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Elliptic relative equilibria in the N-body problem☆

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Abstract

A planar central configuration of the N-body problem gives rise to a solution where each particle moves on a specific Keplerian orbit while the totality of the particles move on a homothety motion. If the Keplerian orbit is elliptic then the solution is an equilibrium in pulsating coordinates so we call this solution an *elliptic relative equilibrium*.

The totality of such solutions forms a four-dimensional symplectic subspace and we give a symplectic coordinate system which is adapted to this subspace and its symplectic complement. In our coordinate system, the linear variational equations of such a solution decouple into three subsystems. One subsystem simply gives the motion of the center of mass, another is Kepler's problem and the third determines the nontrivial characteristic multipliers.

Using these coordinates we study the linear stability of the elliptic relative equilibrium defined by the equilateral triangular central configuration of the three-body problem. We reproduce the analytic studies of G. Roberts. We also study the linear stability of the four- and five-body problem where three or four bodies of unit mass are at the vertices of a equilateral triangle or square and the remaining body is at the center with arbitrary mass μ.

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1. Introduction

Let $q_1, \dots, q_N \in \mathbb{R}^2$ be the position vectors, $p_1, \dots, p_N \in \mathbb{R}^2$ the momentum vectors of N particles of masses $m_1, \dots, m_N$ in an inertial (sidereal) frame. Let the distance between the jth and kth particles be denoted by $d_{jk} = \| q_j - q_k \|$. In these coordinates the *Hamiltonian*, H, and the *self-potential*, S, for the N-body problem are

$$H = \sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} - S(q_1, \dots, q_N), \quad S = \sum_{1 \leqslant j < k \leqslant N} \frac{m_j m_k}{d_{jk}} \tag{1}$$

and the equations of motion are

$$\dot{q}_j = p_j / m_j, \quad \dot{p}_j = \frac{\partial S}{\partial q_j}, \quad j = 1, \dots, N. \tag{2}$$

A *central configuration* is a solution $q_1 = a_1, \dots, q_N = a_N$ of the algebraic equations

$$-\lambda m_j q_j = \frac{\partial S}{\partial q_j}(q_1, \dots, q_N) \tag{3}$$

for some constant λ. One shows that $\lambda = S(a)/2I(a) > 0$ where $I = \frac{1}{2} \sum m_j \|a_j\|^2$ is the moment of inertia.

Only the planar N-body problem is considered here and so sometimes we will think of vectors in $\mathbb{R}^2$ as complex numbers, i.e. we will identify $\mathbb{R}^2$ and $\mathbb{C}$ in the usual way. A classical and elementary result [10,12,15,20] is

Proposition 1.1. *Let $a_1, \dots, a_N$, $a_i \in \mathbb{C}$ be a central configuration with constant λ. Let $(z(t), Z(t)) \in \mathbb{C}^2$ be a solution of the Kepler problem (central force problem) with Hamiltonian*

$$H_{\mathrm{K}} = \frac{1}{2}\|Z\|^2 - \lambda/\|z\|, \quad z, Z \in \mathbb{R}^2. \tag{4}$$

Then

$$q_i = z(t)a_i, \quad p_i = m_i Z(t) a_i, \quad i = 1, \dots, N$$

is a solution of the N-body problem.

Given a central configuration the totality of points in $\mathbb{R}^{4N}$ swept out by such solutions is a four-dimensional, invariant, symplectic subspace [9,10]. We give a symplectic coordinate system which is adapted to this subspace and its complement. This improves and extends the coordinates discussed in [10, Section 4.6].

If $(z(t), Z(t))$ is a circular orbit of the Kepler problem with frequency ω, then it would be an equilibrium solution in a coordinates system which rotates uniformly about the center of mass with frequency ω. Such a solution is often called a *relative equilibrium*. We are interested in the case when the solution $(z(t), Z(t))$ of the Kepler problem is an elliptic orbit in which case the solution is an equilibrium solution in pulsating coordinates (see Section 3.2), and so, we call such a solution an *elliptic relative equilibrium* as in the title of this paper.

