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A New Class of Periodic Solutions in the Restricted Three Body Problem

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0. INTRODUCTION

We consider a conservative Hamiltonian system of two degrees of freedom near an equilibrium where the linear system consists of two harmonic oscillators with rationally related frequencies. This paper investigates phenomena which were first observed in the planar circular restricted problem of three bodies near the triangular Lagrange equilibrium L_4 [5]. In particular the existence of periodic solutions whose periods are near the common period of the two oscillators is investigated.

Section 1 presents numerical evidence of phenomena which suggests the properties under consideration for analysis. The theoretical analysis is set forth in Sections 2 and 3. The principal results are Theorem 1, 2 and 3 which suggest the application to the restricted problem in Section 4.

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1. DESCRIPTION OF NUMERICAL EXPERIMENTS

The planar restricted problem of three bodies is defined by the Hamiltonian function

$$H(x_1, x_2, y_1, y_2, \mu) = \tfrac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) - (1 - \mu)/\rho_1 - \mu/\rho_2$$

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where $\rho_1^2 = (x_1 + \mu)^2 + x_2^2$, $\rho_2^2 = (x_1 + \mu - 1)^2 + x_2^2$; μ, $0 < \mu < 1$, is the mass ratio of the two primary bodies; x_1, x_2 are cartesian coordinates in a barycentric synodical coordinate system which leaves the two primary bodies of masses μ and $1 - \mu$ fixed on the x_1-axis at $(1 - \mu, 0)$ and $(-\mu, 0)$ respectively and y_1, y_2 are the conjugate momenta. ρ_1 and ρ_2 are the distances from the primaries at $(-\mu, 0)$ and $(1 - \mu, 0)$ respectively to the third body.

The substitutions $x_1 \to x_1 + \frac{1}{2}(1 - 2\mu)$, $x_2 \to x_2 + \frac{1}{2}\sqrt{3}$, $y_1 \to y_1 - \sqrt{3/2}$ and $y_2 \to y_2 + \frac{1}{2}(1 - 2\mu)$ change the origin to the triangular equilibrium point L_4. In this new coordinate system one can expand H as a power series to obtain $H = \sum_{i=2}^{\infty} H_i$, where H_i, $i \geqslant 3$, is a homogeneous polynomial in x_1, x_2, of degree i and

$$H_2 = \tfrac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) + x_1^2/8 - 3(1 - 2\mu)\sqrt{3}\, x_1 x_2/4 - 5x_2^2/8.$$

The linearized equations defined by the system whose Hamiltonian is H_2 alone has the characteristic equation $s^4 + s^2 + 27\mu(1 - \mu)/4 = 0$. If $\mu(1 - \mu) \leqslant 1/27$ the characteristic values are pure imaginary and will be denoted by $\pm i\omega_1$, $\pm i\omega_2$ where $\omega_1 \geqslant \omega_2 > 0$. Let μ_r denote the value of μ for which $\omega_1/\omega_2 = r$.

In the interval $0 < \mu < \mu_1$ except for $\mu = \mu_k$ (k an integer) a theorem of Lyapunov asserts the existence of two one parameter families of periodic solutions that lie on invariant surfaces that pass through the origin. Buchanan [2] has shown that the same is true for $\mu = \mu_1$. These one parameter families are locally parameterized by energy h. To distinguish these families they shall be referred to as the short period family (denoted by subscript 1) and the long period family (denoted by subscript 2).

Recently, Deprit and Henrard [4] have found by numerical experimentation a phenomenon that appears at $\mu = \mu_1$. They found families of periodic solutions which connect the long and short period families and are far from the equilibrium point. The phenomenon can be described as follows. As one proceeds away from the equilibrium along the long period family a critical orbit is reached whose nontrivial characteristic multipliers are qth roots of unity. This will occur for some negative energy h_2. A bifurcation is then observed as follows: for energy h slightly greater than h_2 there exist two periodic orbits that are nearly the same as the critical orbit q-times circuited. One orbit is unstable and the other is of stable type, i.e., its characteristic multipliers lie on the unit circle.

As h is increased one finds two similar periodic orbits for all values of h less than some $h_1 > 0$. The qualitative features of these periodic orbits approach a p-times circuited critical orbit in the short period family.

Each of the families examined by Deprit at $\mu = \mu_1$ contains an orbit which lies in the energy level $h = 0$ of the equilibrium.

The second author has investigated numerically the behavior of these families as μ varies. The periodic solution in each family that has the energy of the equilibrium, $h = 0$, was continued to lower mass ratios from $\mu = \mu_1$. It was found that as μ decreased the periodic orbits of each isoenergetic family gradually collapsed into the equilibrium as μ approached μ_r, $r = p/q$, from above. The following facts were observed as $\mu \to \mu_r^+$: 1. The periods of each isoenergetic family approached $T = 2\pi p/\omega_1 = 2\pi q/\omega_2$ as $\mu \to \mu_r^+$ where ω_1 and ω_2 are evaluated at $\mu = \mu_r$. 2. The nontrivial characteristic multipliers of the periodic orbits of each family approached unity as $\mu \to \mu_r^+$. 3. The projection of each orbit into the x_1, x_2 plane verified that the orbit were shrinking in size.

