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On the Convergence of the Zeta Function for Flows and Diffeomorphisms

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I. INTRODUCTION

Let M be a compact differentiable manifold and φ a differentiable mapping of M into itself. In the study of the orbit structure of φ the periodic points, fixed points of some power of φ, are of paramount importance. In [*1*] Artin and Mazur introduce an invariant in the form of a formal power series that measures the isolated periodic points of φ. Since this formal series is analogous to the classical zeta function they call their new formal power series the zeta function for the mapping φ. Let N_ν be the number of isolated fixed points of φ^ν where ν is a positive integer, then the zeta function $\zeta(s, \varphi)$ or $\zeta(s)$ for φ is given by

$$\zeta(s) = \exp\left\{\sum_{\nu=1}^{\infty} \frac{N_\nu t}{\nu}\right\} \tag{1}$$

Let $\mathscr{J}^r$, $r \geqslant 1$, be the set of all C^r mappings of M into itself with the usual C^r topology. In [*1*] Artin and Mazur show that there is a dense set in $\mathscr{J}^r$ such that $N_\nu \leqslant K^\nu$ where K is some constant depending on φ but not on ν. Hence there is a dense set in $\mathscr{J}^r$ such that the zeta function has a nonzero radius of convergence.

The theorem of Artin and Mazur does not indicate the nature of the set of mappings for which this estimate holds since their method of proof is based on algebraic approximation techniques as developed by Nash. In [*2*] the author announced that for a certain general class of diffeomorphisms recently introduced by Smale [*3*, *4*] the above estimate holds. With a closer look at the proof in [*2*] one sees that the theorem is true for mappings also.

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This new class is the most general class presently known that contains all known examples of diffeomorphisms with global stability properties (see [4] for discussion).

In order to define this class of mappings we shall need the following definitions: A point $p \in M$ is called a wandering point of φ if there exists a neighborhood U of p such that $U \cap \varphi^\nu(U) = \varnothing$ for all positive integers ν. A point is a nonwandering point if it is not a wandering point. Let Ω be the set of all nonwandering points of φ. Then Ω is a compact subset of M that contains all the periodic points of φ. In general Ω is only a semi-invariant set, i.e., $\Omega \supset \varphi(\Omega)$.

Let M be endowed with some Riemannian metric structure with norm in the tangent space denoted by $\|\cdot\|$. Let Λ be some semi-invariant set under φ. Then Λ is said to have a hyperbolic structure (U-structure in some references) on Λ if for each point $p \in \Lambda$ there is a splitting of the tangent space at p, T_p, into a direct sum $T_p = E_p{}^u \oplus E_p{}^s$ such that the splitting is a continuous function of p with

$$D\varphi : E_p{}^u \to E^u_{\varphi(p)} \; ; \; D\varphi : E_p{}^s \to E^u_{\varphi(p)}$$

and

$$\| D\varphi^\nu(u)\| \geqslant C\lambda^\nu \| u \| \qquad (u \in E_p{}^u)$$
$$\| D\varphi^\nu(v)\| \leqslant C^{-1}\lambda^{-\nu} \| v \| \qquad (v \in E_p{}^s)$$

for all $p \in \Lambda$ and all positive integers ν where C and λ are constant, $\lambda > 1$.

Throughout differentiable or smooth maybe taken as C^2. The first theorem establishes here is:

THEOREM 1. *Let φ be a smooth mapping of M into itself with a hyperbolic structure on its set of nonwandering points. Then there exists a constant K depending only on φ such that $N_\nu \leqslant K^\nu$ for all positive integers ν.*

COROLLARY 1. *Let φ be as in Theorem 1. Then the zeta function for φ has a nonzero radius of convergence.*

It has been conjectured that the set of mappings φ that have a hyperbolic structure on the set of nonwandering points is dense in $\mathscr{J}^r$. If this were the case, then Theorem 1 would include the theorem of Artin and Mazur, but at present the two theorems are distinct.