We are interested in the linear stability of circular and elliptic relative equilibria, i.e. the characteristic multipliers of these solutions. To that end, we study the linear variational equations. In our coordinate system, the variational equations are block diagonal with one block corresponding to the translational invariance of the problem and one block being the variational equation for the Kepler problem. The first two blocks integrate to give the characteristic multiplier +1 a multiplicity of 8. The last block contains all the information about the remaining (nontrivial) characteristic multipliers.

We study several examples in detail. First, we study the elliptic relative equilibrium when the central configuration is the Lagrange equilateral triangle configuration. In that case, the stability depends on two parameters e the eccentricity of the Kepler solution and the mass parameter

$$\beta^2 = \frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{(m_1 + m_2 + m_3)^2}.$$

We obtain the variational equations in a very simple form and study the stability domains in the e, β parameter space using perturbation methods.

Gascheau [8] in 1843 showed that (circular) Lagrange relative equilibrium solution of the three-body problem is linearly stable if $\beta^2 < \frac{1}{27}$ (also see [23, p. 113ff]). This inequality is also found in Routh's 1875 paper [19].

Next, Danby [5] using numeric methods and Schmidt [22] using analytic methods study the stability of the corresponding Lagrange L_4 equilibrium in the elliptic restricted problem. They find the stability domains in the e, μ plane, where μ is the mass ratio parameter corresponding to β. See [22] for further references on the elliptic restricted problem.

The first to study the elliptic Lagrange relative equilibrium solution in the three-body problem with general masses was Danby [6]. Danby's analysis was incomplete and was completed in the elegant paper by Roberts [18]. Danby and Roberts uses the integrals and symmetries of the three-body problem to reduce the dimension from 12 to 4, whereas, we use a linear symplectic change of variables to isolate the important four-dimensional system. Roberts studies the stability regions in the e, β parameter space using e as a small parameter and by numerical methods for large e. We use normal form theory to reproduce the small parameter expansions of Roberts, but do not study the problem for large e by numerical methods.

Another class of central configurations of the $(N+1)$-body problem has N-bodies of mass 1 at the vertices of a regular N-agon and another body of arbitrary mass μ at the center. The variational equations are $4N+4$ dimensional which by our linear change of coordinates can be reduced to a $4N-4$-dimensional system. We explicitly derive the variational equations for all e, μ when $N = 3, 4$, and we completely analyze the characteristic exponents when $e = 0$ (the circular case).

2. Central configuration coordinates

Vectors will be column vectors, but written as row vectors in the text. Let $Q = (q_1, \ldots, q_N)$ and $P = (p_1, \ldots, p_N)$. Let

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbb{J} = \mathrm{diag}\,(J, J, \ldots, J), \quad \mathbf{J} = \begin{bmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{bmatrix},$$

where $\mathbb{I}$ and $\mathbb{O}$ are the identity and zero matrix, respectively. J will always be 2×2, but the dimensions of $\mathbb{J}$, $\mathbb{I}$ and $\mathbf{J}$ will depend on the context. With this notation the integral of *angular momentum* is

$$C = \sum_{j=1}^{N} q_j^T J p_j = Q^T \mathbb{J} P.$$

If $(a_1, \ldots, a_N)$ is a central configuration then so is $(\alpha A a_1, \ldots, \alpha A a_N)$ where α is a nonzero scalar and $A \in SO(2, \mathbb{R})$ is any 2×2 rotation matrix. Also $\sum m_j a_j = 0$. Thus, a central configuration begets the set of central configurations $\{(\alpha A a_1, \ldots, \alpha A a_N) : \alpha \in \mathbb{R}, A \in SO(2, \mathbb{R})\}$ which is a two-dimensional linear subspace of $\mathbb{R}^{2N}$. (The origin is included for completeness.)

