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Adiabatic Invariants for Linear Hamiltonian Systems

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n independent adiabatic invariants in involution are found for a slowly varying Hamiltonian system of order $2n \times 2n$. The Hamiltonian system considered is $\epsilon\dot{u} = A(t)u$ as $\epsilon \to 0^+$, where $A(t)$ is a $2n \times 2n$ real matrix with distinct, pure imaginary eigen values for each $t \in [-\infty, \infty]$, and $d^{(j)}A/dt^{(j)} \in L_1(-\infty, \infty)$, for all $j > 0$. The adiabatic invariants $I_s(u, t)$, $s = 1,..., n$ are expressed in terms of the eigen vectors of $A(t)$. Approximate solutions for the system to arbitrary order of ϵ are obtained uniformly for $t \in [-\infty, \infty]$.

1. INTRODUCTION

In the classical literature a conservative dynamical system of n degrees of freedom was considered solved when n independent integrals in involution were found. One need only look at the chapters in Whittaker [8] titled "The soluble problems of particle dynamics" and "The soluble problems of rigid dynamics" to see the importance of n integrals in involution. Almost every example is analyzed by such integrals.

In systems which vary slowly with time, integrals must be replaced by quantities which also vary slowly with time, i.e., with adiabatic invariants. Of course the knowledge of n independent adiabatic invariants in involution for a dynamical system does not imply that the system is "solved" as it does in the conservative case. However a great deal of mathematical and physical information can be obtained from adiabatic invariants [3, 4, 1].

In order to illustrate our theorem consider the system

$$\dot{u} = Au; \qquad \dot{u} = d/dt \tag{1.1}$$

where u is a $2n$ dimensional column vector and A is a constant $2n \times 2n$ real Hamiltonian matrix with distinct pure imaginary eigen values $\lambda_1, ..., \lambda_{2n}$. Let the eigen values be ordered so that $\lambda_{s+n} = -\lambda_s = \bar{\lambda}_s$ for $s = 1,..., n$. If

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$c_1, \ldots, c_{2n}$ are row eigen vectors of A corresponding to $\lambda_1, \ldots, \lambda_{2n}$ (i.e., $c_s A = \lambda_s c_s$) which satisfy the reality condition $c_{s+n} = \bar{c}_s$, $s = 1, \ldots, n$ then the n real functions $I_s(u) = (c_s u)(c_{s+n} u) = |c_s u|^2$, $s = 1, \ldots, n$ form a set of n independent integrals in involution for 1.1.

If the matrix A were now allowed to vary slowly with t then one would expect that there would exist n functions close to $I_1, \ldots, I_n$ which also vary slowly with t. This is the general content of our result.

In order to be precise we must make some definitions. A function $f\colon (-\infty, \infty) \to R$ or C will be *gentle* if $(d^s f/dt^s) \in L_1(-\infty, \infty)$ for $s = 0, 1, 2, \ldots$ If f or even df/dt is gentle then

$$\lim_{t \to \pm\infty} \frac{d^s f}{dt^s}, \quad s = 0, 1, 2, \ldots$$

exists and so we may consider f and all its derivatives as defined and continuous on $[-\infty, \infty]$. The assumption that a system varies slowly with t is expressed by considering a system of the form

$$\epsilon \dot{u} = A(t) u \tag{1.2}$$

where A is a $2n \times 2n$ real matrix such that each entry of dA/dt is gentle and ϵ is a small positive parameter. This assumption is more easily understood when one uses the parameter $\tau = \epsilon^{-1} t$ so that 1.2 becomes $du/d\tau = A(\epsilon\tau)u$.