One has a similar set of concepts for differential equations or flows on M. Let $\{\varphi_t\}_{t\in R}$ be the one parameter group of diffeomorphisms defined by a smooth vector field X on M. In order to count the periodic solutions of X Smale [4] has introduced the following candidate for a zeta function for $\{\varphi_t\}$:

$$\Xi(s) = \prod_{\gamma\in\Gamma} \prod_{k=0}^{\infty} \{1 - [\exp \tau(\gamma)]^{-s-k}$$

where Γ is the set of all closed orbits of φ_t excluding singular points and $\tau(\gamma)$ is the minimal period of γ. Of course, this is a formal product since one is not even sure that there are a countable number of periodic orbits in general. However, we shall show that for a certain general class of flows the zeta function is a well-defined analytic function in some right half plane. This class is, of course, the class analogous to the class of Theorem 1.

A point $p \in M$ is called a wandering point if there is some neighborhood U of p and some real number $t_0 > 0$ such that

$$\left\{ \bigcup_{|t| > t_0} \varphi_t(U) \right\} \cap U = \phi.$$

Also a nonwandering point is a point that is not a wandering point. Let Ω be the set of nonwandering points of $\{\varphi_t\}$. Clearly Ω is a compact invariant subset of M.

Let Λ be any invariant set for $\{\varphi_t\}$ then Λ is said to have a hyperbolic structure on the set Λ provided there is a continuous splitting of the tangent spaces of M on Λ, $T_p = E_p{}^u \oplus E_p{}^s \oplus E_p{}^0$ such that

$$D\varphi_t : E_p{}^u \to E^u_{\varphi_t(p)}$$

$$D\varphi_t : E_p{}^s \to E^s_{\varphi_t(p)}$$

and $E_p{}^0$ is the subspace generated by the velocity vector of $\{\varphi_t\}$ at p. Moreover, there must exist constants $C > 0$, $\mu > 0$ such that

$$\| D\varphi_t(u)\| \geqslant Ce^{\mu t} \| u \| \qquad (u \in E_p{}^u)$$

and

$$\| D\varphi_t(v)\| \leqslant C^{-1}e^{-\mu t} \| v \| \qquad (v \in E_p{}^s)$$

for all $p \in \Lambda$.

Again the class of flows introduced by Smale in [4] is the class of all flows with hyperbolic structure on the set of nonwandering points. For flows the theorem corresponding to Theorem 1 is

THEOREM 2. *Let $\{\varphi_t\}$ be the one parameter group of diffeomorphisms generated by a smooth vector field X on M such that $\{\varphi_t\}$ has a hyperbolic structure on the set of nonwandering points. Let N_τ be the number of periodic orbits of $\{\varphi_t\}$ of period less than or equal to τ (singular points excepted). Then there exist constants H and η depending only on $\{\varphi_t\}$ and not on τ such that*

$$N_\tau \leqslant He^{\eta\tau} \text{ for } \tau \geqslant 0$$

COROLLARY 2. *Let $\{\varphi_t\}$ be as in Theorem 2, then the zeta function for $\{\varphi_t\}$ has a right half plane of uniform absolute convergence.*

The proofs of Theorems 1 and 2 are almost the same, yet neither theorem contains the other. Theorem 2 does contain Theorem 1 for diffeomorphisms by using the suspension theorem found in [4]. The basic idea of the proofs is very simple but the necessity to check uniformity at each step has lengthened the arguments considerably.

The first step is to reduce the problem to one of counting the fixed points of a mapping from Euclidean space into itself. For mappings this is done by using a finite number of coordinate systems and for flows by using a finite number of partial cross sections. The next step is to show that the Jacobians at the periodic points grow at most exponentially. This estimate on the Jacobian along with a sharp form of the implicit function theorem allows one to estimate the size of the domain upon which there is a unique fixed point. From this last estimate the theorem follows easily.

II. Proof of Theorem 1

In what follows let $|\cdot|$ denote the usual Euclidean norm in R^m with respect to a fixed basis and also the corresponding matrix norm. The following lemma is a direct result of the implicit function theorem with estimate on the domain of validity as found in Hartman [5] page 12. This lemma is central to the proof of both Theorems 1 and 2.