Let $a = (a_1, \ldots, a_N)$ be a fixed central configuration which is scaled so that $\sum m_j \| a_j \|^2 = 1$. We will define three subspaces: $\mathcal{A}$ which reflects the translational invariance of the problem, $\mathcal{B}$ the space swept out by all rotations and dilation of the central configuration, and $\mathcal{C}$ the complement of the first two spaces. Specifically, define

$$\mathcal{A} = \{(b, b, \ldots, b; m_1 c, m_2 c, \ldots, m_N c) \in \mathbb{R}^{4N} : b, c \in \mathbb{R}^2\},$$

$$\mathcal{B} = \{(\alpha A a_1, \ldots, \alpha A a_N; \beta B m_1 a_1, \ldots, \beta B m_N a_N) : \alpha, \beta \in \mathbb{R}, A, B \in SO(2, \mathbb{R})\}, \tag{5}$$

$$\mathcal{C} = \{x \in \mathbb{R}^{4N} : \{x, \mathcal{A}\} = \{x, \mathcal{B}\} = 0\}.$$

Here $\{\cdot, \cdot\}$ is the usual Poisson bracket defined by $\{x, y\} = x^T \mathbf{J} y$.

Proposition 2.1. *$\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all symplectic linear subspace of $\mathbb{R}^{4N}$. $\mathcal{A}\oplus\mathcal{B}$, $\mathcal{B}$ and $\mathcal{B}\oplus\mathcal{C}$ are invariant. $\mathcal{A}$ and $\mathcal{B}$ are four dimensional, and $\mathcal{C}$ is $4N-8$ dimensional. $\mathcal{R}^{4N}=\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$. $\{\mathcal{A},\mathcal{B}\}=\{\mathcal{B},\mathcal{C}\}=\{\mathcal{C},\mathcal{A}\}=0$.*

Proof. Without loss of generality we normalize the masses so that $\sum m_j = 1$. A symplectic basis for $\mathcal{A}$ is

$$\gamma_1=\begin{bmatrix}1\\0\\\vdots\\1\\0\\0\\0\\\vdots\\0\\0\end{bmatrix},\ \gamma_2=\begin{bmatrix}0\\1\\\vdots\\0\\1\\0\\0\\\vdots\\0\\0\end{bmatrix},\ \delta_1=\begin{bmatrix}0\\0\\\vdots\\0\\0\\m_1\\0\\\vdots\\m_N\\0\end{bmatrix},\ \delta_2=\begin{bmatrix}0\\0\\\vdots\\0\\0\\0\\m_1\\\vdots\\0\\m_N\end{bmatrix} \tag{6}$$

and so $\mathcal{A}$ is a four-dimensional symplectic subspace.

A symplectic basis for $\mathcal{B}$ is

$$\gamma_3=\begin{bmatrix}a_1\\\vdots\\a_N\\0\\\vdots\\0\end{bmatrix},\ \gamma_4=\begin{bmatrix}Ja_1\\\vdots\\Ja_N\\0\\\vdots\\0\end{bmatrix},\ \delta_3=\begin{bmatrix}0\\\vdots\\0\\m_1a_1\\\vdots\\m_Na_N\end{bmatrix},\ \delta_4=\begin{bmatrix}0\\\vdots\\0\\m_1Ja_1\\\vdots\\m_NJa_N\end{bmatrix} \tag{7}$$

and so $\mathcal{B}$ is a four-dimensional symplectic subspace. One sees that $\{\mathcal{A},\mathcal{B}\}=0$ by checking on the basis vectors given above and recalling that $\sum m_j a_j = 0$.

Since $\mathcal{C}$ is the symplectic complement of the eight-dimensional symplectic space $\mathcal{A}\oplus\mathcal{B}$, it is a symplectic subspace of dimension $4N-8$ by Proposition 4, p. 43 of [12]. This proposition also shows that $\mathbb{R}^{4N}=\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$.

