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Asymptotic stability estimates near an equilibrium point

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Abstract

We use the error bounds for adiabatic invariants found in the work of Chartier, Murua and Sanz-Serna [3] to bound the solutions of a Hamiltonian system near an equilibrium over exponentially long times. Our estimates depend only on the linearized system and not on the higher order terms as in KAM theory, nor do we require any steepness or convexity conditions as in Nekhoroshev theory. We require that the equilibrium point where our estimate applies satisfy a type of formal stability called Lie stability.

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1. Introduction

In our previous work [15] we started a search for stability results around an equilibrium point that depend only on the quadratic part of the Hamiltonian, i.e., only on the linearized system. We do not search for stability criteria that depend on the higher order terms as in KAM theory, or on the steepness and convexity conditions found in Nekhoroshev theory. The only complete result of

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this type is Dirichlet's Theorem [4], which gives stability of the equilibrium when the quadratic part is positive (or negative) definite. Stability cannot be determined from the eigenvalues of the linearized system alone as shown by the classical example of Cherry – see [14]. This gave rise to many different formal results, as discussed in [15,21,22] and the references therein.

Here we take a different approach. We establish bounds for exponentially long times on the actual solutions near an equilibrium of an analytic Hamiltonian system following the tradition found in [7,8] – see Lochak [12,13]. First we use the theory developed by dos Santos and coworkers [21,22] on Lie stable systems to prepare the quadratic part of the Hamiltonian to obtain enough proper formal adiabatic invariants. Next the error bounds found in Chartier, Murua and Sanz-Serna [3] are applied to the adiabatic invariants. And finally, a straightforward Liapunov type argument transfers the estimates from the adiabatic invariants to the actual solutions.

Other approaches to get estimates for elliptic equilibria of Hamiltonian systems may be found in [19,6,17,20]. Similarly to Lochak [12,13], the authors of the previous papers apply Nekhoroshev theory and obtain sharp results on stability over exponentially long times. In their papers they require convexity of the Hamiltonian (or related conditions), hence there are no resonances of order less than 5, but they do not assume any Diophantine conditions among the frequencies. Moreover they do not assume any type of formal stability. Our approach is different, since we provide estimates for elliptic equilibria that are formally stable and the kind of stability required is characterized by the quadratic terms of the Hamiltonian. We also require a Diophantine hypothesis among some of the frequencies of the linearized equations of motion.

The present paper has six sections. In Section 2 we state our main theorem and give an example. In Section 3 we deal with the calculation of formal invariants for a resonant Hamiltonian using a normal form approach. In addition we present two propositions which characterize Hamiltonians that are Lie stable by checking only the linearized equation, relating the concept of Lie stability to the existence of a linear combination of the formal integrals for the normal form Hamiltonian. Since our theory relies on the estimates for adiabatic invariants of Chartier et al., in Section 4 we present these authors' result and its connection with our approach. In Section 5 we give the proof of our main theorem. Finally, in Section 6 we relate Lie stability to normal stability and apply the ideas of the previous sections to the spatial case of the circular restricted three body problem.

2. The system

We consider a real analytic Hamiltonian defined in $\mathcal{N}$, a neighborhood of the origin in $\mathbb{R}^{2n}$, of the form

$$\mathcal{H}(x) = \mathbb{H}(x) + \mathcal{K}(x)\,, \tag{1}$$

whose equations of motion are the Hamiltonian system

$$\dot{x} = \mathcal{J}\nabla\mathcal{H}(x)\,, \tag{2}$$

where $\mathcal{J}$ is the standard $2n \times 2n$ symplectic matrix of Hamiltonian theory [14].

In (1) above, $\mathbb{H}$ is the quadratic Hamiltonian

$$\mathbb{H}(x) = \tfrac{1}{2}x^T Sx\,, \tag{3}$$

with $S = S^T$ a $2n \times 2n$ real symmetric matrix. Since $\mathcal{H}(x)$ is analytic, the function $\mathcal{K}(x)$ is a convergent expansion in x that we require to start with terms of degree three.

Depending on context, we denote the general solution of Hamilton's equations by $x(t)$ or $x(t, x_0)$, where x_0 is the initial condition. The linearized equations of motion are

$$\dot{x} = Ax, \quad A = \mathcal{J}S, \tag{4}$$

where A is a $2n \times 2n$ real Hamiltonian matrix. Throughout this paper, we assume that A is nonsingular and that the linearized system is stable, i.e., all the eigenvalues of A are nonzero purely imaginary numbers and A is diagonalizable over the complex numbers. If A is singular then there are nonlinearities that lead to instability. In the nonsingular case, possibly after making a suitable linear symplectic transformation to bring the linear part to diagonal form, one can introduce action-angle variables

$$J_j = \tfrac{1}{2}(u_j^2 + v_j^2), \quad \phi_j = \tan^{-1}\frac{v_j}{u_j}, \quad x = (u_1, \dots, u_n, v_1, \dots, v_n),$$

such that $\mathbb{H}$ takes the form

$$\mathbb{H} = \mu_1 J_1 + \cdots + \mu_n J_n, \tag{5}$$

where $\pm\mu_1 i, \dots, \pm\mu_n i$ are the eigenvalues of A. (Note that we keep the same name for the variable x, and also for A, $\mathbb{H}$ and $\mathcal{H}$.) In [7,8] strong nonresonance conditions are imposed on the eigenvalues which we do not require.

Let $\bar{\mathcal{H}}$ be the normal form of $\mathcal{H}$ defined in (1), i.e., $\bar{\mathcal{H}}$ is a function

$$\bar{\mathcal{H}} = \mathbb{H} + \bar{\mathcal{K}}_3 + \cdots + \bar{\mathcal{K}}_N + \cdots$$

obtained from $\mathcal{H}$ through a symplectic change of coordinates whose series expansion in x starts at degree two, such that each term $\bar{\mathcal{K}}_j$ is a homogeneous polynomial of degree j, and satisfies $\{\bar{\mathcal{H}}, \mathbb{H}\} = 0$, see [14].

