which obviates the relation between the various orders of magnitudes.

Consider the $(N+1)$-body problem in Jacobi coordinates (II.C.16) and so the Hamiltonian is

$$H_{N+1} = \|v_N\|^2/2M_N - u_N^T J v_N - \sum_{i=0}^{N-1} \frac{m_N m_i}{\|d_{Ni}\|} + H_N, \tag{1}$$

where H_N is the Hamiltonian of the N-body problem again in Jacobi coordinates. As before consider H_N as a function of Z and let Z^* be a relative equilibrium of the N-body problem so that

$$H_N(Z) = H_N(Z^*) + \tfrac{1}{2}(Z-Z^*)^T S(Z-Z^*) + O(\|Z-Z^*\|^3). \tag{2}$$

By II.C.12,

$$d_{Ni} = u_N - \sum_{l=1}^{N-1} \alpha_{Nil} u_l \tag{3}$$

and by II.C.6,

$$M_N = m_N(m_0 + \cdots + m_{N-1})/(m_0 + \cdots + m_N). \tag{4}$$

Assume that the mass of the particle indexed by N is small by setting $m_N = \varepsilon^{13}$ and considering ε as a small parameter. Thus $M_N = \varepsilon^{13} + O(\varepsilon^{26})$. We may assume that the total mass is 1 so that $m_0 + \cdots + m_{N-1} = 1 +$

$O(\varepsilon^{13})$ since this can be accomplished by a change in scale. In (1) make the symplectic change of variables

$$u_N = \varepsilon^{-2}\xi, \qquad v_N = \varepsilon^{14}\eta, \qquad Z - Z^* = \varepsilon^6 U. \tag{5}$$

The multiplier for (5) is ε^{12} and so (1) becomes

$$H = \tfrac{1}{2} U^T S U - \xi^T J \eta + \varepsilon^3\{\|\eta\|^2/2 - 1/\|\xi\|\} + O(\varepsilon^6). \tag{6}$$

Let $A = J_{2N-2} S$ so the equations of motion to zeroth order in ε are

$$\dot{U} = AU; \qquad \dot{\xi} = J\xi, \quad \dot{\eta} = J\eta. \tag{7}$$

In all the examples where the characteristic exponents of the relative equilibrium are known the matrix A has the eigenvalues $\pm i$. Thus to this order of approximation there is a 1–1 resonance between the two sets of equations in (7). Changing the exponents in (5) will not eliminate this problem because the requirement that (5) the symplectic forces $U^T S U$ and $\xi^T J \eta$ to appear at the same order in ε. The scaling in (5) introduces correction terms on the approximate periodic solutions period that overcome this resonance as we shall see below.

Angular momentum becomes

$$I = \varepsilon^{12}\xi^T J \eta + I' + \varepsilon^6 I_1 U + O(\varepsilon^{12}), \tag{8}$$

where I' is the value and I_1 is the gradient of the angular momentum of the N-body problem evaluated at Z^*. Thus the system admits the smooth integral

$$\tilde{I} = (I - I')/\varepsilon^6 = I_1 U + O(\varepsilon^6) \tag{9}$$

which depends only on U to lowest order since $I_1 \neq 0$. Thus the reduction to the full reduced space is smooth in ε. The passage to the reduced space does not change the form of the Hamiltonian (6) since this reduction is accomplished by holding $\tilde{I}$ fixed and ignoring a variable conjugate to $\tilde{I}$. Specifically, by Whittaker [34], we may assume that $\tilde{I} = U_1 + O(\varepsilon^6)$, where $U = (U_1, \ldots)$, so that the passage to the reduced space is effected by holding U_1 fixed (=0) and ignoring V_{2N-1}. Setting $V = (U_2, \ldots, U_{2N-1}, U_{2N}, \ldots, U_{4N-4})$ the Hamiltonian (6) becomes

$$H = \tfrac{1}{2} V^T T V - \xi^T J \eta + \varepsilon^3\{\|\eta\|^2/2 - 1/\|\xi\|\} + O(\varepsilon^6), \tag{10}$$

where T is the Hessian of the Hamiltonian of the N-body problem on the reduced space.