For the moment, think of the vectors q_j, p_j, etc. as complex numbers. Then the set $\mathcal{B}$ is the same as

$$\mathcal{B}=\{(za_1,\dots,za_N;Zm_1a_1,\dots,Zm_Na_N):z,Z\in\mathbb{C}\}. \tag{8}$$

Let $(z_0a_1,\dots,z_0a_N;Z_0m_1a_1,\dots,Z_0m_Na_N)$, $z_0,Z_0\in\mathbb{C}$, $z_0\neq 0$ be any point in $\mathcal{B}$ and let $z(t),Z(t)$ be the solution of the Kepler problem (4) starting at z_0,Z_0 when

$t = 0$. Then a direct substitution verifies that

$$V(t) = (Q(t), P(t)) = (z(t)a_1, \ldots, z(t)a_N; Z(t)m_1a_1, \ldots, Z(t)m_Na_N)$$

is a solution of the equations of motion of the N-body problem (2) and clearly $V(t) \in \mathcal{B}$ for all t. This shows that $\mathcal{B}$ is invariant.

In a like manner

$$\begin{aligned}\mathcal{A} \oplus \mathcal{B} = \{(&g + za_1, \ldots, g + za_N,\\ &m_1G + m_1Za_1, \ldots, m_NG + m_NZa_N) : g, G, z, Z \in \mathbb{C}\}.\end{aligned}$$

Consider the Hamiltonian

$$H_+ = \tfrac{1}{2}\|G\|^2 + \tfrac{1}{2}\|Z\|^2 - \lambda/\|z\|,$$

the corresponding equations of motion

$$\dot{g} = G, \quad \dot{G} = 0, \quad \dot{z} = Z, \quad \dot{Z} = -\lambda z/\|z\|^3 \tag{9}$$

(H_+ is the Hamiltonian of the two-body problem in Jacobi coordinates.) Let $(g_0 + z_0a_1, \ldots, g_0 + z_0a_N; m_1G_0 + Z_0m_1a_1, \ldots, m_NG_0 + Z_0m_Na_N)$ be any point in $\mathcal{A} \oplus \mathcal{B}$ and $(g(t), z(t), G(t), Z(t))$ the solution of (9) through that point at $t = 0$. Then a direct substitution verifies that

$$\begin{aligned}V(t) = (&g(t) + z(t)a_1, \ldots, g(t) + z(t)a_N; m_1G(t)\\ &+Z(t)m_1a_1, \ldots, m_NG(t) + Z(t)m_Na_N)\end{aligned}$$

is a solution of the N-body problem and that $V(t) \in \mathcal{A} \oplus \mathcal{B}$, so $\mathcal{A} \oplus \mathcal{B}$ is invariant.

Now

$$\begin{aligned}\mathcal{B} \oplus \mathcal{C} &= \mathcal{A}^{\perp}\\ &= \{(q, p) : bp_1 + \cdots + bp_n - (cm_1q_1 + \cdots + cm_Nq_N) = 0 \text{ for all } b, c \in \mathbb{C}\}\\ &= \{(q, p) : p_1 + \cdots + p_n = 0, m_1q_1 + \cdots + m_Nq_N = 0\}.\end{aligned}$$

In other words $\mathcal{B} \oplus \mathcal{C}$ is the set where the center of mass of the system is at the origin and total linear momentum is zero. This is a well-known invariant space. □

Theorem 2.1. *There exists a linear symplectic transformation from the old coordinates* (Q, P) *to the new coordinates* (g, z, w, G, Z, W) *with* (g, G) *symplectic coordinates for* $\mathcal{A}$, (z, Z) *symplectic coordinates for* $\mathcal{B}$ *and* (w, W) *symplectic coordinates for* $\mathcal{C}$.

The change of coordinates has the following properties:

- *Kinetic energy is*:

$$\sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} = \frac{1}{2}\left\{ \| G \|^2 + \| Z \|^2 + \sum_{j=1}^{N-4} \| W_j \|^2 \right\}.$$