Lemma 1. *Let a, b and c be fixed positive constants and n a positive integer. Let ψ_n be a C^2 map from the closed ball B of radius a^n about the origin in R^m into R^m with $\psi_n(0) = 0$. Let the modulus of the first and second partial derivatives of ψ_n be less than b^n on B. Let $|(D\psi_n(0) - I)| \leqslant c^n$ and $|(D\psi_n(0) - I)^{-1}| \leqslant c^n$ where $D\psi_n(0)$ denotes the Jacobian matrix of ψ_n at the origin and I is the identity matrix. Then there exists a constant d depending only on the dimension m and the constants a, b and c but independent of n such that ψ_n has a unique fixed point in the closed sphere of radius d^n about the origin.*

Proof. Consider the function $g(x) = \psi_n(x) - x$. The fixed points of ψ_n correspond to the zeros of g. Let $A = D\psi_n(0) - I$ and $h(x) = g(x) - Ax$ and so $g(x) = Ax + h(x)$ with $h(0) = Dh(0) = 0$.

Since first and second partials of ψ_n are bounded by b^n there exists a constant k_1 depending only on the dimension m and b such that

$$|Dg(x)| \leqslant k_1{}^n \text{ and } |Dh(x)| \leqslant k_1{}^n |x|$$

for $|x| \leqslant a^n$. Let $k_2 = \min\{a, (2ck_1)^{-1}\}$. Then for $|x| \leqslant k_2{}^n$ one has

$$|Dg(x)^{-1}| = |A^{-1}| \, |(I + A^{-1}Dh(x))^{-1}| \leqslant \frac{c^n}{1 - |A^{-1}Dh|} \leqslant (2c)^n.$$

Now the implicit function theorem cited above can be applied to g to show that g has a unique zero in the sphere of radius d^n where $d = k_2(2k_1c)^{-1}$.

Let (V_i, y_i) and (U_i, x_i), $i = 1,\ldots, r$ be a finite number of coordinate systems for M sucn tnat $V_i \supset \bar{U}_i$, $\cup U_i \supset M$, $x_i = y_i \mid U_i$ and $\overline{y_i(V_i)}$ is compact in R^m. Consider the sets $y_i(V_i)$ and $x_i(U_i)$ in R^m. There exists a constant δ, $0 < \delta < 1$, such that each point of $x_i(\bar{U}_i)$ is contained in a closed sphere of radius δ that is completely contained in $y_i(V_i)$ for $i = 1,\ldots, r$. We shall count the number of fixed points of φ^n in each $x_i(\bar{U}_i)$. Let $\|\cdot\|$ denote the norm in the tangent space of $y_i(V_i)$ induced by the Riemannian metric of M.

Lemma 2. *Let x_0 be a fixed point of $\psi_n = x_i \cdot \varphi^n \cdot x^{-1}$ and let $A = D\psi_n(x_0)$ then there exist constants N and $c > 0$ that are independent of x_0 and n such that $|(A - I)| \leqslant c^n$ and $|(A - I)^{-1}| \leqslant c^n$ for all $n \geqslant N$.*

Proof. Choose coordinates at x_0 so that A has the form $A = \operatorname{diag}(A_1, A_2)$ where A_1 and A_2 are the matrix representations of $D\psi_n(x_0)$ on the spaces $E^u_{x_0}$ and $E^s_{x_0}$ respectively. One has a norm $|||\cdot|||$ in R^m induced by these coordinates. Since the set of nonwandering points is compact and the splitting of the tangent spaces is continuous there exist constants k_3 and $k_4 > 1$ which are independent of n and x_0 such that the matrix norms satisfy

$$k_3^{-1} |||\cdot||| \leqslant \|\cdot\| \leqslant k_3 |||\cdot|||,$$

and

$$k_4^{-1} |||\cdot||| \leqslant |\cdot| \leqslant k_4 |||\cdot|||.$$

It is clear that there exists a constant k_5 independent of x_0 and n such that $|A| \leqslant \leqslant k_5{}^n$ and so $|(A - I)| \leqslant k_5{}^n + |I| \leqslant (k_5 + |I|)^n$.

Now let N be such that $C^{-1}\lambda^{-N} < 1$. Then

$$\begin{aligned}
|(A - I)^{-1}| &\leqslant k_4 |||(A - I)^{-1}||| \leqslant k_4 \{|||(A_1 - I)^{-1}||| + |||(A_2 - I)^{-1}|||\} \\
&\leqslant k_3k_4\{\|(A_1 - I)^{-1}\| + \|(A_2 - I)^{-1}\|\} \\
&\leqslant k_3k_4 \left\{\frac{C^{-1}\lambda^{-n}}{1 - C^{-1}\lambda^{-n}} + \frac{1}{1 - C^{-1}\lambda^{-n}}\right\} \\
&\leqslant \left\{\frac{2k_3k_4}{1 - C^{-1}\lambda^{-N}}\right\}^n
\end{aligned}$$

Let $c = \max\{k_5 + |I|, 2k_3k_4(1 - C^{-1}\lambda^{-N})^{-1}\}$.