Our first assumption is that we can arrange terms so that

$$\mathbb{H} = \sigma_1 Q_1 + \cdots + \sigma_d Q_d, \tag{6}$$

where all the Q_i are formal integrals of $\bar{\mathcal{H}}$ and semidefinite quadratic forms in x and the σ_i are real. The rearrangement process is discussed in Section 3.

Second, we need to impose a *Diophantine condition*[1] on the vector $\sigma = (\sigma_1, \dots, \sigma_d)$; that is, we suppose that there are fixed constants $c > 0$ and $\nu > d - 1$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\}, \quad |k \cdot \sigma| \geq c|k|^{-\nu}. \tag{7}$$

Roughly speaking our main result of the paper is the following.

[1] Analogous (1-dimensional) number theoretic conditions were first used in a dynamical systems setting by C.L. Siegel [24,5].

If the real analytic Hamiltonian (1) has $\mathbb{H}$ of the form (6) as discussed above and the frequency vector σ satisfies the Diophantine condition (7), then there exist $C > 0$, $K > 0$, $a > 1$ and $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$, and for all x_0 with $|x_0| < \rho$ we have

$$|x(t, x_0)| < a\rho \quad \text{for all} \quad 0 \le t \le T = C\,\rho \exp\left(\frac{K}{\rho^{1/(2(\nu+1))}}\right).$$

A more precise statement is given in Theorem 5.1 in Section 5. Notice that in case $\mathbb{H}$ is definite then a classical theorem of Liapunov assures stability.

In Section 3 we treat the following five-degree-of-freedom example

$$\begin{aligned}\mathbb{H} &= 5(\sqrt{5}-1)J_1 + 2(\sqrt{5}-1)J_2 + (\sqrt{5}-1)J_3 - 18J_4 + 18(\sqrt{5}+1)J_5\,. \\ &= \mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3 + \mu_4 J_4 + \mu_5 J_5\,.\end{aligned} \tag{8}$$

Note that $\mathbb{H}$ is an indefinite quadratic form in x since $\mu_4 < 0$, while the other μ_i are positive and there are many relations among the eigenvalues. But after the rearrangement

$$\mathbb{H} = (\sqrt{5}+1)Q_1 - 2Q_2 = \sigma_1 Q_1 + \sigma_2 Q_2$$

where

$$Q_1 = 5J_1 + 2J_2 + J_3 + 18J_5\,, \quad \text{and} \quad Q_2 = 5J_1 + 2J_2 + J_3 + 9J_4\,.$$

Clearly Q_1, Q_2 are positive definite and $(\sigma_1, \sigma_2) = (\sqrt{5}+1, -2)$. Now $|\sigma_1/\sigma_2| = (\sqrt{5}+1)/2$ is the golden mean. It is well known that the golden mean and its equivalents are the irrational numbers that are most badly approximated by rational numbers and thus the Diophantine condition is satisfied.

Therefore if the analytic Hamiltonian $\mathcal{H}$ starts with $\mathbb{H}$ as above then the solutions satisfy the estimates given above.

3. Formal invariants

In this section we use the ideas found in dos Santos et al. [21] to prepare the quadratic part of the Hamiltonian and construct formal invariants. They were interested in a type of formal stability called Lie stability and we refer to their paper for references on various types of formal stability. Concretely, the Hamiltonian $\mathcal{H}$ expanded as

$$\mathcal{H} = \mathbb{H} + \mathcal{K}_3 + \cdots + \mathcal{K}_N + \cdots,$$

where $\mathcal{K}_k$ represents a homogeneous polynomial of degree k in x, is said to be *Lie stable* if there exists an integer $m > 2$ such that the normal form Hamiltonian obtained from it, i.e.

$$\bar{\mathcal{H}}^N = \mathbb{H} + \bar{\mathcal{K}}_3 + \cdots + \bar{\mathcal{K}}_N \quad \text{with} \quad \{\bar{\mathcal{H}}^N, \mathbb{H}\} = 0\,,$$

is stable in the sense of Liapunov for any $N \ge m$, where N is an arbitrary integer. The Hamiltonian $\bar{\mathcal{H}}^N$ is supposed to be truncated at degree N, thus it is obtained from $\mathcal{H}$ through a finite sequence of symplectic transformations.

Following [22], we introduce the $\mathbb{Z}$-module associated with the frequencies μ_i of (5), which is given by

$$\mathcal{M}_\mu = \{k = (k_1, \dots, k_n) \in \mathbb{Z}^n \,\big|\, k \cdot \mu = k_1\mu_1 + \cdots + k_n\mu_n = 0\}.$$

The set $\mathcal{M}_\mu$ is finitely generated, so there exist $k^1, \dots, k^s \in \mathcal{M}_\mu$ such that

$$\mathcal{M}_\mu = k^1\,\mathbb{Z} + \cdots + k^s\,\mathbb{Z} = \{j_1k^1 + \cdots + j_sk^s \,\big|\, j_1, \dots, j_s \in \mathbb{Z}\}.$$

Take a minimal set of generators, so $0 \le s < n$ and the k^j are linearly independent. The case $s = 0$ corresponds to the situation where all the μ_i are independent over the rationals. When $s = 1$ one says that the Hamiltonian $\mathcal{H}$ has a single resonance, whereas the case of multiple resonances corresponds to $s > 1$.