The use of the isoenergetic family, $h = 0$, obviates the fact that were a collapse to occur it would only occur along this family.

Another phenomenon was investigated by the second author. It is illustrated by the results of numerical experiments performed at $\mu = \mu_r$ and for μ near μ_r, $r = 7/2$. For $\mu > \mu_r$ two families of periodic orbits were found to bifurcate from a critical orbit on the long period family. One of these families contains unstable orbits; the other contains orbits of stable type. An orbit in either family is similar to the critical orbit in the long period family circuited twice. As $\mu \to \mu_r^+$ the critical orbit shrinks to the origin. Thus, at $\mu < \mu_r$, these two families of periodic orbits are found to bifurcate from the family of short period orbits. These orbits are nearly the critical orbit in the short period family circuited seven times. For $\mu \to \mu_r^-$, the critical orbit shrinks into the origin. In the above, the periods of all these new periodic solutions are nearly $14\pi/\omega_1 = 4\pi/\omega_2$ where ω_1 and ω_2 are evaluated at $\mu = \mu_r$, $r = 7/2$.

In contrast to the first phenomenon which was supported in a one sided neighborhood of μ_r the above phenomenon occurs at $\mu = \mu_r$ and bifurcations from Lyapunov families appear in a punctured neighborhood of μ_r.

Data and detailed information on the above numerical experiments can be found in [5].

2. Normalization and Approximate Solutions

Let the Hamiltonian H be real analytic in a neighborhood of an equilibrium point in four dimension phase space and also analytic in a real parameter μ. Let the equilibrium be at the origin in four space for all values of μ under discussion. If for some range of μ the eigenvalues of the linearized system are distinct and pure imaginary then there exists a linear sympletic change of variables such that $H = \sum_2^\infty H_i$ where

$$H_2 = \tfrac{1}{2}\omega_1(z_1^2 + z_3^2) \pm \tfrac{1}{2}\omega_2(z_2^2 + z_4^2)$$

and H_i, $i \geqslant 3$, is a homogenous polynomial of degree i in the new phase variables z_1, z_2, z_3, z_4. $\pm\omega_1 i$ and $\pm\omega_2 i$ are the eigenvalues of the linearized system and are real analytic in μ. We shall consider the case when the minus sign occurs in H_2 and shall assume that $\omega_1 > \omega_2 > 0$. The case when H_2 has a plus sign is treated in a similar manner.

If for some range of values of μ the frequencies satisfy $k_1\omega_1 + k_2\omega_2 \neq 0$ for k_1, k_2 integers and $0 < |k_1| + |k_2| \leqslant 5$ then there exists an analytic canonical transformation $(z_1, z_2, z_3, z_4) \to (Z_1, Z_2, Z_3, Z_4)$ which carries the Hamiltonian into the normalized form

$$\begin{aligned} H(Z_1, Z_2, Z_3, Z_4, \mu) = {} & \omega_1(\mu) I_1 - \omega_2(\mu) I_2 \\ & + \tfrac{1}{2}\{A(\mu) I_1^2 + 2B(\mu) I_1 I_2 + C(\mu) I_2^2\} \\ & + K(Z_1, Z_2, Z_3, Z_4, \mu) \end{aligned} \tag{2.1}$$

where $I_1 = \frac{1}{2}(Z_1^2 + Z_3^2)$, $I_2 = \frac{1}{2}(Z_2^2 + Z_4^2)$. A, B, C are real analytic in μ, K is analytic in all variables and has a convergent power series expansion in some neighborhood of the origin in Z space that begins with terms of degree at least six in the Z_i.

Suppose for some $\mu = \mu_r$, $r = p/q$, that $q\omega_1(\mu_r) = p\omega_2(\mu_r)$ where p and q are relatively prime integers. The question to be discussed here and in section 3 is the existence of periodic solutions whose periods are near

$$T = 2\pi p/\omega_1(\mu_r) = 2\pi q/\omega_2(\mu_r). \tag{2.2}$$

In order to discuss this problem let $\mu - \mu_r > 0$ and make the following substitutions $\epsilon = \mu - \mu_r$, $Z_i \to \sqrt{\epsilon}\, Z_i$ in (2.1). The new Hamiltonian takes the form

$$\begin{aligned} H(Z_1, Z_2, Z_3, Z_4, \epsilon) = {} & \omega_1 I_1 - \omega_2 I_2 \\ & + \epsilon\{\lambda_1 I_1 - \lambda_2 I_2 + A I_1^2/2 + B I_1 I_2 + C I_2^2/2\} \\ & + \epsilon^2 L(Z_1, Z_2, Z_3, Z_4, \epsilon) \end{aligned} \tag{2.3}$$

where $\lambda_i = (d/d\mu)\,\omega_i(\mu_r)$ and ω_1, ω_2, A, B, C are the corresponding functions evaluated at $\mu = \mu_r$.