Assume that the relative equilibrium is non-degenerate so that T and $B = J_{2N-3}T$ are non-singular. The full equations on the reduced space are

$$\begin{aligned} \dot{V} &= BV + O(\varepsilon^6), \\ \dot{\xi} &= J\xi + \varepsilon^3\eta + O(\varepsilon^6), \\ \dot{\eta} &= J\eta - \varepsilon^3\xi/\|\xi\| + O(\varepsilon^6). \end{aligned} \tag{11}$$

If we ignore the O-terms in (11) then this system admits a periodic solution of the form

$$\begin{gathered} V \equiv 0, \\ \xi = e^{\omega Jt}a, \qquad \eta = \mp\varepsilon^3 Je^{\omega Jt}a, \end{gathered} \tag{12}$$

where a is any vector such that $\|a\| = 1$ and $\omega = 1 \pm \varepsilon^3$. Since B will have the eigenvalues $\pm i$, in general, and the period of the functions (12) is nearly 2π we must calculate the characteristic multipliers to high order. Assume that the coordinates V have been chosen so that $B = \operatorname{diag}(B_1, B_2)$, where B_1 has eigenvalues of the form $\pm ni$, n a positive integer, and B_2 has no eigenvalues of the form $\pm ni$. For simplicity we shall assume that B_1 is diagonalizable so $\exp 2\pi B_1 = I$; this is true if $N = 2$ or 3. (If this is not true one can place B_1 in Jordan canonical form and scale again so that the off-diagonal terms are of order ε^6.) The matrix $\exp(2\pi B_2) - I$ is non-singular since B_2 does not have eigenvalues of the form $\pm ni$. Let $V = (V_1, V_2)$ be the decomposition of V corresponding to the decompositions of B. Integrate the first equation in (11) from (V_{10}, V_{20}) at $t = 0$ to $t = 2\pi/\omega = 2\pi(1 \mp \varepsilon^3 + O(\varepsilon^6))$ to obtain

$$\begin{aligned} V_1 &= [\exp 2\pi(1 \mp \varepsilon^3 + O(\varepsilon^6))\, B_1]\, V_{10} + O(\varepsilon^6) \\ &= V_{10} \mp \varepsilon^3 2\pi B_1 V_{10} + O(\varepsilon^6), \\ V_2 &= (\exp 2\pi B_2)\, V_{20} + O(\varepsilon^3). \end{aligned} \tag{13}$$

From the first equation in (13) one sees that some of the characteristic multipliers are of the form $1 + \varepsilon^3\beta + O(\varepsilon^6)$, where $\beta \neq 0$. From the second equation one sees that some of the characteristic multipliers are of the form β, $\beta \neq 1$.

For the moment ignore the higher terms in (11) and so the second set of equations is equivalent to

$$\ddot{\xi} - 2J\dot{\xi} - \xi = -\varepsilon^6\xi/\|\xi\|^3. \tag{14}$$

In (14) make the substitution $\xi = e^{\omega Jt}w$ so that

$$\ddot{w} \pm \varepsilon^3 J\dot{w} - \varepsilon^6 w = -\varepsilon^6 w/\|w\|^3. \tag{15}$$

In these coordinates the approximate solution is given by $w = a$. Let $a = (1, 0)$; the linearization of (15) about the equilibrium solution a is

$$\ddot{w} \pm \varepsilon^3 J\dot{w} - \varepsilon^6 w = -\varepsilon^6 R w, \tag{16}$$

where $R = \mathrm{diag}(-2, 1)$. As in the previous section the characteristic equation of (16) is $\lambda^2\{\lambda^2 + \varepsilon^6\}$ and so the remaining four characteristic multipliers of the approximate solution (12) are $1, 1, 1 \pm \varepsilon^3 i$.

As in the previous section we have given approximate solutions of the equations and calculated their approximate characteristic multipliers. We now proceed as in the previous section or as in Henrard [14] to show that these approximate solutions can be continued into the full $(N+1)$-body problem for small ε.

IV. Symmetric Periodic Orbits

A. *Small Mass*

In this section we shall show that a non-degenerate symmetric periodic solution of the restricted $(N+1)$-body problem can be continued into the full $(N+1)$-body problem under mild non-resonance assumptions.