After transforming the Hamiltonian $\mathcal{H}$ to normal form up to degree N, we know that $\bar{\mathcal{H}}^N$ has $n - s$ integrals in involution that are linear combinations of the actions J_i; see Lemma 1 in [22]. They are calculated as follows. One determines the null space of $\{k^1, \dots, k^s\}$, leading to a vector subspace of $\mathbb{R}^n$ spanned by vectors $\{a^1, \dots, a^{n-s}\}$ that satisfy $a^i \cdot k^j = 0$. Then we set $F_l = a^l \cdot J$ with $J = (J_1, \dots, J_n)$. The F_l are independent because the a^l are linearly independent. We may choose the coefficients of the vectors a^l to be integers.

It is possible to arrange $\mathbb{H}$ so that it has the form

$$\mathbb{H} = \xi_1 F_1 + \cdots + \xi_{n-s} F_{n-s}\,, \tag{9}$$

where the ξ_l are linear combinations of the μ_i with the condition that $\xi = (\xi_1, \dots, \xi_{n-s})$ is a nonresonant frequency vector, a feature that is always guaranteed by the construction of the F_i from the set $\mathcal{M}_\mu$. Since ξ corresponds to the vector $(\omega_1, \dots, \omega_d)$ of Chartier et al., the d in [2,3] corresponds to the $n - s$ of dos Santos and Vidal [22]. From now on we use d instead of $n - s$.

The auxiliary set

$$S = \{J = (J_1, \dots, J_n) \,\big|\, F_1(J) = \cdots = F_d(J) = 0\} \tag{10}$$

is introduced in [22] in order to prove Lie stability under some additional hypotheses. In particular, if $S = \{J = 0\}$ one defines the positive definite first integral $W = F_1^2 + \cdots + F_d^2$, which in turn is a Liapunov function for $\bar{\mathcal{H}}^N$, since $W > 0$ except at the origin of $\mathbb{R}^{2n}$, and $\dot{W} = 0$. Thus one proves Liapunov stability for $\bar{\mathcal{H}}^N$ or, in other words, Lie stability for $\mathcal{H}$ (this is Proposition 2 of [22]).

We are ready to state some properties regarding the set S defined in (10).

Proposition 3.1. *Given the Hamiltonian system (2) with $\mathbb{H}$ as in (9), the following statements are equivalent:*

(i) *The set $S = \{J = 0\}$.*
(ii) *There is a linear combination of the d formal integrals F_l for the normal form Hamiltonian $\bar{\mathcal{H}}$ related to $\mathcal{H}$ such that it is a positive definite quadratic form in x.*
(iii) *The Hamiltonian $\mathbb{H}$ can be written as*

$$\mathbb{H} = \sigma_1 Q_1 + \cdots + \sigma_d Q_d\,, \tag{11}$$

where all the Q_l are nonnull formal integrals of $\bar{\mathcal{H}}$ and positive semidefinite quadratic forms in x. The frequency vector $\sigma = (\sigma_1, \dots, \sigma_d)$ is nonresonant.

To prove 3.1, we need to apply a theorem we call the Gordan–Stiemke Alternative [9,25]. Suppose A is a $p \times m$ matrix and for $z \in \mathbb{R}^k$ write $z > 0$ when $z_j > 0$ for each j, and $z \geq 0$ when $z_j \geq 0$ for each j.

Theorem 3.1. *(Gordan–Stiemke Alternative) Exactly one of the following systems has a solution:*

(a) $x^T A > 0$ *for some* $x \in \mathbb{R}^p$,
(b) $A\, y = 0$ *and* $y \geq 0$ *for some nonzero* $y \in \mathbb{R}^m$.

Moreover, when the entries of A are rational numbers, we may choose the vectors x and y to be rational also [23].

Obviously when A has integer entries, x and y may be chosen as integers. We now prove Proposition 3.1.

Proof. (i)$\Longrightarrow$(ii). Let $\mathcal{A}$ be the $d \times n$ matrix whose rows are the vectors $a^l = (a^l_1, \dots, a^l_n)$. We rewrite the system $F_1(J) = \cdots = F_d(J) = 0$ as $\mathcal{A}\, J = 0$, and since (i) holds, its unique solution is $J = 0$. Applying the Gordan–Stiemke Alternative, there is a vector $p = (p_1, \dots, p_d)$ such that $p^T \mathcal{A} > 0$ (option (a)). This implies that the combination

$$\begin{aligned} G(J) &= p_1 F_1 + \cdots + p_d F_d \\ &= p_1(a^1_1 J_1 + \cdots + a^1_n J_n) + \cdots + p_d(a^d_1 J_1 + \cdots + a^d_n J_n) \\ &= (p_1 a^1_1 + \cdots + p_d a^d_1) J_1 + \cdots + (p_1 a^1_n + \cdots + p_d a^d_n) J_n \end{aligned}$$

is an integral of the normal form $\bar{\mathcal{H}}^N$, which is a positive definite function in the coordinates x, since $q_i = p_1 a^1_i + \cdots + p_d a^d_i > 0$ for $i = 1, \dots, n$. Note that $q_i \in \mathbb{Z}^+$ since p_i and a^j_i are integers.