Furthermore the system 1.2 is assumed to be Hamiltonian. Thus the matrix $S(t) = -JA(t)$ is symmetric where J is the usual $2n \times 2n$ matrix of Hamiltonian mechanics given by

$$J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}.$$

The system 1.2 is then written in the Hamiltonian form

$$\dot{u} = J(\partial H/\partial u) \tag{1.3}$$

where

$$H = (1/2\epsilon)\, u^T S(t) u. \tag{1.4}$$

Let $\psi(t, t_0, u_0, \epsilon)$ be the solution of 1.2 which satisfies $\psi(t_0, t_0, u_0, \epsilon) = u_0$. A function $I(u, t)$ will be an *adiabatic invariant* of 1.2 if

$$\mathscr{I}(\infty, t_0, u_0, \epsilon) - \mathscr{I}(-\infty, t_0, u_0, \epsilon) = 0(\epsilon^s) \qquad \text{as} \quad \epsilon \to 0^+$$

for all $s = 0, 1, 2, \ldots$ where $\mathscr{I}(t, t_0, u_0, \epsilon) = I(\psi(t, t_0, u_0, \epsilon), t)$.

Usually in the literature an adiabatic invariant need only satisfy

$$\mathscr{I}(\infty, t_0, u_0, \epsilon) - \mathscr{I}(-\infty, t_0, u_0, \epsilon) = 0(\epsilon^1) \qquad \text{as} \quad \epsilon \to 0^+.$$

Let $I_1, \ldots, I_l$ be l functions of $(u, t) \in R^{2n+1}$. The Poisson bracket of I_r and I_s, $\{I_r, I_s\}$ is defined by

$$\{I_r, I_s\} = (\partial I_r/\partial u)^T J(\partial I_s/\partial u). \tag{1.5}$$

The set of functions $I_1, \ldots, I_l$ are said to be in involution if $\{I_r, I_s\} \equiv 0$ for $1 \leqslant s, r \leqslant l$. The set of functions $I_1, \ldots, I_l$ are said to be independent if $(\partial I_1/\partial u), \ldots, (\partial I_l/\partial u)$ are independent vectors for all (u, t) except for a subset of R^{2n+1} with no interior.

We can now state our main result.

THEOREM 1. *Let the eigen values of $A(t)$ be distinct and pure imaginary for each $t \in [-\infty, \infty]$. Then the system* 1.2 *admits n independent adiabatic invariants in involution.*

In fact the adiabatic invariants are constructed as follows. Let $\lambda_1(t), \ldots, \lambda_{2n}(t)$ be the eigen values of $A(t)$ with the order such that $\lambda_{n+s}(t) = -\lambda_s(t) = \bar{\lambda}_s(t)$ for $s = 1, \ldots, n$. For each t the eigenspace of $A(t)$ corresponding to the eigenvalue $\lambda_i(t)$, $i = 1, \ldots, n$, is one dimesnional and so there exists exactly two eigenvectors of unit length $\pm c_i(t)$. In Lemma 1 we shall show that the choice of unit eigenvector $c_i(t)$ can be made so that $c_i(t)$ is C^∞ for $i = 1, \ldots, n$ and $t \in (-\infty, \infty)$. Let $c_{n+s} = \bar{c}_s$ for $s = 1, \ldots, n$ so that $c_1, \ldots, c_{2n}$ are a full set of eigen vectors of A. Here as before we take the c_s to be row eigen vectors so $c_s(t)A(t) = \lambda_s(t)\, c_s(t)$. Then the n adiabatic invariants of Theorem 1 are

$$\begin{aligned} I_s(u, t) &= | c_s(t)\, J c_{s+n}^T(t)|^{-1} (c_s(t)u)(c_{s+n}(t)u) \\ &= | c_s(t)\, J c_{s+n}^T(t)|^{-1} \, | c_s(t)u |^2 \end{aligned} \tag{1.6}$$

for $s = 1, \ldots, n$.

In order to compare our theorem with similar results in the literature consider the equation

$$\epsilon^2 \ddot{\xi} + \phi^2(t)\xi = 0, \tag{1.7}$$

where ϕ is a positive function of t such that $d\phi/dt$ is gentle and $\phi(\infty) > 0$, $\phi(-\infty) > 0$. Eq. 1.7 can be written as a system in the form of 1.2 by introducing $\epsilon\dot{\xi} = \eta$. In this case the matrix A turns out to be

