JOURNAL OF DIFFERENTIAL EQUATIONS 41, 228–238 (1981)

Hamiltonian Systems with a Discrete Symmetry

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Received October 31, 1980

If $f: M \to M$ is an antisymplectic involution of a symplectic manifold M then the fixed set of f is a Lagrangian submanifold $L \subset M$. Moreover there exist cotangent bundle coordinates in a neighborhood of L in M such that f in these coordinates maps a covector into its negative. Thus classical examples which have a discrete symmetry such as the restricted three-body problems are locally like a reversible system.

1. INTRODUCTION

The equations of motion of many physical systems are invariant under the action of a group due to the fact that the physical systems possess certain symmetries. In a previous paper by the author [5] systems which are invariant under continuous groups were considered (also see [4]). In this paper systems which are invariant under a discrete group (specifically $\mathbb{Z}_2$) are considered. Several typical examples are given in the next section but the main result of the paper is that one example is the prototype for them all. This prototype is a reversible Hamiltonian system on the cotangent bundle of a manifold. The transformation of the cotangent bundle which takes a covector into its negative is an involution which carries the natural symplectic structure into its negative and has the zero section as its fixed set. A reversible Hamiltonian system is invariant under this transformation by definition. In this paper we show that if f is an anti-sympletic involution of a symplectic manifold M then the fixed set $Q \subset M$ is a Lagrangian submanifold (Theorem 1) and moreover there is a neighborhood 0 of Q in T^*Q, a neighborhood N of Q in M, a symplectomorphism $\Phi: 0 \to N$ such that the pull back of the involution to $0 \subset T^*Q$ is just the sign reversing involution (Theorem 2). Thus locally (about the fixed set) there are cotangent bundle coordinates so that a system invariant under an anti-symplectic involution is just a reversible system.

* This research was partially supported by a grant from the Charles P. Taft Foundation and National Science Foundation Grants MSC 78 01425 and MCS 80 01851.

0022-0396/81/080228-11$02.00/0

Since this type of symmetry occurs in many examples, it has been exploited by several authors in their investigations. Hill, Poincaré, Moulton, Arenstorf *et al.*, have found periodic solutions to the equations of celestial mechanics by using the symmetries of the equations. Indeed the results of this paper were suggested by the author's work on the existence of periodic solutions of the N-body problem [6]. In fact some of the elementary results from this earlier paper are reproduced here, but the main result of this paper would have simplified the proof of [6] to some extent.

The proof of the main result of this paper uses the methods developed by Weinstein in [9]. The author like to thank Professor Joel Robbin for his help in adapting the methods of [9] to the present problem.

2. Examples

The restricted three-body problem is defined by the Hamiltonian

$$H = \|y\|^2/2 - x^T Jy - \frac{\mu}{\|a_1 - x\|} - \frac{(1-\mu)}{\|a_2 - x\|}, \tag{2.1}$$

where $x,\ y \in \mathbb{R}^2$, $0 < \mu < 1$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $a_1 = (\mu - 1, 0)$, $a_2 = (\mu, 0)$. The phase space is $(\mathbb{R}^2 - \{a_1, a_2\}) \times \mathbb{R}^2$ and has the usual symplectic structure $\Omega = dx \wedge dy = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. The equations of motion are

$$\begin{aligned} \dot{x} &= Hy = Jx + y, \\ y &= -Hx = Jy + \frac{\mu(a_1 - x)}{\|a_1 - x\|^3} + \frac{(1-\mu)(a_2 - x)}{\|a_2 - x\|^3}. \end{aligned} \tag{2.2}$$

These equations describe the motion of a particle of zero mass whose position is x in a rotating coordinate which is attracted to two fixed bodies at a_1 and a_2 with masses μ and $1 - \mu$ by Newton's law of gravity (see [8] for an elementary discussion of this problem). This Hamiltonian is invariant under the transformation

$$f\colon (x, y) \to (Rx, -Ry),$$

where $R = \operatorname{diag}(1, -1)$; that is, $H \circ f = H$. Clearly $f^2 = \mathrm{id}$ and $f^*\Omega = -\Omega$, i.e., f is an anti-symplectic involution. The fixed point set of this involution is $\{(x^1, 0, 0, y^2)\}$, which consists of all points of orthogonal crossings of the line of masses. A common method of finding periodic solutions to these equations (see [2, 6, 7]) is to find a solution of the equations which crosses the line of masses orthogonally at times $t = 0$ and $t = T > 0$. An easy argument which is repeated in a more general setting in the next section shows that such a solution is $2T$-periodic.