(ii)$\Longrightarrow$(iii). We write $m^1 = (q_1, \dots, q_n)$, where q_i are the positive numbers obtained in (i) above, thus $m^1 = p_1 a^1 + \cdots + p_d a^d$. Without loss of generality, we suppose $p_1 \neq 0$ and replace the basis $\{a^1, \dots, a^d\}$ by $\{m^1, a^2, \dots, a^d\}$, since it is also a basis of the null space of $\{k^1, \dots, k^s\}$. Now, the equation $\alpha_1 m^1 + \alpha_2 a^2 + \cdots + \alpha_d a^d = 0$ becomes

$$\alpha_1 p_1 a^1 + (\alpha_2 + \alpha_1 p_2) a^2 + \cdots + (\alpha_d + \alpha_1 p_d) a^d = 0$$

and since $p_1 \neq 0$, then $\alpha_1 = 0$, so $\alpha_2 = \cdots = \alpha_d = 0$. For $l = 2, \dots, d$, we introduce m^l by $m^l = r_l m^1 + a^l$ where r_l is a positive integer such that all components of m^l are nonnegative. Now $\beta_1 m^1 + \beta_2 m^2 + \cdots + \beta_d m^d = 0$ is rearranged as

$$(\beta_1 + \beta_2 r_2 + \cdots + \beta_d r_d) m^1 + \beta_2 a^2 + \cdots + \beta_d a^d = 0\,.$$

Since $\{m^1, a^2, \dots, a^d\}$ is a basis, we have $\beta_2 = \cdots = \beta_d = 0$ and, finally, $\beta_1 = 0$, so the set $\{m^1, m^2, \dots, m^d\}$ spans the null space of $\{k^1, \dots, k^s\}$. Now define $Q_l = m^l \cdot J$ for $l = 1, \dots, d$. Then since $m^l_j \geq 0$, the Q_l are positive semidefinite quadratic forms in x and are formal integrals of $\bar{\mathcal{H}}$ because the Q_l are linear combinations of the F_k. The σ_l are readily obtained from the ξ_k in (9) and, by construction, the frequency vector $(\sigma_1, \dots, \sigma_d)$ is nonresonant.

(iii)$\Longrightarrow$(i). We take the Q_l as linear combinations of the actions J_k with positive parameters, if necessary after performing a suitable linear symplectic change of variables. Next, the set S is constructed from the functions Q_l as

$$S = \{J = (J_1, \dots, J_n) \,\big|\, Q_1(J) = \dots = Q_d(J) = 0\},$$

but since the Q_l are positive semidefinite and J cannot take negative values, the unique solution to the system above is $S = \{J = 0\}$. □

Another result concerning the set S is the following.

Proposition 3.2. *If the set S related to the Hamiltonian $\mathbb{H}$ given in (9) contains nonzero vectors, then there is a Hamiltonian $\mathcal{H} = \mathbb{H} + \mathcal{K}$, where $\mathcal{K}(x)$ is a convergent expansion in x starting at degree three and the origin of $\mathbb{R}^{2n}$ is unstable.*

Proof. By applying the Gordan–Stiemke Alternative, there is no function $G(J)$ as in item (i) of Proposition 3.1, but there is a vector of nonnegative integers $r = (r_1, \dots, r_n)$ such that $\mathcal{A}r = 0$ and $F_1(r) = \dots = F_d(r) = 0$. (Note that $r \in S$.) Hence r is in the orthogonal complement of the space spanned by $a^1, \dots, a^d$, so $r \in \mathcal{M}_\mu$ with $r \cdot \mu = 0$ and $r \neq 0$. Now we prove that the origin of $\mathbb{R}^{2n}$ is unstable for a certain Hamiltonian $\mathcal{H}$. We choose

$$\mathcal{H}(J, \phi) = \mathbb{H}(J, \phi) + \mathcal{K}(J, \phi) = \mu_1 J_1 + \dots + \mu_n J_n + J_1^{r_1/2} \cdots J_n^{r_n/2} \cos(r_1\phi_1 + \dots + r_n\phi_n)$$

and consider the function

$$\mathcal{C}(J, \phi) = -2J_1^{r_1/2} \cdots J_n^{r_n/2} \sin(r_1\phi_1 + \dots + r_n\phi_n).$$

In terms of x, the function $\mathcal{C}$ is a homogeneous polynomial of degree $r_1 + \dots + r_n$. We have

$$\dot{\mathcal{C}} = J_1^{r_1-1} \cdots J_n^{r_n-1}(r_1^2 J_2 \cdots J_n + r_2^2 J_1 J_3 \cdots J_n + \dots + r_{n-1}^2 J_1 \cdots J_{n-2} J_n + r_n^2 J_1 \cdots J_{n-1}).$$

Let Ω be the region where $\mathcal{C} > 0$, so $J_1 \neq 0, \dots, J_n \neq 0$. But $\dot{\mathcal{C}} > 0$ in Ω and Ω has points arbitrarily close to the origin, so Chetaev's Theorem [14] shows that the origin is unstable. □

According to Propositions 3.1 and 3.2, we may view Proposition 3.1 as a characterization of Lie stability for the case where it is established using only the linearized system around the equilibrium. Specifically, we have proved that a Hamiltonian system for which $S \neq \{J = 0\}$ can lead to instability. However this does not mean that any Hamiltonian system (2) with $\mathbb{H}$ as in (9) and such that $S \neq \{J = 0\}$ is unstable. For example, the origin of $\mathbb{R}^6$ is stable for the Hamiltonian

$$\mathcal{H} = 2\sqrt{5}J_1 + 4\pi J_2 - (\pi + \sqrt{5})J_3 + J_1^4$$

because

$$\mathbb{L} = 2\sqrt{5}J_1 + 4\pi J_2 + (\pi + \sqrt{5})J_3$$

is a Liapunov function for $\mathcal{H}$, although the set S of the quadratic terms of $\mathcal{H}$ is given by $\{(2J_3, J_3, 4J_3) \mid J_3 \geq 0\}$. It is also possible to choose the higher order terms $\mathcal{K}$ so that one gets Lie stability; see Theorem 1.1 of [22].

We now reconsider Hamiltonian (8) and compute the $\mathbb{Z}$-module $\mathcal{M}_\mu$, solving the equation $k \cdot \mu = 0$ with $k = (k_1, \dots, k_5)$, $\mu = (\mu_1, \dots, \mu_5)$ for the integers k_j. We set $s = 3$, $d = 2$ and obtain the linearly independent vectors $k^1 = (0, 0, -18, 2, 1)$, $k^2 = (1, 0, -5, 0, 0)$, $k^3 = (0, 1, -2, 0, 0)$.