- *Angular momentum is preserved, i.e.*

$$C = \sum_{j=1}^{N} q_j^T J p_j = g^T J G + z^T J Z + w^T \mathbb{J} W.$$

- *The self-potential is independent of g, i.e.*

$$S(Q) = S(z, w).$$

- *The space $\mathcal{B}$ is invariant and the Hamiltonian on that space is the Hamiltonian of the Kepler problem, i.e.*

$$\begin{aligned} &\partial H(0, z, 0; 0, Z, 0)/\partial g = 0, \quad \partial H(0, z, 0; 0, Z, 0)/\partial G = 0,\\ &\partial H(0, z, 0; 0, Z, 0)/\partial w = 0, \quad \partial H(0, z, 0; 0, Z, 0)/\partial W = 0,\\ &H(0, z, 0; 0, Z, 0) = H_{\mathrm{K}}(z, Z) = \tfrac{1}{2} \| Z \|^2 - \lambda / \| z \|, \end{aligned} \tag{10}$$

  where

$$\lambda = \sum_{1 \leqslant j < k \leqslant N} \frac{m_j m_k}{\|a_j - a_k\|}. \tag{11}$$

Remark.

- Since S and hence H is independent of g (the center of mass of the system), its conjugate momentum G (total linear momentum) is an integral. As is customary we will set $g = G = 0$ and forget these variables in the subsequent analysis.
- H_{K} is the Hamiltonian of the Kepler problem (the central force problem), and (10) says that $\mathcal{B} = \{g = G = w = W = 0\}$ is invariant and the motion on this invariant subspace is Keplerian.

  Let $z(t), Z(t)$ be any solution of the Kepler problem, then $(g \equiv 0, z(t), w \equiv 0; G \equiv 0, Z(t), W \equiv 0)$ is a solution of the N-body problem. Think of the vectors in $\mathbb{R}^2$ as complex numbers. In this solution the ith particle follows the trajectory $q_i(t) = z(t)a_i$, $p_i(t) = Z(t)a_i$. Thus, each particle moves on a trajectory of the Kepler problem and the configuration of the N particles remains similar to the original central configuration.

- Since the change of variables preserves angular momentum it works well with rotating coordinates also. Thus, if q, p are rotating coordinates so that the Hamiltonian is

$$H = \sum_{j=1}^{N} \left\{ \frac{\| p_j \|^2}{2m_j} - q_j^T J p_j \right\} - S(Q),$$

then after this change of coordinates

$$\begin{aligned} H \; = \; & \frac{1}{2} \left\{ \| G \|^2 + \| Z \|^2 + \sum_{j=1}^{N-2} \| W_j \|^2 \right\} \\ & -\{ g^T J G + z^T J Z + w^T \mathbb{J} W \} - S(z, w). \end{aligned}$$

- A slightly different change of coordinates can be given such that kinetic energy is preserved, i.e.

$$\sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} = \frac{\| G \|^2}{2m_1} + \frac{\| Z \|^2}{2m_2} + \sum_{j=1}^{N-2} \frac{\| W_j \|^2}{2m_{j+2}}$$

and

$$H_{\mathrm{K}} = \frac{\|Z\|^2}{2m_2} - \frac{\hat{\lambda}}{\|z\|},$$

where $\hat{\lambda}$ has the same form as λ, but the a_j's are normalized so that $\sum m_j \|a_j\|^2 = 1/m_2$.

Proof. As before $Q = (q_1, \ldots, q_N) \in \mathbb{R}^{2N}$, $P = (p_1, \ldots, p_N) \in \mathbb{R}^{2N}$ and $\sum m_j = 1$. Let $X = (g, z, w) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2N-4}$, $Y = (G, Z, W) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2N-4}$. Q, P, X, Y are to be considered as column vectors. The linear symplectic change of variables will be of the form

$$Q = AX, \quad P = A^{-T} Y, \tag{12}$$

where A is a $2N \times 2N$ nonsingular matrix with the following properties

$$A^{-1} \mathbb{J} A = \mathbb{J}, \quad A^T M A = \mathbb{I} \tag{13}$$

and M is the $2N \times 2N$ diagonal matrix $M = \text{diag}\,(m_1, m_1, m_2, m_2, \dots, m_N, m_N)$. Here $A^{-T} = (A^T)^{-1} = (A^{-1})^T$.