Classical mechanical systems give another class of examples. Let N be any manifold (usually embedded in $\mathbb{R}^k$ and called the configuration manifold or space), $M = T^*N$ the cotangent bundle of N (called the phase space), Ω the natural symplectic structure on $M = T^*N$, $\pi: T^*M \to M$ the natural projection, $V: M \to \mathbb{R}$ a smooth function (the potential energy), $K: T^*N \to \mathbb{R}$ a Riemannian metric on N (the kenetic energy) and $H = K + V \circ \pi: T^*N \to \mathbb{R}$ and $f: T^*N \to T^*N: v_p \to -v_p$. Clearly $f^2 = \mathrm{id}$ and $f^*\Omega = -\Omega$ so f is an anti-symplectic involution whose fixed set is the zero section $N \subset T^*N$. Since K is a quadratic form in cotangent vectors $K \circ f = K$ and so $H \circ f = H$, i.e., H is invariant under f.

This example is a special case of the more general class of Hamiltonians called reversible Hamiltonians. In a reversible system $H: T^*N \to \mathbb{R}$ is an arbitrary smooth function satisfying $H \circ f = H$, i.e., H is even in the momenta. We shall show later that this example is the prototype.

As another example, consider the N-body problem with Hamiltonian given by

$$H = \sum_{i=1}^{N} \|p_i\|^2/2m_i - \sum_{1 \leqslant i < j \leqslant N} \frac{m_i m_j}{\|q_i - q_j\|},$$

where $q_i \in \mathbb{R}^2$ is the position, $p_i \in \mathbb{R}^2$ is the momentum and $m_i \in \mathbb{R}_+$ is the mass of the ith particle. The usual symplectic structure is $\Omega = \sum_{i=1}^{N} dq_i \wedge dp_i$. This Hamiltonian possesses several symmetries but we shall discuss only one type since it illustrates something new. As in the restricted problem H is invariant under the anti-symplectic involution f: $(q_1,..., q_N, p_1,..., p_N) \to (Rq_1,..., Rq_N, -Rp_N)$. But H is also invariant under an SO_2 action. Consider the action

$$\begin{aligned} F: SO_2 \times \mathbb{R}^{4N} &\to \mathbb{R}^{4N} \\ : (A, (q_1,..., p_N)) &\to (Aq_1,..., Ap_N). \end{aligned}$$

Let $A \in SO_2$ be fixed and $F_A = F(A, \cdot): \mathbb{R}^{4N} \to \mathbb{R}^{4N}$. Then $H \circ F_A = H$ and F_A is symplectic. Thus for each $A \in SO_2$ the Hamiltonian of the N-body problem is invariant under the anti-symplectic involution $f_A = F_A^{-1} \circ f \circ F_A$.

Since H is invariant under the SO_2 action F the equations of motion admit angular momentum $J = \sum_{i=1}^{N} q_1 \times p_i$ as an integral and so if $c \neq 0$ then $J^{-1}(c)$ is an invariant submanifold. Note that both the SO_2 action F and the $\mathbb{Z}_2$ action f leave $J^{-1}(c)$ fixed. Also the flow defined by H and the SO_2 action F commute on $J^{-1}(c)$ and so the flow defined by H is well defined on the orbit space of the SO_2 action. That is, if we define an equivalence relation $\sim$ on $J^{-1}(c)$ by $(q_1,..., p_N) \sim (Aq_1,..., Ap_N)$ for $A \in SO_2$ then the function H and the flow defined by H are well defined on the quotient space $B = J^{-1}(c)/\sim$. In fact B is in a natural way a symplectic manifold and the flow on B is Hamiltonian [5]. Since f_A leaves J fixed $f_A: J(c) \to J(c)$ and the family $\{f_A\}$ of $\mathbb{Z}_2$ actions becomes a single $\mathbb{Z}_2$ action on B.