The first integrals are obtained from the null space of $\{k^1, k^2, k^3\}$. Concretely, a basis for the null space is given by the vectors a^1, a^2 with $a^1 = (5, 2, 1, 0, 18)$, $a^2 = (5, 2, 1, 9, 0)$. Therefore two positive semi-definite formal integrals of a normalized Hamiltonian, $\bar{\mathcal{H}} = \mathbb{H} + \bar{\mathcal{K}}$, with $\bar{\mathcal{K}}$ an arbitrary perturbation, are

$$Q_1 = a^1 \cdot J = 5J_1 + 2J_2 + J_3 + 18J_5\,, \qquad Q_2 = a^2 \cdot J = 5J_1 + 2J_2 + J_3 + 9J_4\,.$$

By Proposition 3.1 one gets $S = \{J = 0\}$, thus the origin in $\mathbb{R}^{10}$ is Lie stable. Moreover

$$\mathbb{H} = (\sqrt{5} + 1)Q_1 - 2Q_2\,.$$

The function $G = (\sqrt{5} + 1)Q_1 + 2Q_2$ is a positive definite integral of the truncated normal form $\bar{\mathcal{H}}^N$. Also $(\sigma_1, \sigma_2) = (\sqrt{5} + 1, -2)$ and $|\sigma_1/\sigma_2| = (\sqrt{5} + 1)/2$. As a vector in $\mathbb{R}^2$, $\omega = (\sqrt{5} + 1, -2)$ is not only Diophantine, but belongs to the class of "most Diophantine" vectors in the sense that, for every $\nu > 1$, there is a $c > 0$ such that for all $k \in \mathbb{Z}^2\backslash\{0\}$, $|k \cdot \omega| > c|k|^{-\nu}$. (Other vectors in $\mathbb{R}^2$ may satisfy a Diophantine condition only for larger values of ν.) For discussions and proofs of these facts, see e.g. [10] or [11].

4. Adiabatic invariants

In references [1–3] and [15] the authors construct d formal integrals in involution for the Hamiltonian (1). Although conceptually the same, they are constructed by different algorithms, so there may be slight differences. Since we use the estimates of [3], we will take their definition of the invariants. Specifically, we are going to apply the estimates on the invariants found in Corollary 3.6 of Theorem 3.5 in [3].

By the theory of Hamiltonian normal forms [14] we know that for a Hamiltonian $\bar{\mathcal{H}} = \mathbb{H} + \bar{\mathcal{K}}$ with $\mathbb{H}$ given in (9), the functions F_j, with F_j as in Section 3, are d formal integrals of motion. Applying the inverse of the Lie transformation that brings (1) into its normal form, the functions F_j are transformed back to functions P_j which correspond to d formal integrals of $\mathcal{H}$. The P_j are formal series and we can choose them so that their principal terms are given by $\xi_l F_l$; see [18].

We need to rearrange the Hamiltonian (1) to apply the results of [3]. Since we are interested in the stability of the origin for $\mathcal{H}$, we stretch coordinates by $x = \varepsilon y$. This change of coordinates is symplectic with multiplier ε^{-2}. The resulting Hamiltonian reads

$$\mathcal{F}(y, \varepsilon) = \mathbb{H}(y) + \mathcal{G}(y, \varepsilon) = \mathbb{H}(y) + \sum_{k=3}^{\infty} \varepsilon^{k-2} \mathcal{G}_k(y)\,, \tag{12}$$

where $\mathcal{G}$ is obtained from $\mathcal{K}$ in (1) as $\varepsilon^2 \mathcal{G}(y, \varepsilon) = \mathcal{K}(\varepsilon y)$ and each $\mathcal{G}_k$ is a homogeneous polynomial of degree k in y.

Let $I_1(y,\varepsilon),\dots,I_d(y,\varepsilon)$ be the d formal series in y given in [3]. They are formal integrals for the Hamiltonian $\mathcal{F}$ given in (12); that is,

$$\{\mathcal{F}, I_i\} = 0\,, \quad i = 1,\dots,d\,,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket operator. The integrals I_i are in involution, so

$$\{I_i, I_j\} = 0\,, \quad i,j = 1,\dots,d\,, \quad i \neq j\,.$$

Moreover they are of the form

$$I_i(y,\varepsilon) = \sum_{k=2}^{\infty} \varepsilon^{k-2} I_{ik}(y)\,,$$

where each $I_{ik}(y)$ is a homogeneous polynomial of degree k in y and

$$I_{i2}(y) = F_i(y)\,.$$

Each I_i is related in a straightforward way with P_i; specifically, we have

$$\varepsilon^2 \xi_i I_i(y,\varepsilon) = P_i(\varepsilon y)\,.$$

The I_i are only formal series, so to get estimates we truncate the series, thereby obtaining adiabatic invariants. The truncation I_i^N is

$$I_i^N(y,\varepsilon) = \sum_{k=2}^{N} \varepsilon^{k-2} I_{ik}(y)\,.$$

We now introduce a few more notations before stating the results of Chartier et al. on adiabatic invariants in a form suited to our needs.

Take $\mathcal{N} = B_R$ to be the open ball of radius $R > 0$ centered on 0 in $\mathbb{R}^{2n}$. Given a solution $y = y(t, y_0, \varepsilon)$ of the system (12) with initial condition y_0 in $\mathcal{N}$, let $\gamma = \gamma(y_0,\varepsilon)$ be the solution's first time of escape from $\mathcal{N}$, i.e.

$$\gamma = \inf\{t > 0 \,\big|\, |y(t, y_0, \varepsilon)| \geq R\}\,. \tag{13}$$

Given $\gamma > 0$ and $T > 0$, we set

$$D = [0,\gamma) \cap [0,T]\,, \tag{14}$$

i.e., D is the shorter of the two intervals.