Let $a_1, \ldots, a_N$ be the position vectors of a symmetric central configuration. Specifically assume that there is a reflection R and a permutation σ of $(1, \ldots, N)$ such that $\sigma^2 = \text{identity}$, $a_j = Ra_{\sigma(j)}$ and $m_j = m_{\sigma(j)}$. By rotating the axes, if necessary, we may assume that the fixed line of the reflection is the abscissa and so $R = \mathrm{diag}(1, -1)$. Consider the Hamiltonian H_N of the N-body problem in rotating coordinates (II.A.8), let $Z = (x_1, \ldots, x_N, y_1, \ldots, y_N)$, $Z^* = (a_1, \ldots, a_N, -m_1 Ja_1, \ldots, -m_N Ja_N)$ and F the matrix of the transformation $x_i \to Rx_{\sigma(i)}$, $y_i \to -Ry_{\sigma(i)}$. Thus F is an anti-symplectic involution on $\mathbb{R}^{4N}$, $FZ^* = Z^*$ and $H_N(FZ) = H_N(Z)$. As before $H_N(Z) = H_N(Z^*) + \frac{1}{2}(Z - Z^*)^T S(Z - Z^*) + O(\|Z - Z^*\|^3)$, where S is the Hessian of H_N at Z^*. Since $H_N(FZ) = H_N(Z)$ the matrix S satisfies $FSF = S$ or S is block diagonal.

Now proceed as in Section III.A and change variables by $x_0 = \xi$, $y_0 = \varepsilon^2 \eta$ and $Z = Z^* - \varepsilon U$ so that the Hamiltonian becomes

$$H_{N+1} = \left\{ \|\eta\|^2/2 - \xi^T J\eta - \sum_1^N m_j/\|a_j - \xi_j\| \right\} + \frac{1}{2} U^T S U + O(\varepsilon). \tag{1}$$

This Hamiltonian is invariant under the anti-symplectic involution

$$\begin{gathered} \xi \to R\xi, \qquad \eta \to -R\eta, \\ U \to FU. \end{gathered} \tag{2}$$

Now consider this system on the reduced space. When $\varepsilon = 0$ the reduction does not depend on the ξ, η variables and so only the U variables are affected. Choose local coordinates V at the image of the relative equilibrium on the reduced space with $V=0$ corresponding to the relative equilibrium. Then when $\varepsilon=0$ the Hamiltonian (1) is of the form

$$\tilde{H}=\left\{\|\eta\|^2/2-\xi^TJ\eta-\sum_1^N m_j/\|a_j-\xi\|\right\}+\frac{1}{2}V^T\tilde{S}V+O(\|U\|^3), \tag{3}$$

where $\tilde{S}$ is the Hessian of the reduced Hamiltonian at $V=0$. The map (2) projects to

$$\begin{gathered}\xi\to R\xi, \qquad \eta\to -R\eta,\\ V\to g(V),\end{gathered} \tag{4}$$

where g is an anti-symplectic involution. By the results of Section II.D we can choose symplectic coordinates (ζ, ν) at $V=0$ such that $V^TSV=\zeta^TA\zeta+\nu^TB\nu$ and the symmetry manifold Q is locally given by $\nu=0$. In these coordinates the equations become

$$\begin{aligned}
\dot{\xi}&=J\xi+\eta,\\
\dot{\eta}&=J\eta-\sum_1^N m_j(a_j-\xi)/\|a_j-\xi\|^3,\\
\dot{\zeta}&=B\nu+O(\|\zeta\|^2+\|\nu\|^2),\\
\dot{\nu}&=-A\zeta+O(\|\zeta\|^2+\|\nu\|^2).
\end{aligned} \tag{5}$$

Let $\xi=\phi(t)$, $\eta=\psi(t)$ be a non-degenerate $2T$-periodic solution of the restricted problem, i.e., they satisfy the first pair of equations in (5). Assume that the characteristic exponents of the relative equilibrium are

$$0, 0, \pm i, \pm i, \pm\alpha_4,\dots, \pm\alpha_N,$$

where $\alpha_j T\not\equiv 0 \bmod \pi i$ for $j=4,\dots, N$. Thus by the results of Section II.D the eigenvalues of $\left(\begin{smallmatrix}0 & B\\ -A & 0\end{smallmatrix}\right)$ are $\pm\alpha_4,\dots, \pm\alpha_N$.