The constant a^n maybe taken as δ in the case under consideration.

Hence we have established the existence of the constants a, b and c of Lemma 1 and so the existence of d if $n > N$.

Let L be the total volumn of all the $\overline{y_i(V_i)}$ in R^m. The fixed points of φ^n in $x_i(\bar{U}_i)$ can be covered by disjoint balls of radius $d^n/3$. And since by our construction $d < \delta < 1$ these balls are interior to one of the $\overline{y_i(V_i)}$. From elementary calculus there is a constant k_7 depending only on the dimension m such that the volumn of one of these balls is greater than or equal to $k_7(d^n/3)^m$. Hence for $n > N$

$$N_n k_7 \left(\frac{d^n}{3}\right)^m \leqslant L$$

from which it follows that $N_n \leqslant K^n$ where $K = (1 + Lk_7^{-1})3^m d^{-m}$. A standard argument hields the theorem for all $n \geqslant 0$.

III. The Proof of Theorem 2

The proof of Theorem 2 is essentially the same as the proof of Theorem 1. In order to apply Lemmas 1 and 2 to the case of a flow one most make a reduction to a problem related to mappings and this is done by choosing a finite number of cross sections. These mappings are not defined on the intersection of a cross section and the stable or unstable manifold of a saddle point. Therefore, one must obtain an estimate on the rate of growth of the derivatives of these maps near these intersections. This new estimate is the only major difference between Theorems 1 and 2. (see Lemma 3 below).

Cover all the singular points of X with coordinate systems (U_i, x_i) and (V_i, y_i), $i = 1,..., r$ as follows. Let (W_i, z_i) be some coordinate system that contains a singular point and is such that z_i takes the local stable and local unstable manifolds into the coordinate planes $R^p \times \{0\}$ and $\{0\} \times R^q$ respectively in $R^p \times R^q = R^m$ (see [5] Chapter IX). Moreover, assume that $z_i(W_i)$ is an open ball of radius one about the origin in R^m.

Henceforth we shall omit the index i until the two neighborhoods are constructed. Take spheres S_1 and S_2 in $R^p \times \{0\}$ and $\{0\} \times R^q$ respectively of radius 3/4. Consider the cylinders

$$Z_1 = \{(\xi, \eta) \in R^p \times R^q : (\xi, 0) \in S_1\} \cap z(W)$$

and

$$Z_2 = \{(\xi, \eta) \in R^p \times R^q : (0, \eta) \in S_2\} \cap z(W).$$

If we choose $\mathscr{E}$ sufficiently small all trajectories of X that start on Z_1 with $|\eta| \leqslant \mathscr{E}$ at $t = 0$ intersect Z_2 once and only once before leaving $z_i(W_i)$. Put $P = Z_1 \cap \{(\xi, \eta) | |\eta| \leqslant \mathscr{E}\}$. Let Q be the set of all points Z_2 that are images of points of P under the flow plus S_2. Let $x(U)$ be the closure of the union of orbits joining P to Q and $U = z^{-1}(x(U))$ and $x = z | U$. Let V

and y be constructed in the same way only using $\mathscr{E}/2$. Let the sets corresponding to P and Q be denoted by $\tilde{P}$ and $\tilde{Q}$. (Henceforth indexes will be replaced).

Cover $M - \bigcup_{i=1}^{r} V_i$ with a finite number of closed flow boxes (see [6]) (U_i, x_i) and (V_i, y_i) $i = r + 1,..., s$ such that $U_i^0 \supset V_i$, $y_i = x_i \mid U_i$ and such that the interiors of the V_i cover $M - \bigcup_{i=1}^{r} V_i^0$. In each flow box (U_i, x_i) pick a cross section $P_i \subset x_i(U_i)$ and let $\tilde{P}_i = P_i \cap y_i(V_i)$. Although not every orbit of X crosses one of these cross sections $\tilde{P}_i$; $i = 1,..., r,..., s$; every periodic orbit does.