Before proceeding with the formal analysis, consider the solutions to the first approximation. Assume for the moment that $L \equiv 0$ and change to action-angle variables by $Z_1 = (2I_1)^{1/2}\cos\varphi_1$, $Z_3 = (2I_1)^{1/2}\sin\varphi_1$, $Z_2 = (2I_2)^{1/2}\cos\varphi_2$, $Z_4 = (2I_2)^{1/2}\sin\varphi_2$. To order ϵ, the solutions of (2.3) satisfying $I_i(0) = J_i$ and $\varphi_i(0) = \theta_i$ are

$$\begin{aligned} I_i(t) &= J_i \qquad (i = 1, 2) \\ \varphi_1(t) &= \theta_1 + \omega_1 t + \epsilon\{\lambda_1 + A J_1 + B J_2\} t \\ \varphi_2(t) &= \theta_2 - \omega_2 t + \epsilon\{-\lambda_2 + B J_1 + C J_2\} t. \end{aligned} \tag{2.4}$$

To this order of approximation the short period family lies on the surface $J_1 > 0$, $J_2 = 0$ and the long period family lies on the surface $J_2 > 0$, $J_1 = 0$. Solutions (2.4) are periodic of period $T + \epsilon\beta$ provided

$$\begin{aligned} \omega_1\beta + T(\lambda_1 + AJ_1 + BJ_2) &= 0 \\ -\omega_2\beta + T(-\lambda_2 + BJ_1 + CJ_2) &= 0 \end{aligned} \tag{2.5}$$

where $T = 2\pi p/\omega_1 = 2\pi q/\omega_2$ and $J_1, J_2 \geqslant 0$.

With β as parameter, equations (2.5) give a parametric representation of a straight line in the plane (J_1, J_2) if $AC - B^2 \neq 0$. Energy h is given by $h = \omega_1 J_1 - \omega_2 J_2$ to the first approximation and so represents a straight line also. These two lines will have a unique point of intersection at J_{10}, J_{20} if

$$D = \{A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2\} \neq 0 \tag{2.6}$$

and the corresponding value of β is given by

$$\beta = -TD^{-1}\{A\omega_2\lambda_2 + B(\omega_1\lambda_2 + \omega_2\lambda_1) + C\omega_1\lambda_1 + (AC - B^2)h\}. \tag{2.7}$$

If $J_{10} > 0$ and $J_{20} > 0$ then to the first approximation the solutions of (2.3) with $I_1(0) = J_{10}$ and $I_2(0) = J_{20}$ are periodic of period $T + \epsilon\beta$. Since the two angles θ_1 and θ_2 are arbitrary these solutions fill a torus in the energy level h.

The intercepts of the straight line (2.5) are

$$M_1 = \frac{\omega_1\lambda_2 - \omega_2\lambda_1}{A\omega_2 + B\omega_1} \quad \text{and} \quad M_2 = \frac{\omega_1\lambda_2 - \omega_2\lambda_1}{B\omega_2 + C\omega_1} \tag{2.8}$$

and the corresponding values of h are

$$h_1 = \omega_1 M_1 \quad \text{and} \quad h_2 = -\omega_2 M_2 . \tag{2.9}$$

If M_1 and M_2 are positive then the energy line will intersect the line (2.5) in the first quadrant for all h, $h_2 < h < h_1$. In this case the first approximation of (2.3) has a one parameter family of tori each filled with $T + \epsilon\beta$ periodic solutions for $h_2 < h < h_1$. As $h \to h_1$ these tori approach a periodic solution on the short period family and as $h \to h_2$ these tori approach a periodic solution on the long period family. In other words, one has a family of tori filled with periodic solutions connecting the long and short period families. For the system (2.1) one has the same phenomena except the family collapses into the origin as $\mu \to \mu_r^+$, $r = p/q$.

If M_1 and M_2 are both negative the same result follows by defining $\epsilon = \mu_r - \mu$.

If $M_1 < 0$ and $M_2 > 0$ then the line (2.5) and the energy line have a point of intersection in the first quadrant for $h < h_2$ if $\omega_1 M_1 + \omega_2 M_2 < 0$ and for $h > h_2$ if $\omega_1 M_1 + \omega_2 M_2 > 0$. Thus to the first approximation (2.3) has a one parameter family of tori each filled with $T + \epsilon\beta$ periodic solutions for all $h < h_2$. As $h \to h_2$ these tori approach an orbit on the long period family.

If $M_1 > 0$ and $M_2 < 0$ we have a similar bifurcation off the short period family.

If $M_1 < 0$ and $M_2 > 0$ the system (2.1) will have a family of tori bifurcating off the long period family if $\mu > \mu_r$ and a family bifurcating off the short period family for $\mu < \mu_r$. The bifurcation orbit on the long (short) period family tends to the origin as $\mu \to \mu_r^+$ ($\mu \to \mu_r^-$).