$$\begin{pmatrix} 0 & 1 \\ -\phi^2 & 0 \end{pmatrix}$$

which satisfies the hypothesis of our theorem. The eigen values of A are $\pm i\phi(t)$ and the eigen vectors are $(\pm i\phi, 1)$. The quantity $| c_s(t)\, J c_{s+n}^T(t)|$ in this case is 2ϕ and so the adiabatic invariant is $I = (1/2\phi)\,(i\phi\xi + \eta)(-i\phi\xi + \eta) = (1/2\phi)\,(\phi^2\xi^2 + \eta^2)$. This complete result was first obtained by Littlewood [5] and the reader is referred to this paper for a discussion of earlier partial results. Recently Wasow [6] has given an eloquent proof of Littlewood's theorem by fully describing the form of the fundamental matrix solution of Eq. 1.7. Wasow has even obtained the precise asymptotic order of the adiabatic invariant under further mild assumptions on ϕ by using turning point theory [7]. Indeed the present authors were first stimulated by the results of Professor Wasow and wish to thank him for several enlighting conversations on the subject of adiabatic invariants.

2. First Order Diagonalization

In this section we shall show how to diagonalize Eqs. 1.2 to first order by a linear symplectic change of variables.

Lemma 1. *Let $\lambda_1(t),..., \lambda_{2n}(t)$ be the eigen values of $A(t)$ and $c_1(t),..., c_{2n}(t)$ the corresponding smooth row eigen vectors of unit length. Then $\dot\lambda_s(t)$ and the entries of $\dot c_s(t)$ are gentle for $s = 1,..., 2n$.*

Proof. Let $p(t, \lambda) = \det(A(t) - \lambda I)$. $p(t, \lambda)$ is a polynomial of degree $2n$ in λ with coefficients with gentle derivatives. By assumption the zeros of $p(t, \lambda)$ are distinct for $-\infty \leqslant t \leqslant \infty$. Let Γ_s be a circle in the complex λ plane centered at $\lambda_s(\infty)$ and sufficiently small that $\lambda_r(\infty)$, $r \neq s$, lies in the exterior of Γ_s. Then there exists a $T > 0$ such that for all $t \geqslant T$, Γ_s contains $\lambda_s(t)$ in its interior and $\lambda_r(t)$, $r \neq s$, lies in the exterior of Γ_s. Thus for $t \geqslant T$

$$\lambda_s(t) = \frac{1}{2\pi i} \int_{\Gamma_s} \zeta \frac{(\partial p/\partial\lambda)(t, \zeta)}{p(t, \zeta)} \, d\zeta.$$

So

$$\frac{d^l\lambda_s(t)}{dt^l} = \frac{1}{2\pi i} \int_{\Gamma_s} R_l(t, \zeta) \, d\zeta$$

where $R_l(t, \zeta)$ is a rational function of ζ whose denominator is bounded away from zero for $t \geqslant T$, $\zeta \in \Gamma_s$ and whose numerator is a polynomial in ζ with gentle coefficients for $l = 1, 2,...$. Thus we can interchange the order of integration to show that $(d^l\lambda_s/dt^l) \in L_1(T, \infty)$. In a similar way $(d^l\lambda_s/dt^l) \in L_1(-\infty, T')$ for $l = 1, 2,...$ and so $\dot\lambda_s$ is gentle.

Now let $B(t) = A(t) - \lambda_s(t)I$ so the entries of $B(t)$ have gentle derivatives. Since $(adj\, B(t))\, B(t) = \det B(t)I = 0$ the nonzero rows of adj $B(t)$ are eigen vectors of $A(t)$ corresponding to the eigen value $\lambda_s(t)$. Since rank $B(t) = n - 1$ for $-\infty \leqslant t \leqslant \infty$ there exist closed intervals $O_1, ..., O_r$ such that $\cup O_i = [-\infty, \infty]$, int $O_i \cap$ int $O_{i+1} \neq \varnothing$, $O_i \cap O_j = \varnothing$ if $| i - j | > 1$ and rows $b_i(t)$ of adj $B(t)$ such that $b_i(t) \neq \varnothing$ on O_i. Thus on O_i the unit eigenvectors of $A(t)$ corresponding to $\lambda_s(t)$ is $\pm b_i(t)\{\| b_i(t)\|\}^{-1}$. Now it is clear that we can choose the signs so that $c_s(t)$ is C^∞ on $(-\infty, \infty)$. Also since $c_s(t) = \pm b_i(t) \{\| b_i(t)\|\}^{-1}$ on O_i it is clear that $(d^l/dt^l)\, c_s(t) \in L_1(O_i)$ for $l = 1, 2,\ldots$ and $i = 1,\ldots, r$. Thus $\dot{c}_s(t)$ has gentle entries.