3. The Symmetry Manifold

In this section we show that the fixed point set of an anti-symplectic involution is a Lagrangian submanifold and we establish some basic results about symmetric periodic solutions. The notation and elementary facts of symplectic geometry used below can be found in [1]. Let (M, Ω) be a symplectic manifold, where M is a smooth $2n$ dimensional manifold and Ω is a closed, non-degenerate two-form on M. For each $p \in M$, let Ω_p denote the skew symmetric bilinear form on T_pM defined by restricting Ω to $T_pM \times T_pM$. Since Ω is non-degenerate the map $\flat: T_pM \to T_p^*M: v_p \to \Omega_p(v_p, \cdot)$ is an isomorphism. Let $\#$ denote the inverse of $\flat$ and write $\flat: v_p \to v_p^\flat$, $\#: \alpha_p \to \alpha_p^\#$. A subspace $W \subset T_pM$ of dimension n such that $\Omega_p | W \equiv 0$ is called a Lagrangian subspace and a submanifold $N \subset M$ such that T_pN is a Lagrangian subspace of T_pM for all $p \in N$ is called a Lagrangian submanifold.

If $H: M \to \mathbb{R}$ is a smooth function then dH is a covariant vector field on M and $dH^\#$ is a contravariant vector field. The vector field $dH^\#$ is called a Hamiltonian vector field and H the Hamiltonian.

Let $f: M \to M$ be a smooth involution, i.e., $f^2 = f \circ f = \mathrm{id}$, where id is the identity map of M. Also let $f^*\Omega = -\Omega$, where f^* is the derivative operator on two-forms induced by f; so f is anti-symplectic. If H is invariant under f, i.e., $H \circ f = H$, then we shall say that H admits f as a symmetry. Examples of such anti-symplectic involutions have been given in the previous section.

Let $Q = \{p \in M: f(p) = p\}$, the fixed set of f, and call it the symmetry manifold of f. We shall always assume that Q is non-empty. In the first example of the previous section the set Q is the set of all initial conditions which give rise to an orthogonal crossing of the line of masses of the restricted three-body problem. In a reversible system Q is the set of zero vectors or the zero section. The first set of lemmas prove that Q is a Lagrangian submanifold.

Let (V, ω) be a symplectic linear space where V is a linear space of dimension $2n$ and ω is a skew symmetric, non-degenerate, bilinear two-form on V. Let $L: V \to V$ be a linear map and define $(L^*\omega)\ (u, v) = \omega(Lu, Lv)$.

Lemma 1. *Let $L: V \to V$ be a linear map such that $L^2 = \mathrm{id}$ and $L^*\omega = -\omega$. Then $\tilde{Q} = \{v \in V: Lv = v\}$ is a Lagrangian subspace. Moreover there exists a symplectic basis $q_1, \dots, q_N, p_1, \dots, p_N$ for V such that $\tilde{Q} =$*

$\text{span}\{q_1,\ldots,q_N\}$, $Lq_l = q_l, Lp_l = -p_l$ *and the matrix representing* L *in this basis is* $\text{diag}(I_n, -I_n)$.

Proof. Since $L^2 = \text{id}$ the only eigenvalues of L are ± 1 and L is semi-simple. Thus $V = N_+ \oplus N_-$, where N_+ (resp. N_-) is the space of all eigenvectors of L corresponding to the eigenvalue $+1$ (resp. -1). Let $L\alpha = \varepsilon\alpha$, $L\beta = \varepsilon\beta$, where $\varepsilon = \pm 1$. $\omega(\alpha, \beta) = -(L^*\omega)(\alpha, \beta) = -\omega(L\alpha, L\beta) = -\varepsilon^2\omega(\alpha, \beta)$. Thus $\omega(\alpha, \beta) = 0$ so $\omega | N_+ = 0$ and $\omega | N_- = 0$. Since V is the direct sum of N_+ and N_- these two subspaces are Lagrangian. By the preliminary results of [3] there exists a symplectic basis $q_1,\ldots,q_N, p_1,\ldots,p_N$ such that $N_+ = \text{span}(q_1,\ldots,q_N)$ and $N_- = \text{span}(p_1,\ldots,p_N)$; therefore the lemma is established. (Also see Lemma 5 below.)