In the case when M_1 and M_2 have different signs the families described above persist at $\mu = \mu_r$. Assume that $K \equiv 0$ and $\mu = \mu_r$ in system (2.1). Let $\epsilon > 0$, make the substitution $Z_i \to \epsilon Z_i$ and change to action angle variables. Then the approximate solutions are

$$\begin{aligned} I_i(t) &= J_i \qquad (i = 1, 2) \\ \varphi_1(t) &= \theta_1 + \omega_1 t + \epsilon\{AJ_1 + BJ_2\}t \\ \varphi_2(t) &= \theta_2 - \omega_2 t + \epsilon\{BJ_1 + CJ_2\}t \end{aligned} \tag{2.10}$$

These solutions are $T + \epsilon\beta$ periodic if

$$\begin{aligned} \omega_1\beta + T\{AJ_1 + BJ_2\} &= 0 \\ -\omega_2\beta + T\{BJ_1 + CJ_2\} &= 0. \end{aligned} \tag{2.11}$$

Under the assumption that M_1 and M_2 are of different sign and $D \neq 0$ the equations (2.11), when β is eliminated, represent a straight line in J_1, J_2 plane with positive slope through the origin which intersects the energy line in a single point. The value of β is

$$\beta = TD^{-1}\{B^2 - AC\}h \tag{2.12}$$

Thus to the first approximation system (2.1) has a one parameter, h, family of tori filled with $T + \epsilon\beta$ periodic solutions that approaches the origin at $\mu = \mu_r$ as $h \to 0$.

3. Main Theorems

Lemma 1. *Let p and q be relatively prime integers, $p > 2$, and d and g constants, $dg < 0$. Let $P(x, y; \epsilon)$ be analytic in all variables for $x^2 + y^2 < -3dg^{-1}$, $|\epsilon| < \epsilon_0$ and be an area preserving diffeomorphism for each fixed ϵ.*

Let P have the form $P = F + \epsilon^2 G$ where

$$F : (r, \theta) \to (r, \theta + 2\pi q/p + \epsilon\{d + gr\}) \qquad r = x^2 + y^2, \qquad \theta = \tan^{-1} y/x \tag{3.1}$$

Then (1) *there exist an $\epsilon_1 > 0$ and an analytic function $k(\epsilon) = (k_1(\epsilon), k_2(\epsilon))$, $|\epsilon| \leqslant \epsilon_1$, such that $k(0) = (0, 0)$ and $(x, y) = k(\epsilon)$ is the unique fixed point of P^j for $x^2 + y^2 \leqslant -2dg^{-1}$, $|\epsilon| \leqslant \epsilon_1$, $1 \leqslant j < p$.*

(2) *there exists a closed analytic curve S_ϵ in the x, y plane which is analytic in ϵ, diffeomorphic to a circle and contains $k(\epsilon)$ in its interior for $|\epsilon| \leqslant \epsilon_1$ such that $S_0 = \{(r, \theta) : r = -dg^{-1}, \theta \text{ arbitrary}\}$ and either* (a) *P^p leaves S_ϵ fixed or* (b) *P^p has ℓ fixed points on S_ϵ, $\infty > \ell \geqslant 2p$. In case* (b) *at least p points of P^p have index $+1$ and at least p points of P^p have index -1.*

(3) *P^p has no other fixed points than those described in* (1) *and* (2) *for $x^2 + y^2 \leqslant 2dg^{-1}$, $0 < |\epsilon| \leqslant \epsilon_1$.*

Proof. By (3.1) the Jacobian of P with respect to x and y at $x = y = \epsilon = 0$ has eigenvalues $\exp(\pm 2\pi qi/p)$. Thus the Jacobian of P^j, $0 < j < p$, at $x = y = \epsilon = 0$ does not have an eigen value of plus one and part (1) follows from the implicit function theorem.

Change variables by $x \to x - k_1(\epsilon)$, $y \to y - k_2(\epsilon)$ so that the form of P is unchanged except now $G(0, 0, \epsilon) = 0$. Let $P^p : (r, \theta) \to (R, \Theta)$ and consider $Q(r, \theta, \epsilon) = \epsilon^{-1}\{\Theta(r, \theta, \epsilon) - \theta\} = p\{d + gr\} + \epsilon K(r, \theta, \epsilon)$, where K is analytic in all variables provided $r \neq 0$. Since $g \neq 0$ the implicit function theorem yields the existence of a function $\zeta(\theta, \epsilon)$ with $\zeta(\theta, 0) = -dg^{-1}$ and $Q(\zeta(\theta, \epsilon), \theta, \epsilon) \equiv 0$. The curve S_ϵ is given by $S_\epsilon = \{(r, \theta) : r = \zeta(\theta, \epsilon), \theta$ arbitrary$\}$. Since P^p is area preserving and leaves the origin fixed, $P^p(S_\epsilon) \cap S_\epsilon \neq \varnothing$. Each point of $P^p(S_\epsilon) \cap S_\epsilon$ is a fixed point of P^p. Since $P^p(S_\epsilon)$ and S_ϵ are defined by analytic functions either (a) they coincide or (b) they intersect at a finite number of points. In case (b) the same argument as found in [1] pp. 215–18 gives at least one fixed point of index $+1$ and one fixed point of index -1. There are no other fixed points of P^p since $\Theta(r, \theta, \epsilon) - \theta$ is not zero off of S_ϵ for ϵ small.