With the information from Lemma 1 we shall construct a symplectic change of variables. Let the ordering be such that $\lambda_{s+n}(t) = -\lambda_s(t) = \bar{\lambda}_s(t)$ and $c_{s+n}(t) = \bar{c}_s(t)$ for $s = 1,\ldots, n$.

Since A is Hamiltonian $AJ + JA^T = 0$ so $\lambda_r c_r J c_s^T = c_r A J c_s^T = -c_r J A^T c_s^T = -\lambda_s c_r J c_s^T$. Thus $c_r J c_s^T = 0$ unless $\lambda_r + \lambda_s = 0$ or $| r - s | = n$. Since $c_r J c_s^T = 0$ for all s is impossible we have that $c_r J c_s^T \neq 0$ when $| r - s | = n$. Now let $1 \leqslant r \leqslant n$.

$$c_r J c_{r+n}^T = c_r J \bar{c}_r^{\,T} = \overline{\bar{c}_r J c_r^T} = \overline{c_r J^T \bar{c}_r^{\,T}} = -\overline{c_r J c_{r+n}^T}$$

and so $c_r J c_{r+n}^T$ is pure imaginary. By interchanging λ_r, λ_{r+n} and c_r, c_{r+n} if necessary, we may assume $c_r J c_{r+n}^T = ai$ with $a_r > 0$. Now define $d_r(t) = | c_r(t)\, J c_{r+n}(t)^T |^{-1/2}\, c_r(t)$ for $r = 1,\ldots, n$ and $d_{r+n}(t) = \bar{d}_r(t)$ for $r = 1,\ldots, n$. Thus we have $d_r J d_s^T = 0$ for $| r - s | \neq n$ and $d_r J d_{r+n}^T = +i$.

Remark. Note that the adiabatic invariants defined in the introduction are just $I_r(t, u) = (d_r u)(d_{r+n} u)$.

Let $P(t)$ be the $2n \times 2n$ matrix whose rth row is d_r. Then from the above $P(t)\, J P(t)^T = iJ$ and $P(t)\, A(t) P^{-1}(t) = A_0(t) = \operatorname{diag}(\lambda_1(t),\ldots, \lambda_{2n}(t))$. Note that by Lemma 1 the matrix $\dot{P}$ has gentle entries. It will be important in the argument that follows to keep track of the fact that Eq. 1.1 is real. Let Q be the $2n \times 2n$ matrix defined by

$$Q = \begin{pmatrix} O_n & I_n \\ I_n & O_n \end{pmatrix}.$$

Now by construction $\bar{P} = QP$ and so $Q\bar{A}_0 Q = A_0$.

We are now ready to make the change of variables $x = P(t)u$ in Eq. 1.2 to get

$$\epsilon \dot{x} = [A_0(t) + \epsilon A_1(t)]x \tag{2.1}$$

where

$$A_0(t) = P(t)\, A(t)\, P(t)^{-1} \tag{2.2}$$

$$A_1(t) = \dot{P}(t) P^{-1}\,(t). \tag{2.3}$$

Since $PJP^T = iJ$ and $| \det P(t)| = 1$, $A_1(t)$ is gentle A_0 and A_1 satisfy the reality condition $QA_sQ = \bar{A}_s$, $s = 0, 1$. The change of variables $x = P(t)u$ is a symplectic change of variables with multiplier i and so Eqs. 2.1 are Hamiltonian.

The new Hamiltonian is of the form

$$\epsilon^{-1}\{H_0(x, t) + \epsilon H_1(x, t)\} \tag{2.4}$$

where $H_0(x, t) = \frac{1}{2}x^T S_0(t)x$, $H_1(x, t) = \frac{1}{2}x^T S_1(t)x$. Here, both the matrices $S_0 = -JA_0$ and $S_1 = -J\dot{P}P^{-1}$ are symmetric, because $PJP^T = iJ$. Since $QJ = -JQ$ one sees that S_0 and S_1 satisfy the condition $QS_sQ = -\bar{S}_s$, $s = 0, 1$.