Let 0_1 and 0_2 be open neighborhoods of the origin in $\mathbb{R}^k$ and $F: 0_1 \to 0_2$ a C^1 map which fixes the origin and $F^2 = \text{id}$. Let $DF(0) = A$, the $k \times k$ Jacobian matrix of F at the origin, so $F(x) = Ax + \Phi(x)$, where $\Phi(0) = 0$ and $D\Phi(0) = 0$.

LEMMA 2. *The change of coordinates* $y = g(x) = x + \frac{1}{2}A\Phi(x)$ *reduces* F *to a linear map. That is* $g \circ F \circ g^{-1}: y \to Ay$.

Proof. Since $F^2 = \text{id}$, $A\Phi(x) + \Phi(Ax + \Phi(x)) = 0$. Using this one checks directly that $g \circ F = Ag$, where g is defined in the lemma. Since $F^2 = \text{id}$, A is non-singular and so by the inverse function theorem g^{-1} exists in a neighborhood of the origin.

LEMMA 3. *In a neighborhood of the origin the fixed point set of* F *is an* l*-dimensional manifold, where* l *is the dimension of the eigenspace of* A *corresponding to the eigenvalue* $+1$.

Proof. The lemma is obvious for linear maps and the general case is reduced to the linear use by Lemma 2.

THEOREM 1. *Let* f *be an anti-symplectic involution of a symplectic manifold* M, *then the symmetry manifold* Q *is a Lagrangian submanifold.*

Proof. By Lemma 1 the linearization of f at a fixed point has an n dimensional eigenspace corresponding to the eigenvalue $+1$ so by Lemma 3 the symmetry manifold is locally an n dimensional submanifold. Clearly the tangent space to the symmetry manifold is the subspace of the total tangent space which is fixed under the derivative of f which by Lemma 1 is Lagrangian.

Now let H be an f invariant Hamiltonian. The next lemma shows that Q is the natural generalization of the set of orthogonal crossings of the line of masses in the restricted problem.

LEMMA 4. *If $\gamma(t)$ is a solution of $dH^{\#}$ such that $\gamma(0) \in Q$ and $\gamma(T) \in Q$ for some $T > 0$ then $\gamma(t)$ is $2T$-periodic and the orbit of γ is invariant under f.*

Proof. Since H is f invariant and f is anti-symplectic $dH(f(x))^{\#} = Df(x)\, dH(x)^{\#}$ for all $x \in M$. Let $\delta(t) = f(\gamma(2T-t))$ so

$$\begin{aligned}\dot{\delta}(t) &= -Df\,(\gamma(2T-t))\,\dot{\gamma}(2T-t)\\ &= -Df(\gamma(2T-t))\, dH(\gamma(2T-t))^{\#}\\ &= dH(f(\gamma(2T-t)))^{\#}\\ &= dH(\delta(t))^{\#}.\end{aligned}$$

Thus $\delta(t)$ and $\gamma(t)$ are both solutions of $dH^{\#}$ and are equal when $t = T$; therefore by the uniqueness theorem for ordinary differential equations $f(\gamma(2T-t)) = \delta(t) \equiv \gamma(t)$. Thus $f(\gamma(0)) = \gamma(0) = \gamma(2T)$, which implies γ is $2T$-periodic.

4. COTANGENT BUNDLE COORDINATES

In the previous section we showed that an involution can be linearized locally, but the change of coordinates was not symplectic even when the involution was anti-symplectic. In this section we show that an anti-symplectic involution can be globally linearized by introducing cotangent bundle coordinates. Weinstein [9] showed that there exists a neighborhood N of a Lagrangian submanifold Q of M and a neighborhood 0 of the zero section Q in T^*Q and a symplectomorphism $\Phi: 0 \to N$. 0 inherits its symplectic structure from the naturally defined one on T^*Q. We prove that this construction can be carried out in such a manner that $\Phi \circ f \circ \Phi: 0 \to 0$: $\alpha_p \to -\alpha_p$. Thus the antisymplectic involution when written in these cotangent bundle coordinates takes a covector into its negative. This means that reversible systems are the prototype of systems admitting the discrete symmetries considered in this paper. The specific result of this section is