Theorem 4.1 *(Chartier, Murua and Sanz-Serna). Let the real analytic system (1) satisfy the Diophantine condition (7) and let $y_0 \in \mathcal{N}$. Then there are constants $C > 0$ and $K > 0$ such that for small enough $\varepsilon > 0$, there is a positive integer N such that for sufficiently small $\kappa > 0$ and for $i = 1,\dots,d$,*

$$|I_i^N(y(t, y_0, \varepsilon), \varepsilon) - I_i^N(y_0, \varepsilon)| < \kappa^2 \quad \textit{for all} \quad t \in D = [0, \gamma) \cap [0, T],$$

where

$$T = C\kappa^2 \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right).$$

Here we have taken the liberty of changing the notation of Corollary 3.6 of [3]. We changed δ to κ^2 and $1/C'$ to C. Also the corollary contains the proviso "for any solution $x(t)$ of (47) that remains in $\mathcal{N}$ for $0 \le t \le T$" which we incorporate in the definition of D after replacing x by y. Finally, when we use this result below, we replace "for small enough $\varepsilon > 0$" by "for all $\varepsilon \in (0, \varepsilon_1)$, where $\varepsilon_1 > 0$ is an appropriate threshold."

The fact that all the invariants remain small does not by itself imply that the solutions remain small. In Cherry's classic example (cf. §13.1 of [14]) the Hamiltonian

$$\mathcal{H} = 2J_1 - J_2 + J_1 J_2^{1/2} \cos(\phi_1 + 2\phi_2)$$

is an integral but the origin is unstable. Thus, no matter how small $\mathcal{H}$ is, some small initial conditions generate solutions tending to infinity.

5. Main theorem

Assume now that the origin of the Hamiltonian system (2) is Lie stable and that this stability character is deduced using only the quadratic terms of $\mathbb{H}$. According to Proposition 3.1, the Hamiltonian $\mathbb{H}$ can be written as in (11). Thus, there are d formal integrals for the normal form $\bar{\mathcal{H}}$ associated to $\mathcal{H}$ introduced in (1), and they are given by Q_l. These functions are positive semidefinite quadratic forms in x. Applying the inverse of the Lie transformation that brings $\mathcal{H}$ to the form $\bar{\mathcal{H}}$, the Q_l are transformed back, becoming formal integrals for $\mathcal{H}$ that we call M_l. The M_l are formal series that can be selected in such a way that their principal terms are $\sigma_l Q_l$. Introducing y in place of x as in §4, we define the functions I_i as

$$\varepsilon^2 \sigma_i I_i(y, \varepsilon) = M_i(\varepsilon y) \quad \text{for} \quad i = 1, \dots, d.$$

Furthermore we have

$$I_i(y, \varepsilon) = \sum_{k=2}^{\infty} \varepsilon^{k-2} I_{ik}(y),$$

where each $I_{ik}(y)$ is a homogeneous polynomial in y of degree k and $I_{i2}(y) = Q_i(y)$. Note that we can assume that the Q_l are linear combinations of the J_k with positive constants. We are now ready to state our main result.

Theorem 5.1. *If the real analytic Hamiltonian (1) has $\mathbb{H}$ in the form (9) satisfying conditions (i), (ii) or (iii) of Proposition 3.1, while the frequency vector σ satisfies the Diophantine condition (7), then there exist $C > 0$, $K > 0$, $a > 1$ and $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$, and for all x_0 with $|x_0| < \rho$ we have*

$$|x(t,x_0)| < a\rho \quad \textit{for all} \quad 0 \le t \le T = C\,\rho \exp\left(\frac{K}{\rho^{1/(2(\nu+1))}}\right).$$

Proof. Let C, K, ν, ε, κ, T and N be as given in the statement of Theorem 4.1. Define $\lambda = |\sigma_1| + \cdots + |\sigma_d|$. If $I_j^N(y,\varepsilon)$ is the truncation of the function $I_j(y,\varepsilon)$ at degree N in y, that is, it is an adiabatic invariant, we introduce V^N as

$$V^N(y,\varepsilon) = |\sigma_1| I_1^N(y,\varepsilon) + \cdots + |\sigma_d| I_d^N(y,\varepsilon).$$

The quadratic part of V^N is $V_2 = |\sigma_1| I_{12}(y) + \cdots + |\sigma_d| I_{d\,2}(y)$, and it is a positive definite quadratic form in terms of y because $I_{i2}(y) = Q_i(y)$ and the Q_i are positive semidefinite since item (iii) of Proposition 3.1 holds.

First we prove that if $\tilde{\varepsilon}_0$ is small enough, there are positive constants α and β, independent of ε, such that whenever $\varepsilon|y| \le \tilde{\varepsilon}_0$, we have $\alpha|y|^2 \le V^N(y,\varepsilon) \le \beta|y|^2$. Defining $M_j^N(x)$ to be the truncation of the formal integral M_j at degree N in x, we define

$$U^N(x) = \operatorname{sign}(\sigma_1) M_1^N(x) + \cdots + \operatorname{sign}(\sigma_d) M_d^N(x).$$

The quadratic part of U^N is the positive definite form

$$U_2(x) = \operatorname{sign}(\sigma_1)\mathbb{H}_1(x) + \cdots + \operatorname{sign}(\sigma_d)\mathbb{H}_d(x).$$

Thus, given small enough $\tilde{\varepsilon}_0 > 0$, we know that there are positive constants such that whenever $|x| \le \tilde{\varepsilon}_0$, we have $\alpha|x|^2 \le U^N(x) \le \beta|x|^2$. Therefore we have $\alpha\varepsilon^2|y|^2 \le U^N(y,\varepsilon) \le \beta\varepsilon^2|y|^2$, so $\alpha|y|^2 \le V^N(y,\varepsilon) \le \beta|y|^2$ when $\varepsilon|y| \le \tilde{\varepsilon}_0$.