Remark. In the new coordinates the adiabatic invariants of our theorem now take the simple form $I_s(x, t) = x_s x_{n+s}$. Thus it is clear that they are independent and in involution.

3. Formal Diagonalization

In the preceding section we showed how to diagonalize the equations of motion to first order. In this section we shall show how to formally diagonalize the equations to all orders. In order to do this we shall deal extensively with a class of functions which we shall now define. A function $K(x, t)$ will be called a GR function if $K(x, t) = x^T S(t)x$ where $S(t)$ is a $2n \times 2n$ symmetric matrix with gentle entries which satisfies the reality condition $\bar{S} = -QSQ$. This last condition is usually called a reality condition. GR stands for gentle-real.

For notational reasons which will become clear as we proceed we shall consider a slightly more general Hamiltonian. Consider now

$$\epsilon^{-1}H_*(x, t, \epsilon) = \epsilon^{-1} \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} H_j^0(x, t) \tag{3.1}$$

where

$$H_0^0(x, t) = H_0(x, t) = \sum_{s=1}^{n} \lambda_s(t)\, x_s x_{s+n}$$

and $H_j^0(x, t)$ is a GR function for $j \geqslant 1$. In the previous section we reduced the Hamiltonian to this form with $H_0^0 = H_0$, $H_1^0 = H_1$, $H_j^0 = 0$ for $j \geqslant 2$. We shall use the method of Lie transforms developed by Deprit [2] to reduce the Hamiltonian 3.1 by a formal linear symplectic change of variables to a Hamiltonian whose equations of motion are diagonal. The change of variable $x = \phi(y, t, \epsilon)$ will be constructed by constructing a function $W(x, t, \epsilon)$ and requiring ϕ to be the solution of

$$dx/d\epsilon = J(\partial W/\partial x)(x, t, \epsilon) \tag{3.2}$$

subject to the initial condition $x(t, 0) = y$. The function ϕ will be linear in y since the function W will be constructed quadratic in x. For fixed t, ϵ the function ϕ will be symplectic since Eq. 3.2 is Hamiltonian.

The new Hamiltonian $\epsilon^{-1}K(y, t, \epsilon)$ is the sum of $\epsilon^{-1}H^*(y, t, \epsilon) = \epsilon^{-1}H_*[\phi(y, t, \epsilon), t, \epsilon]$ and a remainder function R given by the formula

$$R(y, t, \epsilon) = \int_0^\epsilon \frac{\partial}{\partial t} W[\phi(y, t, s), t, s]\, ds. \tag{3.3}$$

The Hamiltonian $\epsilon^{-1}K$ will have a formal expansion

$$\epsilon^{-1}K(y, t, \epsilon) = \epsilon^{-1} \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} K^j(y, t) \tag{3.4}$$

where $K^0 = H_0{}^0$ and for $j \geqslant 1$

$$K^j(y, t) = \sum_{s=1}^{n} a_s{}^j y_s y_{s+n}\,. \tag{3.5}$$

The functions a^j will be gentle and pure imaginary so K^j is a GR function.

The change of variables will be constructed by defining W order by order. Specifically let

$$W(x, t, \epsilon) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} W_{j+1}(x, t). \tag{3.6}$$

We will construct W so that W_j is a GR function and so that the new Hamiltonian is given in 3.4 and 3.5.

Let

$$f_*(x, t, \epsilon) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} f_j{}^0(x, t)$$

be any formal expansion in ϵ. The method of Lie transforms gives a recursive method for computing the function $f^*(y, t, \epsilon) = f_*[\phi(y, t, \epsilon), t, \epsilon]$, the Lie transform of f_*. The computation is done by computing a double indexed set of functions $\{f_k{}^l\}$ which agree with our previous definition when $l = 0$ and are related by the formula

$$f_k{}^l = f_{k+1}^{l-1} + \sum_{s=0}^{k} \binom{k}{s} \{f_{k-s}^{l-1}, W_{s+1}\} \tag{3.7}$$

where $\{\ ,\ \}$ is the Poisson bracket operator. The function f^* then has the expansion

$$f^*(y, t, \epsilon) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} f_0{}^j(y, t). \tag{3.8}$$