THEOREM 2. *Let f be an anti-symplectic involution of a symplectic manifold (M, Ω) and Q the symmetric manifold of f as given by Theorem 1. Let Γ be the natural symplectic structure on T^*Q and $g: T^*Q \to T^*Q$: $\alpha_p \to -\alpha_p$. Then there exists a neighborhood 0 of Q in T^*Q, a neighborhood N of Q in M and a diffeomorphism $\Phi: 0 \to N$ such that*

(1) *Φ is symplectic, i.e., $\Phi_*(\Omega) = \Gamma$,*

(2) *the following diagram commutes*:

$$\begin{tikzcd} 0 \arrow[r, "g"] \arrow[d, "\Phi"'] & 0 \arrow[d, "\Phi"] \\ N \arrow[r, "f"] & N \end{tikzcd}$$

Proof. Let P' be a Riemannian metric on M and define a new Riemannian metric P on M by $P_p(x, y) = \frac{1}{2}\{P'_p(x, y) + P'_{f(p)}(f_*(x), f_*(y))\}$ where $p \in M$; x, $y \in T_pM$ and $f_*: T_pM \to T_{f(p)}M$ is the derivative of f. Clearly P is f invariant and so f maps the geodesics of P into geodesics. Let exp be the expotential mapping defined by P; so

$$f(\exp_p(v_p)) = \exp_{f(p)}(f_*(v_p)), \tag{4.1}$$

where $v_p \in T_pM$.

There exists a fiber preserving mapping $J: TM \to TM$ such that $J^2 = -\mathrm{id}$ and J is skew symmetric with respect to P, i.e.,

$$P(Jx, y) = -P(x, Jy), \tag{4.2}$$

and also

$$P(Jx, y) = \Omega(x, y). \tag{4.3}$$

(See [1, p. 173.)

LEMMA 4. $Jf_* + f_*J = 0$.

Proof. $P(f_*Jx, f_*y) = P(Jx, y) = \Omega(x, y) = -\Omega(f_*x, f_*y) = -P(Jf_*x, f_*y)$. Since P is non-degenerate and f_* is onto, the first and last terms being equal imply $f_*J = -Jf_*$.

At each $p \in Q$ we have shown that T_pQ is a Lagrangian subspace. Define $D_p = J(T_pQ)$ and $D = \bigcup_{p \in Q} D_p$ with the obvious vector bundle structure.

LEMMA 5. *D_p is the Lagrangian complement of T_pQ in T_pM. Moreover T_pQ (resp. D_p) consists of all the eigenvectors of f_* corresponding to the eigenvalue* $+1$ (*resp.* -1).

Proof. Let $x_1, \ldots, x_N$ be an orthonormal basis of T_pQ so that $P(x_i, x_j) = \delta_{ij}$. Now $y_1 = Jx_1, \ldots, y_N = Jx_N$ is a basis for D_p. Since T_pQ is Lagrangian $\Omega(x_i, x_j) = 0$, so $\Omega(y_i, y_j) = \Omega(Jx_i, Jx_j) = P(J^2x_i, Jx_j) = -P(x_i, Jx_j) = -P(Jx_j, x_i) = -\Omega(x_j, x_i) = 0$. Thus D_p is Lagrangian also. Moreover since $\Omega(x_i, y_j) = \Omega(x_i, Jx_j) = P(Jx_i, Jx_j) = P(x_i, x_j) = \delta_{ij}$ the vectors $x_1, \ldots, x_N, y_1, \ldots, y_N$ form a symplectic basis for T_pM. Clearly

$f_* x_i = x_i$ by definition of Q and by Lemma 4 $f_* y_i = f_* J x_i = J f_* x_i = -J x_i = -y_i$.

Define $\psi_1 : D \to M$: $d_p \to \exp_p(d_p)$ and $g_1 : D \to D$: $d_p \to -d_p$.