The form of the change of variables in (12) insures the transformation is symplectic. Kinetic energy is

$$\sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} = \tfrac{1}{2} P^T M^{-1} P = \tfrac{1}{2} Y^T A^{-1} M^{-1} A^{-T} Y$$

$$= \frac{1}{2} Y^T \mathbb{I} Y = \frac{1}{2} \left\{ \| G \|^2 + \| Z \|^2 + \sum_{j=1}^{N-2} \| W_j \|^2 \right\}.$$

(**Remark.** If we replace A by $\tilde{A} = AM^{1/2}$ then $\tilde{A}^T M \tilde{A} = M$ and kinetic energy would be preserved as stated in the remark given above.)

Angular momentum is preserved because

$$\sum_{j=1}^{N} q_j^T J p_j = Q^T \mathbb{J} P = X^T A^T \mathbb{J} A^{-T} Y$$

$$= X^T \mathbb{J} Y = g^T J G + z^T J Z + w^T \mathbb{J} W.$$

The matrix A will be constructed by a modified Gram–Schmidt method to insure that A satisfies $A^T M A = \mathbb{I}$, which by an abuse of terminology we will call M-orthogonal. Think of A as a block matrix made up of 2×2 matrices, that is

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

where A_{ij} is a 2×2 matrix. Each of these submatrices will have the special form $A_{ij} = [b, -Jb]$ where b is any 2-column vector. A direct computation shows that if A_{ij} has this special form then $JA_{ij} = A_{ij}J$, and if each of the submatrices in A have this special form then $A^{-1}\mathbb{J}A = \mathbb{J}$.

Let $A = [c_1, c_2, \dots, c_{2N}]$ were c_j is the jth column of A. The first four columns are

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad c_4 = \begin{bmatrix} -Ja_1 \\ -Ja_2 \\ \vdots \\ -Ja_N \end{bmatrix}.$$

By recalling that $\sum m_i = \sum m_i \|a_i\|^2 = 1$, $\sum m_i a_i = 0$ one sees that these four vectors are *M*-orthogonal, and the special form of the 2×2 submatrices holds. Moreover, $c_1^T M c_1 = c_2^T M c_2 = C_3^T M c_3 = c_4^T M c_4 = 1$.

If $A^T M A = \mathbb{I}$ then $A^{-1} = A^T M$. By the definition of c_1 and c_2 this implies $X = (g, z, w) = (m_1 q_1 + \cdots + m_N q_N, \cdot, \cdot)$ or $g, = m_1 q_1 + \cdots + m_N q_N$ is the center of mass of the system. Thus *S* is independent of *g*. In a like manner *G* is total linear momentum.

We now use induction to construct the remaining column vectors by pairs. The general step is the same as the first. Let *d* be any vector independent of c_1, c_2, c_3, c_4 and *e* the *M*-projection of *d* onto the span of the first four, so $d-e$ is *M*-orthogonal to the first four. Let α be the scale constant so that $c_5 = \alpha(d-e)$ satisfies $c_5^T M c_5 = 1$. The construction of c_5 is just the same as the Gram–Schmidt procedure. Let $c_5 = [\eta_1, \eta_2, \ldots, \eta_N]$ where each η_j is a 2-column vector. Define $c_6 = [-J\eta_1, -J\eta_2, \ldots, -J\eta_N]$. By construction $c_6^T M c_6 = 1$ and $c_6^T M c_5 = 0$.

We claim that c_6 is *M*-orthogonal to the first four also, since if it were not then c_5 would not be *M*-orthogonal to one of the first four. Say for example that $c_6^T M c_4 \neq 0$ then since

$$c_5^T M c_3 = \sum m_i \eta_i a_i = \sum m_i (-J\eta_i)^T (-J a_i) = c_6^T M c_4$$

this would imply that c_5 is not *M*-orthogonal to the first four which contradicts the construction of c_5. The other cases are similar.