It is convenient to display the functions $\{f_k^l\}$ in the triangular array

$$\begin{array}{ccccc} f_0^0 & & & & \\ \downarrow & & & & \\ f_1^0 & \to & f_0^1 & & \\ \downarrow & & \downarrow & & \\ f_2^0 & \to & f_1^1 & \to & f_0^2 \\ \downarrow & & \downarrow & & \downarrow \end{array} \tag{3.9}$$

Note that the coefficients of $\epsilon^j/j!$ in f_* appear in the far left column of 3.9 and the coefficients of $\epsilon^j/j!$ of f^* appear on the main diagonal of 3.9.

From the above we see that to compute H^* and R we need compute two sets $\{H_k^l\}$ and $\{R_k^l\}$ where $R_j^0 = \partial W_{j+1}/\partial t$, H_j^0 is as given before and then

$$H^*(y, t, \epsilon) = \sum_{j=0}^{\infty} (\epsilon^j/j!)\, H_0^j(y, t) \tag{3.10}$$

$$R(y, t, \epsilon) = \sum_{j=1}^{\infty} (\epsilon^j/j!)\, R_0^{j-1}(y, t). \tag{3.11}$$

Before proceeding we require a technical result.

LEMMA 2. *Let L be a* GR *function. Then there exists a* GR *function D and a* GR *function F of the form*

$$F(x, t) = \sum_{s=1}^{n} f_s(t)\, x_s x_{s+n}\,,$$

f_s pure imaginary and gentle, such that

$$F = L + \{H_0^0, D\}. \tag{3.12}$$

Proof. Let $L = x^T M x$, $D = x^T E x$, $M = (m_{sr})$, $E = (e_{rs})$. Since $(\partial H_0^0/\partial x)^T J = -x^T A_0$ we have $\{H_0^0, D\} = -x^T\{A_0 E + E A_0\}x$. The coefficient of $x_r x_s$ in $L + \{H_0^0, D\}$ is $m_{rs} - (\lambda_r + \lambda_s)e_{rs}$. If $|r - s| \neq n$ then $\lambda_r + \lambda_s$ is a function with gentle derivative which is bounded away from zero so we may choose $e_{rs} = (\lambda_r + \lambda_s)^{-1} m_{rs}$. Thus e_{rs} is gentle and $e_{rs} = e_{sr}$. If $|r - s| = n$ let $e_{rs} = 0$. Define $f_r = 2m_{r,r+n}$. With these definitions we clearly have 3.12 and so we need only check the reality conditions. Since L is a GR function $\bar{m}_{rs} = -m_{\alpha\beta}$ when $|r - \alpha| = |s - \beta| = n$. If $|r - \alpha| = |s - \beta| = n$ then $\bar{\lambda}_r = \lambda_\alpha$, $\bar{\lambda}_s = \lambda_\beta$ and so $\bar{e}_{rs} = (\bar{\lambda}_r + \bar{\lambda}_s)^{-1}\bar{m}_{rs} = (\lambda_\alpha + \lambda_\beta)^{-1}(-m_{\alpha\beta}) = -e_{\alpha\beta}$. Thus D is a GR function. Also $f_r = 2m_{r,r+n} = 2m_{r+n,r} = -2\bar{m}_{r,r+n} = -f_r$ so f_r is pure imaginary.

Before proceeding with the inductive construction of ϕ we note that the class of GR functions is an algebra over the reals where addition is taken as pointwise addition of functions and the product of two functions is taken as their Poisson bracket. In fact if K is a GR function then so is $\{H_0^0, K\}$. (Note that H_0^0 is not a GR function but its derivative is.)

INDUCTION HYPOTHESIS I_m. Let H_k^l, R_α^β and W_s be determined for $l, k \geqslant 0$; $l + k \leqslant m$; $1 \leqslant s \leqslant m$; $\alpha, \beta \geqslant 0$, $\alpha + \beta \leqslant m - 2$ such that (1) all except H_0^0 are GR functions, and (2) $K_0^m = H_0^m + R_0^{m-2}$ is of the form

$$K_0^m(y) = \sum_{j=1}^{n} a_j^m(t)\, y_j y_{j+n} \tag{3.13}$$

where the coefficients a_j^m are gentle and pure imaginary.