THEOREM 1. *Let μ_r, $r = p/q$, be such that $q\omega_1(\mu_r) = p\omega_2(\mu_r)$ where p and q are relatively prime integers $p > q \geqslant 1$, $p > 3$. Let $T = 2\pi p/\omega_1 = 2\pi q/\omega_2$ and $\gamma = TD^{-1}\{B^2 - AC\}$. Assume $A\omega_2 + B\omega_1 < 0$ and $B\omega_2 + C\omega_1 > 0$ and $D > 0$ (resp. $D < 0$). Then there exist an $h_0 > 0$ and a neighborhood N of the origin such that for each $h \in (0, h_0)$ (resp. $h \in (-h_0, 0)$) the system whose Hamiltonian is* (2.1) *with $\mu = \mu_r$ has either* (a) *a torus filled with periodic solutions or* (b) *ℓ periodic solutions, $\infty > \ell \geqslant 2$ in the level surface $H = h$.*

These periodic solutions have period $T + h\gamma + 0(h)$ and tend to the origin as $h \to 0$. Moreover there are no other periodic solutions with period in a neighborhood of T in N.

Proof. Change variables in (2.1) by $Z_i \to \sqrt{\epsilon} Z_i$ where $\epsilon > 0$ so that

$$H(Z_1, Z_2, Z_3, Z_4, \epsilon) = \omega_1 I_1 - \omega_2 I_2 + \frac{\epsilon}{2}\{AI_1^2 + 2BI_1I_2 + CI_2^2\} + \epsilon^2 L(Z_1, Z_2, Z_3, Z_4, \epsilon). \tag{3.2}$$

Consider the approximate equation obtained from (3.2) by letting $L \equiv 0$ (call this system (3.3)). If time is restricted to a compact set the solutions of (3.2) and (3.3) differ by terms that are $0(\epsilon^2)$. In (3.3) change to action angle variables so that (2.10) are the solutions of (3.3). Let $D > 0$ and $h > 0$ be fixed. In the energy level h consider the transversal cross section $\varphi_1 \equiv 0$ for systems (3.2) and (3.3). The cross section map P for (3.2) differs from the cross section map F for (3.3) by terms that are $0(\epsilon^2)$ so in light of Lemma 1 it is sufficient to compute F. The time required for F is $2\pi/\omega_1 + \epsilon\xi + 0(\epsilon^2)$ where

$$\xi = -2\pi\{AJ_1 + BJ_2\}/\omega_1^2 \tag{3.2}$$

Which is easily computed from the component φ_1 in (2.10). Eliminating J_1 by using the relation $h = \omega_1 J_1 - \omega_2 J_2 + 0(\epsilon)$ one finds that

$$F : (J_2, \theta_2) \to (J_2 + 0(\epsilon^2), \theta_2 - 2\pi q/p + \epsilon\{d + gJ_2\} + 0(\epsilon^2))$$

where $d = T\{A\omega_2 + B\omega_1\}\, h/\omega_1^2$, $g = TD/\omega_1^2 p$

Under the assumptions of the theorem $dg < 0$ and so Lemma 1 can be applied to yield the existence of fixed points of P^p. Thus the existence of the periodic solutions is assured. The fact that in case (b) of the lemma one fixed points has index $+1$ and one has index -1 guarantees that there are at least two distinct periodic solutions. The fixed points lie on S_ϵ which has J_2 coordinate given by $J_2 = -dg^{-1} + 0(\epsilon)$. Substituting this and $h = \omega_1 J_1 - \omega_2 J_2 + 0(\epsilon)$ in (3.4) yields the desired formula for the period.

LEMMA 2. *Let $P(x, y)$ be an area preserving diffeomorphism of a neighborhood of the origin into itself with the origin fixed. Let the Jacobian of P at $x = y = 0$ $x = y = 0$ have eigenvalues* $\exp \pm 2\pi \ell i/k$, *k and ℓ are relatively prime integers $k \geqslant 3$. Then there exists a symplectic change of variables $(x, y) \to (u, v)$ such that*

$$P : (r, \theta) \to (R, \Theta)$$

where

$$r = u^2 + v^2, \qquad \theta = \tan^{-1} u/v$$
$$R = r + 2b(\sin k\theta)\, r^{k/2} + R_1(\rho, \theta)$$
$$\Theta = \theta + \psi + ar + b(\cos k\theta)\, r^{(k-2)/2} + \Theta_1(\rho, \theta)$$

a and b constants, $\psi = 2\ell\pi/k$

R_1 and Θ_1 are analytic in $r^{1/2}$ and θ, 2π periodic in θ

$$\Theta_1 = 0(\rho^{3/2}), \quad R_1 = 0(\rho^2), \quad \rho = r^{1/2}$$

Remark. Observe that if $k > 4$ the terms in $r^{k/2}$ and $r^{k-2/2}$ can be absorbed in R_1 and Θ_1 respectively. The constants a and b are invariants and do not depend on the particular choice of transformation that reduces P to the above form. If P depends on a parameter ϵ such that P at $\epsilon = 0$ satisfies the above conditions then the transformation can be constructed in the same way except now a, b, ψ, R_1 and Θ_1 depend on ϵ.

The proof of this lemma is just a minor modification of the Birkhoff normalization theorem.