LEMMA 6. *There exist open neighborhoods* 0_1 *of* Q *in* D *and* N_1 *of* Q *in* M *such that* (1) $g_1(0_1) = 0_1$ *and* $f(N_1) = N_1$, (2) ψ_1: $0_1 \to N_1$ *is a diffeomorphism and* (3) *the diagram*

$$\begin{tikzcd} 0_1 \arrow[r, "g_1"] \arrow[d, "\psi_1"'] & 0_1 \arrow[d, "\psi_1"] \\ N_1 \arrow[r, "f"] & N_1 \end{tikzcd}$$

commutes.

Proof.

$$f \circ \psi(d_p) = f \circ \exp_p(d_p) = \exp_{f(p)}(f_*(d_p))$$
$$= \exp_p(-d_p) = \psi(-d_p) = \psi \circ g_1(d_p).$$

By the implicit function theorem there exists neighborhoods $0_1'$ of Q in D and N_1' of Q in M such that $\psi: 0_1' \to N_1'$ is a diffeomorphism. Define $N_1 = N_1' \cap f(N_1')$ and $0_1 = 0_1' \cap g_1(0_1')$. Clearly the three properties of the lemma now hold.

Define ν: $T^*Q \to TQ$ as the vector bundle isomorphism induced by the Riemannian metric P. Thus $D_p = J(T_p Q) = J(\nu(T_p^* Q)) = J \circ \nu(T_p^* Q)$. Since J and ν are vector bundle isomorphism $\Xi = (J \circ \nu)^{-1}$ is a vector bundle isomorphism of T^*Q onto D. Define $\psi_2 = \Xi \circ \psi_1$ and g: $T^*Q \to T^*Q$: $\alpha_p \to -\alpha_p$.

LEMMA 7. *There exists open neighborhoods* 0_2 *of* Q *in* T^*Q *and* N_2 *of* Q *in* M *such that* (1) $g_1(0_2) = 0_2$ *and* $f(N_2) = N_2$, (2) $\psi_2 : 0_2 \to N_2$ *is a diffeomorphism and* (3) *the diagram*

$$\begin{tikzcd} 0_2 \arrow[r, "g"] \arrow[d, "\psi_2"'] & 0_2 \arrow[d, "\psi_2"] \\ N_2 \arrow[r, "f"] & N_2 \end{tikzcd}$$

commutes.

Proof. Define $0_2 = \Xi^{-1}(0_1)$, $N_2 = N_1$. The lemma then follows from Lemma 6 and the fact that Ξ is a linear map on fibers.

The map ψ_2 of Lemma 7 may not be symplectic and so we follow the arguments of Weinstein [9] to overcome this difficulty. Let Ω_0 be the pull back of the symplectic form Ω of M to T^*Q by the diffeomorphism ψ_2 so $\psi_2^*(\Omega)=\Omega_0$. Since $f^*\Omega=-\Omega$ by Lemma 7, $g^*(\Omega_0)=-\Omega_0$. Let Ω_1 be the standard symplectic structure on T^*Q and since g reverses covectors $g^*(\Omega_1)=-\Omega_1$. Note that $Q\subset T^*Q$ is a Lagrangian submanifold with respect to both symplectic structure. Define $\Omega_t=\Omega_0+t(\Omega_1-\Omega_0)$ and note that Ω_t is a symplectic structure on some neighborhood of Q in T^*Q for $0\leqslant t\leqslant 1$.

LEMMA 8. *There exists a one-form* β *such that* $d\beta=\Omega_1-\Omega_0$ *and* $g^*\beta=-\beta$.

Proof. Ω_1 is closed and Ω_0 is exact by construction. But Ω_0 is actually closed since $\Omega_0|Q\equiv 0$ (hence Ω_0 is closed on Q) and Q is a homotopic retract of T^*Q. Thus there exists a one form α such that $d\alpha=\Omega_1-\Omega_0$. Define $\beta=\frac{1}{2}(\alpha-g^*\alpha)$ so

$$\begin{aligned} d\beta &= \tfrac{1}{2}(d\alpha-dg^*\alpha)=\tfrac{1}{2}(d\alpha-g^*d\alpha)\\ &= \tfrac{1}{2}(\Omega_1-\Omega_0-g^*[\Omega_1-\Omega_0])=\Omega_1-\Omega_0. \end{aligned}$$

Clearly $g^*\beta=-\beta$.