Assuming the estimates of Theorem 4.1 hold for $t \in D$ and letting $0 < \kappa < 1$, we have

$$\begin{gathered}
|V^N(y(t),\varepsilon) - V^N(y_0,\varepsilon)| \le |\sigma_1||I_1^N(y,\varepsilon) - I_1^N(y_0,\varepsilon)| + \cdots + |\sigma_d||I_d^N(y,\varepsilon) - I_d^N(y_0,\varepsilon)|,\\
|V^N(y(t),\varepsilon) - V^N(y_0,\varepsilon)| < \lambda\kappa^2,\\
|V^N(y(t),\varepsilon)| - |V^N(y_0,\varepsilon)| < \lambda\kappa^2,\\
|V^N(y(t),\varepsilon)| < \lambda\kappa^2 + |V^N(y_0,\varepsilon)| \le \lambda\kappa^2 + \beta|y_0|^2.
\end{gathered}$$

At this point we observe that κ and ε are independent small parameters in Theorem 4.1, and it will be useful to relate them by choosing $\varepsilon = \kappa$. We also need to express κ in terms of ρ. We set $\kappa = \sqrt{\rho}$ so that, from the inequality $|x_0| < \rho$ we deduce that $|y_0|^2 < \rho$. Thus

$$\alpha|y(t)|^2 \le V^N(y(t),\varepsilon) < (\lambda+\beta)\rho,$$

and therefore

$$|y(t)| < a\sqrt{\rho} \quad \text{for} \quad t \in D, \tag{15}$$

where

$$a = \sqrt{\frac{\lambda+\beta}{\alpha}}.$$

We show next that $\gamma > T$ and thus $D = [0, T]$. Assume the contrary, that is, assume $\gamma \le T$, so that $D = [0, \gamma)$. We choose $\rho_0 = \min\{\sqrt{\tilde{\varepsilon}_0}, \sqrt{\varepsilon_1}\}$ (where $\tilde{\varepsilon}_0$ is given above and $\varepsilon_1 > 0$ is the threshold guaranteed by Theorem 4.1), and we take $\rho < \min\{1, R^2/(4a^2)\}$. Now by assumption, $|y(t, y_0, \varepsilon)| \nearrow R$ as $t \nearrow \gamma$. But by Theorem 4.1 and estimate (15) above, we have $|y(t, y_0, \varepsilon)| < a\sqrt{\rho} < R/2$ for all $t \in [0, \gamma)$, which is a contradiction. It follows that $\gamma > T$, so $D = [0, T]$ as desired.

Finally, we return to the Hamiltonian $\mathcal{H}$ defined in (1). First we observe that $|y(t)| < a\sqrt{\rho}$ leads to $|x(t)| < a\rho$. Since

$$T = C\,\kappa^2 \exp\left(\frac{K}{\varepsilon^{1/(\nu+1)}}\right),$$

expressing κ and ε in terms of ρ, we see that $|x_0| < \rho$ implies $|x(t)| < a\rho$ for all t in the interval $[0, T]$, where

$$T = C\,\rho \exp\left(\frac{K}{\rho^{1/(2(\nu+1))}}\right). \quad \square$$

Consider a Hamiltonian of the form $\mathcal{H} = \mathbb{H} + \mathcal{K}$ where $\mathbb{H}$ is given in (8) and $\mathcal{K}$ represents an arbitrary higher order perturbation. According to our analysis in Section 3, $\mathcal{H}$ satisfies the hypotheses of Theorem 5.1 and the corresponding estimates hold.

As a second application of Theorem 5.1, let us consider the Hamiltonian

$$\mathbb{H} = (1 - \sqrt{2})J_1 - \sqrt{2}J_2 + (2 - \sqrt{2})J_3\,.$$

We readily deduce that the normal form Hamiltonian $\bar{\mathcal{H}}$ related to the Hamiltonian $\mathcal{H} = \mathbb{H} + \mathcal{K}$, where $\mathcal{K}$ is an arbitrary perturbation starting at degree three when it is written in rectangular coordinates, has two formal integrals, namely, $Q_1 = J_1 + 2J_3$ and $Q_2 = J_1 + J_2 + J_3$. Setting $G = Q_1 + \sqrt{2}Q_2$, then formally, $\{\bar{\mathcal{H}}, G\} = 0$ and G is a positive definite quadratic form in x. Therefore the origin of $\mathbb{R}^6$ is Lie stable for the equation associated to $\mathcal{H}$. Alternatively, one can apply Proposition 3.1 and check that $S = \{0\}$ to conclude Lie stability. On the other hand, since $\mathbb{H} = Q_1 - \sqrt{2}Q_2$, we have $(\sigma_1, \sigma_2) = (1, -\sqrt{2})$, which is a Diophantine vector in $\mathbb{R}^2$, thus Theorem 5.1 applies and one gets the asymptotic estimates for the equilibrium point. However, the lowest order of resonance is four, for instance the term $J_1 J_2^{1/2} J_3^{1/2} \cos(2\phi_1 - \phi_2 - \phi_3)$ can appear in the normal form $\mathcal{K}$, and the standard theory of asymptotic estimates for elliptic equilibria cannot be applied.