LEMMA 3. *Let p and q be relatively prime integers, $p \geqslant 4$, and let d, e and g be nonzero constants. Let $P(x, y; \epsilon, h)$ be analytic in all variables, $x^2 + y^2 \leqslant \alpha$, $\alpha > 0$, $|\epsilon| < \epsilon_0$, $\epsilon_0 > 0$, $|h + de^{-1}| < \beta$, $\beta > 0$. For fixed ϵ and h let P be an area preserving diffeomorphism. Let P have the form $P = F + \epsilon^2 G$ where*

$$F : (r, \theta) \to (r, \theta + 2\pi q/p + \epsilon\{d + eh + gr\})$$
$$r = x^2 + y^2, \qquad \theta = \tan^{-1} y/x$$

Then (1) *there exist $\epsilon_1 > 0$, $\beta_1 > 0$ and an analytic function $k(\epsilon, h)$, $|\epsilon| \leqslant \epsilon_1$, $|h + de^{-1}| \leqslant \beta_1$ such that $k(0, h) = 0$ and $(x, y) = k(\epsilon, h)$ is the unique fixed point of P^j for $x^2 + y^2 \leqslant \alpha/2$ and $1 \leqslant j < p$.*

(2) *there exists an analytic function $h(\epsilon)$, $|\epsilon| \leqslant \epsilon_1$, such that $h(0) = -de^{-1}$ and the eigenvalues of the Jacobian of P at $k(\epsilon, h(\epsilon))$ are $\exp(\pm 2\pi qi/p)$.*

(3) *For $h > h(\epsilon)$ when $eg < 0$ (or $h < h(\epsilon)$ when $eg > 0$) there exists an analytic curve $S_{\epsilon,h}$ in the x, y plane such that $S_{\epsilon,h}$ is diffeomorphic to a cricle, $k(\epsilon, h(\epsilon))$ lies in the interior of $S_{\epsilon,h}$, $S_{0,h} = \{r = -(d + eh)/g\}$ and $S_{\epsilon,h} \to k(\epsilon, h(\epsilon))$ as $h \to h(\epsilon)$.*

(4) *P^p either leaves $S_{\epsilon,h}$ fixed or there are ℓ fixed points, $\infty > \ell \geqslant 2p$, of P^p on $S_{\epsilon,h}$. In the second case at least p points have index $+1$ and at least p fixed points have index -1.*

(5) *P^p has no other fixed points than those described above for $x^2 + y^2 \leqslant \alpha/2$, $|\epsilon| \leqslant \epsilon_1$, $|h + de^{-1}| \leqslant \beta_1$.*

Proof. Part 1 is proven the same way as part 1 of Lemma 1. Change variables by $x \to x - k_1(\epsilon, h)$ and $y \to y - k_2(\epsilon, h)$ so that the form of P is unchanged except now $G(0, 0; \epsilon, h) \equiv 0$. Let $P : (r, \theta) \to (R, \Theta)$. Observe that the eigenvalues of P at $x = y = 0$ are $\exp(\pm\eta i)$ where $\eta = \Theta(0, \theta; \epsilon, h) - \theta$. (Note $\Theta(0, \theta; \epsilon, h)$ is independent of θ). Since $e \neq 0$ there exists, by the implicit function theorem, a function $h(\epsilon)$ such that $h(0) = -d/e$ and $\Theta(0, \theta, \epsilon, h(\epsilon)) - \theta = 2\pi q/p$. Let $\zeta = h - h(\epsilon)$. Apply Lemma 2 to P so that P has the form $P : (r, \theta) \to (R, \Theta)$ and

$$\Theta(r, \theta, \epsilon, \zeta) = \theta + 2\pi q/p + \epsilon\{e\zeta + gr\} + \epsilon^2\Theta_2(r, \theta, \epsilon, \zeta)$$

where Θ_2 is analytic in $r^{1/2}$, θ, ϵ, ζ and $\Theta_2 = 0(r)$.

Thus $\partial\Theta/\partial r$ is continuous at $r = 0$. Now let $P^p : (r, \theta) \to (R', \Theta')$ where $\Theta'(r, \theta, \epsilon, \zeta) = \theta + p\epsilon\{e\zeta + gr\} + \epsilon^2\Theta_2'(r, \theta, \epsilon, \zeta)$ and Θ_2' has the same properties as Θ_2. Since $g \neq 0$ the implicit function theorem yields the existence of a function $\rho(\theta, \epsilon, \zeta) = -e\zeta/g + 0(\epsilon)$ such that

$$\Theta'(\rho(\theta, \epsilon, \zeta), \theta, \epsilon, \zeta) - \theta \equiv 0.$$

Moreover ρ is C^1 and analytic for $\zeta \neq 0$. The curve $S_{\epsilon,h}$ is defined as the set $\{(r, \theta) : \theta$ arbitrary and $r = \rho(\theta, \epsilon, \zeta)\}$.

The rest of the proof proceeds in the same manner as the proof of Lemma 1.

THEOREM 2. *Let μ_r, $r = p/q$, be such that $q\omega_1(\mu_r) = p\omega_2(\mu_r)$ where p and q are relatively prime integers $p > q > 3$. Let $T = 2\pi p/\omega_1 = 2\pi q/\omega_2$, $\epsilon = \mu - \mu_r$ and assume $M_1 < 0$, $M_2 > 0$ and $D \neq 0$. Then there exist a neighborhood N of the origin, and $\epsilon_0 > 0$ and continuous functions $h_1(\cdot)$ and $h_2(\cdot)$ such that*

(a) $h_2(\cdot) : (0, \epsilon_0) \to (-\infty, 0)$, $h_1(\cdot) : (-\epsilon_0, 0) \to (0, \infty)$ *and* $h_1(\epsilon) = \epsilon h_i + 0(\epsilon)$ *as* $\epsilon \to 0$.