As noted before Ω_t is non-degenerate on a neighborhood of Q for all $0\leqslant t\leqslant 1$ and so for each $0\leqslant t\leqslant 1$ the form Ω_t defines a linear operator (the sharp operator) from the cotangent space to the tangent space for each point in this neighborhood of Q. Following Weinstein [9] we define a time dependent vector field Y_t by

$$\Omega_t(Y_t,\cdot)=-\beta$$

or

$$Y_t=\tilde{\Omega}_t^{-1}(\beta),$$

where $\tilde{\Omega}_t^{-1}$ is the sharp operator defined by Ω_t. This time dependent vector field is zero on Q and so the solutions can be extended to the interval $0\leqslant t\leqslant 1$ by further restricting the neighborhood of Q. Let $\xi_t(\zeta)$ be the solution of Y_t such that $\xi\circ(\zeta)=\zeta$, i.e.,

$$\frac{d\xi_t}{dt}(\zeta)=Y_t(\xi_t(\zeta)),\qquad \xi_0(\zeta)=\zeta.$$

By the remark given above and the theorems on the smoothness of solutions of ordinary differential equations the map $\xi=\xi_1$ is a diffeomorphism of a neighborhood of Q onto a neighborhood of Q which fixes Q. By the same computation as founded in Weinstein [9] we have $\xi^*(\Omega_1)=\Omega_0$.

LEMMA 9. $g_* Y_t = Y_t$, $\xi \circ g = g \circ \xi$.

Proof. By definition of Y_t

$$\Omega_t(Y_t, u) = -\beta(u),$$

where u is an arbitrary vector field defined near Q. Applying g to this formula gives

$$\begin{aligned} g_* \Omega_t(g^* Y_t, g^* u) &= -g_* \beta(g^* u), \\ -\Omega_t(g^* Y_t, g^* u) &= \beta(g^* u), \\ \Omega_t(g^* Y_t, g^* u) &= \Omega_t(Y_t g^* u). \end{aligned}$$

But since Ω_t is non-degenerate, g is a diffeomorphism and u is arbitrary this last formula implies $g^* Y_t = Y_t$.

Now $d\xi_t/dt = Y_t \circ \xi_t$ and $\xi_0 = \text{id}$. Define $N_t = g \circ \xi_t \circ g$ so $n_0 = \text{id}$ and

$$\begin{aligned} \frac{dN_t}{dt} &= Dg \cdot \frac{d\xi_t}{dt} \circ g \\ &= Dg \cdot Y_t \circ \xi_t \circ g \\ &= g_* Y_t \circ g \circ \xi_t \circ g \\ &= Y_t \circ N_t. \end{aligned}$$

Thus ξ_t and N_t satisfy the same differential equation and the same initial condition so $\xi_t \equiv N_t$ and $g \circ \xi = \xi \circ g$. Let 0_3 be a neighborhood of Q in T^*Q, where ξ^{-1} is a diffeomorphism into 0_2 of Lemma 7 and define $0 = 0_3 \cap g(0_3)$. Thus $\xi^{-1}: 0 \to 0_2$, $g(0) = 0$ and the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{g} & 0 \\ {\scriptstyle \xi^{-1}}\downarrow & & \downarrow{\scriptstyle \xi^{-1}} \\ 0_2 & \xrightarrow{g} & 0_2 \end{array}$$

commutes. Let $\Phi = \psi_2 \circ \xi^{-1}$ and $N = \Phi(0) \subset N_2 \subset M$ so Φ is a diffeomorphism of 0 onto N, the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{g} & 0 \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ N & \xrightarrow{f} & N \end{array}$$

commutes and $\Phi^*(\Omega) = \xi^{-1^*}(\psi_2^*(\Omega)) = \xi^{-1^*}(\Omega_1$. Since $\Omega_1 = \Gamma$ of the statement of the theorem the proof is finished.

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