Another example is the following. Consider the Hamiltonian

$$\mathcal{H}(J, \phi) = \sqrt{3}J_1 + (1 + \sqrt{3})J_2 - 3J_3 + J_1^2 - J_2^2 + 4J_3^2 + J_1^{3/2} J_2^{3/2} J_3^{1/2} \sin(3\phi_1 - 3\phi_2 - \phi_3)\,,$$

which is already in normal form. Here $\mathcal{H}$ has formal integrals $Q_1 = J_1 + J_2$ and $Q_2 = J_1 + 3J_3$, and the set $S = \{0\}$. Hence, the origin of $\mathbb{R}^6$ is Lie stable for the equation associated to $\mathcal{H}$, the quadratic part of $\mathcal{H}$ is written as $\mathbb{H} = (1 + \sqrt{3})Q_1 - Q_2 = \sigma_1 Q_1 + \sigma_2 Q_2$, and $(\sigma_1, \sigma_2) = (1 + \sqrt{3}, -1)$ is Diophantine, hence Theorem 5.1 is satisfied. On the other hand, although the lowest order of resonance is seven, since the quadratic terms in the actions may be written as $J_1^2 - J_2^2 + 4J_3^2 = J^T A J$ and $A = \mathrm{diag}\{1, -1, 4\}$ is not sign-definite, the convexity condition does not hold (nor does the weaker quasi-convexity condition), and the estimates of [19,6,17,20, 12,13] do not apply.

As mentioned in the introduction, our main result Theorem 5 does not rely on the hypotheses of KAM or Nekhoroshev theory, but yields conclusions similar to those of Nekhoroshev theory applied at an equilibrium. The preceding two examples show that our Theorem 5 is indeed complementary to versions of Nekhoroshev theory requiring convexity, since it applies in these examples where the latter methods do not.

6. Lie and normal stability

In [15] we introduced a criterion which guarantees the formal stability of an equilibrium point of a Hamiltonian system. The criterion, which we call the Moser–Weinstein condition (MWC), see [26,16,27], is based only on the linearized system and not on higher order terms in the expansion of the Hamiltonian. This type of formal stability, which we call normal stability, applies to all normalized systems with the same linear part.

In [22] the authors prove that Lie stability generalizes normal stability, showing that the resonance condition on the eigenvalues corresponding to the matrix of the linearized system can be relaxed. Thus the theory developed in the previous sections applies to normally stable Hamiltonians, and the estimates provided in Theorem 5.1 apply for the examples handled in [15].

In particular, we applied the theory of normal stability to the spatial circular restricted three body problem with the aim of obtaining conditions on the mass parameter μ so that the points L_4 and L_5 are normally (and then Lie and formally) stable equilibria. The coordinates of L_4 and L_5 in the six-dimensional phase space are

$$\big(1/2-\mu\,,\,\pm\sqrt{3}/2\,,\,0\,,\,\mp\sqrt{3}/2\,,\,1/2-\mu\,,\,0\big)\,,$$

where the upper signs apply for L_4 and the lower signs for L_5. We shift the origin to L_4 or to L_5, linearize around the origin and drop the constant terms, so the resulting Hamiltonian has associated eigenvalues $\pm\lambda_1$, $\pm\lambda_2$ and $\pm\lambda_3$ with

$$\lambda_1=\frac{\sqrt{-1-\sqrt{27\mu^2-27\mu+1}}}{\sqrt{2}}\,,\quad \lambda_2=\frac{\sqrt{-1+\sqrt{27\mu^2-27\mu+1}}}{\sqrt{2}}\,,\quad \lambda_3=i\,.$$

When $0<\mu<\mu_R=\frac{1}{2}(1-\sqrt{69}/9)\approx 0.0385\ldots$ the eigenvalues λ_i are all pure imaginary, the corresponding eigenvectors form a basis of $\mathbb{R}^6$ and one can build a symplectic variable change that transforms the quadratic terms of the Hamiltonian to

$$\mathbb{H}=-\omega_1 J_1+\omega_2 J_2+\omega_3 J_3\,,$$

where each $\omega_j=-\lambda_j i>0$.

In Proposition 1 of [15], we should exclude the values of the frequencies ω_i where $-k_1\omega_1+k_2\omega_2+k_3\omega_3=0$ with k_i integers, since the condition on the eigenvalues of the linear system is not fulfilled in this case. However, for certain values of μ we can obtain Lie stability by analyzing only $\mathbb{H}$.

Setting $\mu = (81 - \sqrt{5973})/162 \approx 0.0229\ldots$ then

$$\mathbb{H} = -\tfrac{1}{6}(4+\sqrt{2})J_1 + \tfrac{1}{6}(4-\sqrt{2})J_2 + J_3\,,$$

and normal stability does not hold (indeed $-k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0$ with $k_1 = 3$, $k_2 = -3$, $k_3 = 4$), but L_4 and L_5 are Lie stable because in this case $S = \{0\}$. Using the theory of the previous sections, $\mathbb{H}$ is expressed as $\mathbb{H} = -(4+\sqrt{2})(J_1+J_2)/6 + (4J_2+3J_3)/3$ with $(-(4+\sqrt{2})/6, 1/3)$ Diophantine. Thus, the estimates of Theorem 5.1 apply for L_4 and L_5.

On the other hand, setting $\mu = (75 - \sqrt{5101 - 64\sqrt{6}})/150 \approx 0.0312\ldots$, then

$$\mathbb{H} = -\tfrac{1}{10}(6+\sqrt{6})J_1 + \tfrac{1}{10}(-2+3\sqrt{6})J_2 + J_3\,,$$

the condition for normal stability is not fulfilled. In this case $-k_1\omega_1 + k_2\omega_2 + k_3\omega_3 = 0$ with $k_1 = 3$, $k_2 = 1$, $k_3 = 2$ and $S = \{(3J_1, J_1, 2J_1) \,|\, J_1 \geq 0\}$. Thus one needs to calculate higher order terms in the normal form to distinguish between Lie stability or instability after applying some of the criteria in [22]. In this case we cannot apply Theorem 5.1.

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