The vector field X in $z_i(W_i) \subset R^p \times R^q$ is given by the ordinary differential equations

$$\dot{\xi} = A\xi + G(\xi, \eta)$$
$$\dot{\eta} = B\eta + H(\xi, \eta)$$

where ξ and η are p and q vectors respectively, A and B are $p \times p$ and $q \times q$ matrices respectively and G and H are C^2 functions defined on $z_i(W_i) \subset R^p \times R^q$ and map into R^p and R^q respectively. The eigenvalues of $A(B)$ have negative (positive) real parts. Also $G(0, y) = 0$, $H(x, 0) = 0$, $DG(0, 0) = 0$ and $DH(0, 0) = 0$.

Let ψ be the map that sends a point $\zeta \in P$ to the point on Q where the trajectory of X through ζ meets Q. ψ is well defined on all of P except on S_1. We need to estimate the size of the derivative of ψ near S_1. Let $T(\xi_0, \eta_0)$ be the time required for trajectory passing through $(\xi_0, \eta_0) \in P$ to meet Q.

LEMMA 3. *Let $(\xi_0, \eta_0) \in P$, $\eta_0 \neq 0$. There exist the constants f and g such that the modulus of the first partial derivatives of ψ are bounded by $f^{T(\xi_0,\eta_0)}$ on the subset of P defined by*

$$\{(\xi, \eta) | \, |(\xi_0, \eta_0) - (\xi, \eta)| \leq g^{T(\xi_0,\eta_0)}\} \cap P.$$

Remark. This lemma asserts that there is a neighborhood of (ξ_0, η_0) that decreases at most exponentially with T upon which the derivatives of ψ increase at most exponentially with T.

Proof. Since all the eigenvalues of B have positive real part there exist K_1 and $K_2 > 0$ such that $\| e^{-Bt} \| \leq K_1 e^{-K_2 t}$ for $t \geq 0$. Let us reverse time by taking $\tau = -t$. Then $\eta' = -B\eta - H(\xi, \eta)$ where prime represents derivative with respect to τ.

Consider the solution $(\xi(\tau), \eta(\tau))$ that goes through a point $(\xi_1, \eta_1) \in Q$ at $\tau = 0$ and intersects P at (ξ_0, η_0). Then

$$\eta(\tau) = e^{-B\tau}\eta_1 - \int_0^{\tau} e^{-B(\tau-s)} H(\xi(s), \eta(s)) \, ds.$$

Without loss in generality we can assume that the neighborhood $z_i(W_i)$ is sufficiently small that $|H(\xi, \eta)| \leqslant (K_2)(2K_1)^{-1} |\eta|$ and then

$$|\eta(\tau)| \leqslant K_1 e^{-K_2\tau} |\eta_1| + \frac{K_2}{2} \int_0^\tau e^{-K_2(\tau-s)} |\eta(s)| \, ds,$$

and by Granwalls inequality

$$|\eta(\tau)| \leqslant K_2 |\eta_1| e^{-K_2\tau/2},$$

and letting $\tau = T(\xi_0, \eta_0)$ we have

$$|\eta_0| \leqslant K_2 |\eta_1| \exp\left[-\frac{K_2}{2} T(\xi_0, \eta_0)\right].$$

Thus if we take a neighborhood N of radius $\frac{1}{2}|\eta_0|$ about (ξ_0, η_0) in P then the maximum time $T(N)$ for any trajectory through this neighborhood to reach Q is bounded by the relation $|\eta_0|(2|\eta_1|)^{-1} \leqslant \exp\{-\frac{1}{2} K_2 T(N)\}$. Thus the spherical neighborhood N of (ξ_0, η_0) on P has a radius bounded below by $f^{T(\xi_0 \eta_0)}$ where f is some positive constant.

The Jacobian of ψ satisfies the first variational equation which is a linear homogeneous differential equation with bounded coefficients. It is well known that the solutions of such an equation grow at most exponentially with time. Thus the lemma follows.

With this lemma the proof of Theorem 2 follows in the same way as the proof of Theorem 1. One considers the fixed points of the mappings of the partial cross sections $\tilde{P}_i$, $i = 1,..., r,..., s$ into themselves defined by the flow.

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