Now we claim that $\xi=\phi(t)$, $\eta=\psi(t)$, $\zeta=\nu=0$ is a non-degenerate periodic solution of (5) and hence can be continued into the full $(N+1)$-body problem. Since the first and second pairs of equations in (5) are independent and we have assumed that (ϕ, ψ) is a non-degenerate symmetric periodic solution of the first pair in (5), it is enough to show that $\zeta=\nu=0$ is a non-degenerate $2T$-periodic solution of the second pair of equations.

To prove the non-degeneracy of the solution $\zeta=\nu=0$ we can linearize the equations to calculate the necessary Jacobian. The solution of $\dot{\zeta}=B\nu$, $\dot{\nu}=-A\zeta$ which satisfies $\zeta=\zeta_0$, $\nu=0$ at $t=0$ is

$$\zeta(t, \zeta_0) = (\cos \sqrt{BA}\, t)\, \zeta_0,$$
$$\eta(t, \zeta_0) = -B^{-1}(\sin \sqrt{BA}\, t)\, \zeta_0.$$

The eigenvalues of BA are $\alpha_4^2,..., \alpha_N^2$ and so the eigenvalues of $\sqrt{BA}$ are taken from $\pm\alpha_4,..., \pm\alpha_N$. Thus this solution is non-degenerate since

$$\frac{\partial v}{\partial \zeta_0}(T, 0) = \pm\det B(\sin \alpha_4 T) \cdots (\sin \alpha_N T)$$

is non-zero.

An immediate consequence is: *Any non-degenerate symmetric periodic solution of the restricted three-body problem whose period is not a multiple of 2π can be continued into the three-body problem.*

B. *Bifurcation of a Primary*

In this section we consider symmetric periodic solutions when two of the bodies are close. Introduce the scale parameter and change variables as discussed in Section III.B so that the Hamiltonian becomes

$$K = \{\|\eta\|^2/2 - 1/\|\xi\|^2\} + \varepsilon^6\{\tfrac{1}{2} U^T SU - \xi^T J\eta\} + O(\varepsilon^7). \tag{1}$$

Note that we have also scaled the variables so that $M_1 = m_0 m_1 = 1$. As in the previous section we assume here that the relative equilibrium of the N-body problem is symmetric in a line and so K is invariant under the anti-symplectic involution

$$\begin{gathered} \xi \to R\xi, \qquad \eta \to -R\eta, \\ U \to FU, \end{gathered} \tag{2}$$

where R and F are as in the previous section.

In order to calculate the necessary Jacobian it is convenient to change from the rectangular coordinates ξ, η to the Delaunay elements l, g, L, G so that

$$K = -\frac{1}{2L^2} + \varepsilon^6 \left\{ \frac{1}{2} U^T SU - G \right\} + O(\varepsilon^7). \tag{3}$$

In the Kepler problem l is the mean anomaly, g is the argument of the perihelion and G is angular momentum. In these coordinates an orthogonal crossing of the line of symmetry occurs when g and l are multiples of π. See Szebehely [31] for a complete discussion of Delaunay's elements and the symmetry condition.

As in the previous section we consider this Hamiltonian on the reduced space and use the coordinate system (ζ, v) introduced here. The Hamiltonian becomes

$$K = -\frac{1}{2L^2} + \frac{\varepsilon^6}{2}\{\zeta^T A\zeta + \nu^T B\nu - 2G\} + O(\varepsilon^7). \tag{4}$$

Assume that the relative equilibrium is non-degenerate so that A and B are non-singular. The equations of motion are

$$\begin{aligned} l' &= 1/L^3, & L' &= 0, \\ g' &= -\varepsilon^6, & G' &= 0, \\ \zeta' &= \varepsilon^6 B\nu, & \nu' &= -\varepsilon^6 A\zeta \end{aligned} \tag{5}$$

plus terms of order $O(\varepsilon^7)$.

These equations are autonomous and so we may take the fast angle as the independent variable so that the equations of motion become

$$\begin{aligned} &\frac{dL}{dl} = 0, \\ &\frac{dg}{dl} = -\varepsilon^6 L^3, & &\frac{dG}{dl} = 0, \\ &\frac{d\zeta}{dl} = \varepsilon^6 L^3 B\nu, & &\frac{d\nu}{dl} = -\varepsilon^6 L^3 A\zeta \end{aligned} \tag{6}$$

plus terms of order $O(\varepsilon^7)$.