(b) *There exists a unique orbit $\Gamma_1(\epsilon)$ for $\epsilon \in (-\epsilon_0, 0)$ (resp. $\Gamma_2(\epsilon)$ for $\epsilon \in (0, \epsilon_0)$) in N on the short (long) period family with non trivial characteristic multipliers that are* pth *roots* (qth *roots*) *of unity. $\Gamma_i(\epsilon)$ tends to the origin as $\epsilon \to 0$. The value of energy for $\Gamma_i(\epsilon)$ is $h_i(\epsilon)$ for system* (2.1).

(c) *For fixed ϵ, $0 < \epsilon < \epsilon_0$ (resp. $-\epsilon_0 < \epsilon < 0$) there exists an $\eta > 0$ such that for each $h \in (h_2(\epsilon), h_2(\epsilon) + \eta)$ or $h \in (h_1(\epsilon) - \eta, h_1(\epsilon))$ the system whose Hamiltonian is* (2.1) *has either a torus filled with periodic solutions or ℓ, $\infty > \ell \geqslant 2$, periodic solutions with energy h in N.*

(d) *The periods of these periodic solutions are given by*

$$T + \epsilon\delta + h\gamma + 0(\epsilon, h)$$

where

$$\delta = -TD^{-1}\{A\omega_2\lambda_2 + B(\omega_1\lambda_2 + \omega_2\lambda_1) + C\omega_1\lambda_1\} \quad \textit{and} \quad \gamma = TD^{-1}\{B^2 - AC\}$$

(e) *There exist no other periodic solutions in N with least period in a neighborhood of T.*

(f) *For ϵ fixed, $0 < \epsilon < \epsilon_0$, (resp. $-\epsilon_0 < \epsilon < 0$) these periodic solutions as given in* (c) *tend to $\Gamma_2(\epsilon)$ (resp. $\Gamma_1(\epsilon)$) as $h \to h_2(\epsilon)^+$ (resp. $h \to h_1(\epsilon)^-$).*

Proof. Change coordinates so that (2.3) results from (2.1). The solutions of (2.3) and the approximate equation obtained from (2.3) by letting $L \equiv 0$ differ by terms $0(\epsilon^2)$ provided time is restricted to a compact set. As in Theorem 1 we compute the section map for the approximate equations given in this case by the formulas (2.4) and then apply Lemma 3.

THEOREM 3. *Let μ_r, $r = p/q$, be such that $q\omega_1(\mu_r) = p\omega_2(\mu_r)$ where p and q are relatively prime integers $p > q > 3$. Let $\epsilon = \mu - \mu_r > 0$ and $T = 2\pi p/\omega_1(\mu_r) = 2\pi q/\omega_2(\mu_r)$. Assume $M_i > 0$ $(i = 1, 2)$ and let h_1 and h_2 be defined by* (2.9). *Then there exist a neighborhood N of the origin, an $\epsilon_0 > 0$, and continuous functions $h_1(\cdot)$, $h_2(\cdot)$ such that:*

(a) $h_1(\cdot) : (0, \epsilon_0) \to (0, \infty)$, $h_2(\cdot) : (0, \epsilon_0) \to (-\infty, 0)$ *and* $h_i(\epsilon) = \epsilon h_i + 0(\epsilon)$ *as* $\epsilon \to 0^+$.

(b) *The system which is defined by the Hamiltonian* (2.1) *has either a torus filled with periodic solutions or has ℓ, $\infty > \ell \geqslant 2$, perioric solutions in N for each ϵ, $0 < \epsilon < \epsilon_0$ and for each value of energy h in $(h_2(\epsilon), h_1(\epsilon))$.*

(c) *As $\epsilon \to 0^+$ these periodic solutions tend to the origin.*

(d) *The periods of these periodic solutions are given by*

$$T + \epsilon\delta + h\gamma + 0(\epsilon, h)$$

where

$$\delta = -TD^{-1}\{A\omega_2\lambda_2 + B(\omega_1\lambda_2 + \omega_2\lambda_1) + C\omega_1\lambda_1\}$$

and

$$\gamma = TD^{-1}\{B^2 - AC\}$$

(e) *There exist no other periodic solutions in N with least period in a neighborhood of T.*

(f) *There exist a unique orbit $\Gamma_1(\epsilon)$ (resp. $\Gamma_2(\epsilon)$) in N on the short (long) period family with non trivial characteristic multipliers that are* pth *roots* (qth *roots*) *of unity. $\Gamma_i(\epsilon)$ tends to the origin as $\epsilon \to 0^+$. The value of energy for $\Gamma_i(\epsilon)$ is $h_i(\epsilon)$.*

(g) *For ϵ fixed, $0 < \epsilon < \epsilon_0$, these periodic solution as given by* (b) *tend to $\Gamma_1(\epsilon)$ (resp. $\Gamma_2(\epsilon)$) as $h \to h_1(\epsilon)^-$ (resp. $h \to h_2(\epsilon)^+$).*

The proof of this theorem is similar to the proof of Theorems 1 and 2.