I_0 is trivially satisfied. Now assume I_{m-1}. First we note that R_0^{m-2} can be calculated from $W_1, \ldots, W_{m-1}$ which are given as GR functions by I_{m-1}. To see this consider the array 3.9 and the formula 3.7 with the f's replaced by R's. Note that the far left column is known down to R_{m-2}^0 since $R_j^0 = \partial W_{j+1}/\partial t$. The formula 3.7 shows that successive columns can be computed as far down as the row containing R_α^β with $\alpha + \beta = m - 2$, i.e., the $m - 1$ row. Since the GR functions form an algebra it is clear that each R_α^β will be a GR function.

Now we turn to the computation of the H's. from I_{m-1} we know H_k^l for $l + k \leqslant m - 1$ and H_m^0 is given. From 3.7 we have

$$H_{m-1}^1 = H_m^0 + \sum_{s=0}^{m-2} \binom{m-1}{s} \{H_{m-1-s}^0, W_{s+1}\} + \{H_0^0, W_m\} \tag{3.14}$$

In the above we have separated off the one term which contains W_m. Thus

$$H_{m-1}^1 = E_{m-1}^1 + \{H_0^0, W_m\} \tag{3.15}$$

where E_{m-1}^1 is a GR function computed from functions given by I_{m-1} and so is known. A simple induction argument on l gives

$$H_{m-l}^l = E_{m-l}^l + \{H_0^0, W_m\} \tag{3.16}$$

for $1 \leqslant l \leqslant m$ where E_{m-l}^l is a GR function computed from functions given by I_{m-1}. Thus

$$K_0^m = H_0^m + R_0^{m-2} = E_0^m + R_0^{m-2} + \{H_0^0, W_m\}. \tag{3.17}$$

Now in Lemma 2 take L as $E_0^m + R_0^{m-2}$ and let W_m be the solution D and K_0^m the solution F. This proves I_m.

Now $\phi(y, t, \epsilon)$ has a formal expansion of the form

$$\phi(y, t, \epsilon) = \sum_{j=0}^{\infty} (\epsilon^j/j!)\, \phi^j(y, t)$$

where $\phi^0(y, t) = y$. For $j \geqslant 1$ the functions ϕ^j are of the form $\phi^j(y, t) = \Phi^j(t)y$ where $\Phi^j(t)$ is a $2n \times 2n$ matrix with gentle entries. In order to see this let $f_*(x, t, \epsilon) = x_s$, the s-component of the vector x, then $f^*(y, t, \epsilon)$ is the s-component of the vector $\phi(y, t, \epsilon)$. A simple induction argument using formula 3.7 yields the above claim.

In summary we have.

LEMMA 3. *Under the above assumption there exists a formal linear symplectic change of variables*

$$x = \left\{ \sum_{j=0}^{\infty} (\epsilon^j/j!)\, \Phi^j(t) \right\} y \qquad 3.18$$

which reduces 3.1 *and hence* 2.4 *to the Hamiltonian* 3.4. *In the above* $\Phi^0 = I$ *and the entries of* Φ^j, $j \geqslant 1$ *are gentle.*

4. THE ESTIMATES

By the remark at the end of Section 2 it is enough to prove that $I_s(x) = x_s x_{s+n}$ is an adiabatic invariant for the system 2.1 for $s = 1,\ldots, n$. Let m be a positive integer and consider the truncated change of variables

$$x = \left\{ \sum_{j=0}^{m} (\epsilon^j/j!)\, \Phi^j(t) \right\} v \qquad 4.1$$

(cf. Eq. 3.18). Then Eq. 21.1 becomes

$$\epsilon \dot{v} = \left\{ \sum_{j=0}^{m} (\epsilon^j/j!)\, A^j(t) \right\} v + \epsilon^{m+1} D(t, \epsilon) v \qquad 4.2$$

where $A^j = \mathrm{diag.}(a_1{}^j,\ldots, a_n{}^j, -a_1{}^j,\ldots, -a_n{}^j)$, $a_s{}^j$ are gentle and pure imaginary and $D(t, \epsilon)$ is a $2n \times 2n$ matrix with gentle entries such that $\int_{-\infty}^{\infty} \| D(t, \epsilon)\| \, dt \leqslant B$ for $0 \leqslant \epsilon \leqslant \epsilon_0$ a sufficiently small number.