In these coordinates the symmetry manifold Q is given by l, $g \equiv 0 \bmod \pi$ and $\nu = 0$.

For the moment ignore the higher order terms and seek a symmetric periodic solution of the approximate equations. Let α and β be relatively prime integers and set $\varepsilon^6 = \alpha/\beta$. Integrate the approximate equations (6) from $l = \pi$ to $l = (1+\beta)\pi$ with initial conditions $L = 1$, $\zeta = \zeta_0$, $\nu = 0$, $g = -\pi$ and $G = G_0$. Let the subscript a stand for approximate solution. Thus

$$\begin{aligned} &L_a = 1, \\ &g_a = -\pi - \varepsilon^6\beta\pi = -(1+\alpha)\pi, & &\Gamma_a = \Gamma_0, \\ &\zeta_a = 0, & &\nu_a = -\varepsilon^6 A\zeta_0\beta = -\alpha A\zeta_0. \end{aligned} \tag{7}$$

Thus if $\zeta_0 = 0$ this approximate solution satisfies the symmetry condition. By holding α fixed and taking β large the scale parameter ε is small and so one might expect that these approximate solutions persist. As Arenstorf [2, 3, 5] has pointed out the usual implicit function theorem does not apply since one cannot set $\varepsilon = 0$ and find the approximate solution. Thus we must proceed along a path suggested by Arenstorf.

Fix the integer α and the initial condition for G once and for all. Let the subscript f denote the full, a the approximate and e the error. Integrate the full equations (7) from $l=\pi$ to $l=(1+\beta)\pi$ starting with initial conditions $L=L_0$, $\zeta=\zeta_0$, $\nu=0$, $g=-\pi$ to obtain

$$\begin{aligned} g_f(L_0,\zeta_0,\varepsilon,\beta) &= g_a(L_0,\zeta_0,\varepsilon,\beta)+g_l(L_0,\zeta_0,\varepsilon,\beta),\\ g_a(L_0,\zeta_0,\varepsilon,\beta) &= -\pi-\varepsilon^6\beta\pi L_0^3,\\ \nu_f(L_0,\zeta_0,\varepsilon,\beta) &= \nu_a(L_0,\zeta_0,\varepsilon,\beta)+\nu_l(L_0,\zeta_0,\varepsilon,\beta),\\ \nu_a(L_0,\zeta_0,\varepsilon,\beta) &= -\varepsilon^6 A\zeta_0\beta L_0^3. \end{aligned} \tag{8}$$

The error terms g_l and ν_l are due to the $O(\varepsilon^7)$ terms which must be added to Eqs. (6). The Lipshitz constants for these equations is $O(\varepsilon^6)$ and the error term is $O(\varepsilon^7)$ and so by standard Grownwall estimates, see Hartman [13] or Coddington and Levinson [8], there are constants c_1 and c_2 such that

$$\begin{aligned} |g_l(L_0,\zeta_0,\varepsilon,\beta)| &\leqslant c_1\varepsilon^7\{e^{c_2\varepsilon^6\beta}-1\},\\ |\nu_l(L_0,\zeta_0,\varepsilon,\beta)| &\leqslant c_1\varepsilon^7\{e^{c_2\varepsilon^6\beta}-1\}. \end{aligned} \tag{9}$$

In these estimates the full solution must remain in a compact neighborhood of the approximate solution. This can be assured by bounding $\varepsilon^6\beta\leqslant\alpha$ and taking ε small. Similar estimates hold for the partial derivatives of g_l and ν_l. Because A is non-singular and because of estimate (9) the equation $\nu_f(L_0,\zeta_0,\varepsilon,\beta)=0$ can be solved for ζ_0; i.e., there exists a function $\bar{\zeta}(L_0,\varepsilon,\beta)$ such that $\bar{\zeta}(L_0,0,\beta)=0$ and $\gamma_f(L_0,\bar{\zeta}(L_0,\varepsilon,\beta),\varepsilon,\beta)=0$. $\bar{\zeta}=O(\varepsilon^{7/6})$, as does its partials. Thus we must solve

$$q_f(L_0,\bar{\zeta}(L_0,\varepsilon,\beta),\varepsilon,\beta)=-(1+\alpha)\pi. \tag{10}$$