4. APPLICATIONS TO THE LAGRANGE EQUILIBRIUM L_4 IN RESTRICTED PROBLEM

In this section the restricted problem is examined to determine the values of the mass ratio for which the theorems of section 3 apply. It is only necessary to check the signs and zeros of the function

$$\omega_1\lambda_2 - \omega_2\lambda_1, \quad A\omega_2 + B\omega_1, \quad B\omega_2 + C\omega_1 \quad \text{and} \quad A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2.$$

The Hamiltonian function for the restricted problem has been normalized through terms of fourth order by Deprit and Deprit [3] for $0 < \mu < \mu_2$; $\mu_2 < \mu < \mu_1$. The coefficients A, B, C of sections 2 and 3 are

$$A = \frac{\omega_2^2(81 - 696\omega_1^2 + 124\omega_1^4)}{72(1 - 2\omega_1^2)^2(1 - 5\omega_1^2)}$$
$$B = -\frac{\omega_1\omega_2(43 + 64\omega_1^2\omega_2^2)}{6(4\omega_1^2\omega_2^2 - 1)(25\omega_1^2\omega_2^2 - 4)} \tag{4.1}$$

and

$$C(\omega_1, \omega_2) = A(\omega_2, \omega_1)$$

The function $D = A\omega_2^2 + 2B\omega_1\omega_2 + C\omega_1^2$ is given by

$$D = \frac{36 - 541\omega_1^2\omega_2^2 + 644\omega_1^4\omega_2^4}{8(4\omega_1^2\omega_2^2 - 1)(25\omega_1^2\omega_2^2 - 4)} \tag{4.2}$$

It is found that $\omega_1\lambda_2 - \omega_2\lambda_1 > 0$ for all μ in the interval $0 < \mu < \mu_1$ by using the relations $\omega_1/\omega_2 = -\lambda_2/\lambda_1$ and noting that $\lambda_1 < 0$ and $\lambda_2 > 0$ hold throughout this interval. By direct evaluation using (4.1) one finds that

$$\begin{array}{ll} A\omega_2 + B\omega_1 > 0 & \mu_2 < \mu < \mu_1 \\ A\omega_2 + B\omega_1 < 0 & 0 < \mu < \mu_2 \\ B\omega_2 + C\omega_1 > 0 & \mu_2 < \mu < \mu_1, \quad 0 < \mu < \mu_d \\ B\omega_2 + C\omega_1 < 0 & \mu_d < \mu < \mu_2 \end{array}$$

where μ_d is the unique number defined by $B\omega_2 + C\omega_1 = 0$, $0 < \mu_d < \mu_1$ μ_d has the approximate value $\mu_d = 0.0127$.

The function D has a unique zero at $\mu = \mu_c$ in $0 < \mu < \mu_2$ and no zero in $\mu_2 < \mu < \mu_1 \cdot \mu_c$ has the approximate value $\mu_c = 0.0109$.

The ordering of these endpoints of intervals is given by

$$0 < \mu_4 < \mu_c < \mu_d < \mu_3 < \mu_2 < \mu_1$$

Specifically one has

(a) Theorem 3 applies to the restricted problem for all μ_r, $r = p/q$, $p > q > 3$, in the intervals $\mu_2 < \mu_r < \mu_1$ and $\mu_d < \mu_r < \mu_2$. For the interval $\mu_2 < \mu_r < \mu_1$ the periodic solutions appear for $\mu > \mu_r$ and for the interval $\mu_d < \mu_r < \mu_2$ the periodic solutions appear for $\mu < \mu_r$.

(b) Theorems 1 and 2 apply to the restricted problem for all μ_r, $r = p/q$, $p > q > 3$, in the interval $0 < \mu < \mu_d$. For $\mu < \mu_r$, the bifurcating family terminates in the family of short period orbits and for $\mu > \mu_r$ the bifurcating family terminates in the family of long period orbits.

One can verify that neither μ_d nor μ_c is such that $q\omega_1 = p\omega_2$ where p and q are relatively prime integers, $p > q \geqslant 1$.

Therefore, for any μ_r, $r = p/q$, $r > q > 3$, *in the restricted problem either Theorems* 1 *and* 2 *apply or Theorem* 3 *applies.*

Note added in proof. At several points in this paper the phrase "analytic in ϵ" should be changed to "analytic in $\epsilon^{1/2}$."

References

1. George D. Birkhoff, *Dynamical Systems*, Amer. Math. Soc. Coll. Publ., IX (1927), pp. 165–169.
2. Daniel Buchanan, "Trojan Satellites-Limiting Case," *Transactions Royal Society of Canada*, Vol. 31 (1941), Series III, pp. 9–25.
3. André Deprit and Andrée Deprit-Bartholomé, "Stability of the Triangular Lagrangian Points," *Astronomical Journal*, Vol. **72** (1967), pp. 173–179.
4. André Deprit and Jacques Henrard, "A Manifold of Periodic Orbits," in *Advances in Astronomy and Astrophysics*, Vol. **6** (1968), Academic Press, New York.
5. Julian Palmore, "Bridges and Natural Centers in the Restricted Three Body Problem," University of Minnesota Report, Control Science Center, (1968).