Now consider the truncated system

$$\epsilon \dot{w} = \left\{ \sum_{j=0}^{m} (\epsilon^j/j!)\, A^j(t) \right\} w. \qquad 4.3$$

Because all the matrices A^j are diagonal with pure imaginary entries the fundamental matrix solution of 4.3 which is the identity at $t = 0$ is uniformly bounded for $t \in [-\infty, \infty]$ and $\epsilon \in (0, 1]$. Moreover the functions $w_s w_{s+n}$, $s = 1,\dots, n$ are integrals for 4.3.

Let x_0 be fixed and $v_0(\epsilon)$ be computed from

$$x_0 = \left\{ \sum_{j=0}^{m} (\epsilon^j/j!)\, \Phi^j(0) \right\} v_0(\epsilon) \qquad \text{so} \qquad v_0(0) = x_0 .$$

Let $x(t) = x(t, \epsilon, x_0)$ be the solution of 2.1 satisfying $x(0) = x_0$ and let $v(t) = v[t, \epsilon, v_0(\epsilon)]$ and $w[t, \epsilon, v_0(\epsilon)]$ be the solutions of 4.2 and 4.3 respectively which are equal to $v_0(\epsilon)$ when $t = 0$.

Thus $x(t)$ is carried into $v(t)$ by 4.1 and since $\Phi^0 = I$, $\Phi^j(\pm\infty) = 0$ for $j \geqslant 1$ we have $x(\pm\infty) = v(\pm\infty)$. Represent v in 4.2 by the variation of parameters formula, with $\epsilon^{m+1} D(t, \epsilon) v$ as the inhomogeneous term. Then, by a standard Gronwall inequality estimate we have $v(t) - w(t) = 0(\epsilon^m)$, uniformly for $t \in [-\infty, \infty]$. Thus

$$\begin{aligned} &x_s(\infty)x_{s+n}(\infty) - x_s(-\infty)x_{s+n}(-\infty) \\ &\quad = v_s(\infty)v_{s+n}(\infty) - v_s(-\infty)v_{s+n}(-\infty) \\ &\quad = w_s(\infty)w_{s+n}(\infty) - w_s(-\infty)w_{s+n}(-\infty) + O(\epsilon^m) = O(\epsilon^m). \end{aligned} \tag{4.4}$$

Since m is arbitrary we have shown that $x_s(\infty)x_{s+n}(\infty) - x_s(-\infty)x_{s+n}(-\infty)$ is asymptotic to zero as $\epsilon \to 0^+$ and so $I_s = x_s x_{s+n}$ is an adiabatic invariant for 2.1.

Remarks. (1) An explicit form for the solution w in 4.3 can be obtained readily because the coefficient matrix is diagonal. Since $v(t, \epsilon) - w(t, \epsilon) = O(\epsilon^m)$, we can easily calculate an approximate solution for v to the order of ϵ^m uniformly for $t \in [-\infty, \infty]$.

(2) In this paper $JA(t)$ is assumed to be symmetric. If this assumption is removed, we can still find a formal transformation which diagonalizes the system 2.1. Thus, when we perform a change of variables analogous to 4.1, the system 2.1 is changed to 4.2 with $A^j = \text{diag.}\,(a_1{}^j,\dots, a_n{}^j, a_n{}^j)$ and $D(t, \epsilon)$ satisfying the same properties. The $a_s{}^j(t)$ are gentle, but are not longer pure imaginary for $j \geqslant 1$. $A^0(t)$ is still pure imaginary, therefore the estimates $v(t, \epsilon) - w(t, \epsilon) = O(\epsilon^m)$ remains valid. However, the quantities $w_s w_{s+n}(t, \epsilon)$, $1 \leqslant s \leqslant n$ is now no longer constant for $t \in [-\infty, \infty]$.

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