The approximate equation $g_a(L_0,\bar{\zeta},\varepsilon,\beta)=-\pi-\varepsilon^6\beta\pi L_0^3$ has a solution when $\varepsilon^6=\alpha/\beta$ and $L_0=1$. Moreover $(\partial g_a/\partial L_0)(1,\bar{\zeta},\varepsilon,\beta)=-3\alpha\pi$ when $\varepsilon^6=\alpha/\beta$ which is a fixed non-zero number. From estimate (9) the error function g_l can be made arbitrarily small by taking β large and fixing $\varepsilon^6=\alpha/\beta$. Similar estimates on the particles of g_l hold. Thus the implicit function theorem of Arenstorf [2, 3, 5] applies and here a constant β_0 exists such that if $\beta>\beta_0$ there is a solution $L_s(\beta)$ such that

$$g_f(L_s(\beta),\bar{\zeta}(L_s(\beta),\varepsilon,\beta),\varepsilon,\beta)=-(1+\alpha)\pi,$$

where $\varepsilon^6=\alpha/\beta$. Thus the solution of the $(N+1)$-body problem with these initial conditions is a symmetric periodic solution.

The relative equilibrium of the two-body problem is non-degenerate and so this result contains the main result of Arenstorf [3]. Palmore [23] has shown that the collinear relative equilibrium of the N-body problem is non-degenerate and so this result contains the main result of Arenstorf [5].

C. *Orbits at Infinity*

In this section we shall show that there are periodic solutions of the $(N+1)$-body problem where one small body and the center of mass of the other N bodies move approximately on a large elliptic solution of the two-body problem and the N bodies move approximately on a relative equilibrium solution when the relative equilibrium admits a line of symmetry.

As before assume that the relative equilibrium is symmetric in a line and is non-degenerate. Scale the Hamiltonian as in Section III.C so that the Hamiltonian on the reduced space becomes III.C.5 or

$$H=\frac{1}{2}U^TTU-\xi^TJ\eta+\varepsilon^3\left\{\|\eta\|^2/2-\frac{1}{\|\xi\|}\right\}+O(\varepsilon^6). \tag{1}$$

As in the previous section introduce Delaunay's elements l, g, L, G so that (1) becomes

$$H=\tfrac{1}{2}U^TTU-G-\varepsilon^3/2L^2+O(\varepsilon^6). \tag{2}$$

Also as in the previous section introduce coordinates (ζ, ν) so that the symmetry manifold becomes $\nu=0$, $l\equiv 0 \bmod \pi$, $g\equiv 0 \bmod \pi$. Thus the Hamiltonian becomes

$$H=\tfrac{1}{2}\{\zeta^TA\zeta+\nu^TB\nu\}-G-\varepsilon^3/2L^2+O(\varepsilon^6) \tag{3}$$

and the equations of motion are

$$\begin{aligned} \dot{l}&=\varepsilon^3/L^3, & \dot{L}&=0,\\ \dot{g}&=-1, & \dot{G}&=0,\\ \dot{\zeta}&=B\nu, & \dot{\nu}&=-A\zeta \end{aligned} \tag{4}$$

plus terms of order ε^6. In this case g is the fast angle and so we use it as the independent variable so that the equations become

$$\begin{gathered} \frac{dl}{dg}=-\varepsilon^3/L^3, \qquad \frac{dL}{dg}=0,\\ \frac{dG}{dg}=0,\\ \frac{d\zeta}{dg}=-B\nu, \qquad \frac{d\nu}{dg}=A\zeta \end{gathered} \tag{5}$$

plus terms of order ε^6.

Let α and β be relatively prime integers and set $\varepsilon^3 = \alpha/\beta$. Integrate these approximate equations from $g = \pi$ to $g = (1 + \beta)\pi$ starting with the initial conditions $L = 1$, G arbitrary, $l = \pi$, $\zeta = \nu = 0$ to obtain $l = (1 - \alpha)\pi$. Thus this approximate solution satisfies the symmetry condition. If α is held fixed and β is chosen large then these solutions can be shown to persist in the full $(N + 1)$-body problem by the method of the previous section.

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