By (13) $A^{-T} = MA$ and so the first 4 columns of A^{-T} are

$$d_1 = \begin{bmatrix} m_1 \\ 0 \\ m_2 \\ 0 \\ \vdots \\ m_N \\ 0 \end{bmatrix}, d_2 = \begin{bmatrix} 0 \\ m_1 \\ 0 \\ m_2 \\ \vdots \\ 0 \\ m_N \end{bmatrix}, d_3 = \begin{bmatrix} m_1 a_1 \\ m_2 a_2 \\ \vdots \\ m_N a_N \end{bmatrix}, d_4 = \begin{bmatrix} -m_1 J a_1 \\ -m_2 J a_2 \\ \vdots \\ -m_N J a_N \end{bmatrix}.$$

The space $\mathcal{A}$ is spanned by $\gamma_1, \gamma_2, \delta_1, \delta_2$ in (6) which is the same as the space given by $Q = g_1 c_1 + g_2 c_2$, $P = G_1 d_1 + G_2 d_2$ where $g_1, g_2, G_1, G_2 \in \mathbb{R}$. Similarly, the space $\mathcal{B}$ is spanned by $\gamma_3, \gamma_4, \delta_3, \delta_4$ in (7) which is the same as the space given by $Q = z_1 c_3 + z_2 c_4$, $P = Z_1 d_3 + Z_2 d_4$ where $z_1, z_2, Z_1, Z_2 \in \mathbb{R}$.

Again think of the various vectors as complex numbers. The space $\mathcal{B}$ has coordinates z, Z see (8) and the constructed change of coordinates is $q_j = z a_j$, $p_j = Z m_j a_j$. Substituting these into the Hamiltonian (1) gives

$$H|\mathcal{B} = \sum_{j=1}^{N} \frac{\| Z m_j a_j \|^2}{2 m_j} - \sum_{1 \leqslant j < k \leqslant N} \frac{m_j m_k}{\| z a_j - z a_k \|}$$

$$= \sum_{j=1}^{N} \left\{ m_j \parallel a_j \parallel^2 \right\} \frac{\|Z\|^2}{2} - \left\{ \sum_{1 \leqslant j < k \leqslant N} \frac{m_j m_k}{\|a_j - a_k\|} \right\} \frac{1}{\|z\|}$$

$$= \frac{\|Z\|^2}{2} - \frac{\lambda}{\|z\|} = H_{\mathrm{K}}. \qquad \square$$

Corollary 2.1. *Let $(z(t), Z(t))$ be a T-periodic elliptic solution of the Kepler problem, i.e. of the system whose Hamiltonian is H_{K} in (10). Then $(g, z, w, G, Z, W) = (0, z(t), 0, 0, Z(t), 0)$ is a T-periodic solution of the N-body problem. The linear variational equation of this periodic solution is of the form*

$$\begin{bmatrix} \dot{g} \\ \dot{G} \\ \dot{z} \\ \dot{Z} \\ \dot{w} \\ \dot{W} \end{bmatrix} = V(t) \begin{bmatrix} g \\ G \\ z \\ Z \\ w \\ W \end{bmatrix},$$

where $V(t)$ is the block-diagonal, T-periodic, $4N \times 4N$ matrix $V(t) = \mathrm{diag}(V_1(t), V_2(t), V_3(t))$ with

$$V_1(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad V_2(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda \dfrac{2z_1^2 - z_2^2}{r^5} & \lambda \dfrac{3z_1 z_2}{r^5} & 0 & 0 \\ \lambda \dfrac{3z_1 z_2}{r^5} & \lambda \dfrac{2z_2^2 - z_1^2}{r^5} & 0 & 0 \end{bmatrix},$$

$r(t) = \sqrt{z_1^2 + z_2^2}$. $V_3(T)$ is a T-periodic, $(4N - 8) \times (4N - 8)$ matrix which depends on the particular central configuration.

Proof. This is a straight forward computation, see for example Lemma 3.1. $\square$

Corollary 2.2. *The characteristic multiplier* $+1$ *of an elliptic central configuration solution has algebraic multiplicity at least* 8.

Proof. This is known in various special cases [10,18,23]. By Corollary 2.1 the variational equation decouples into three parts. The first part with coefficient matrix V_1 is autonomous and

$$e^{V_1 T} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, $e^{V_1 T}$ has +1 as an eigenvalue of multiplicity 4.

The second part with coefficient matrix V_2 is the variational equation of the Kepler problem which is a Hamiltonian system with two integrals (energy and angular momentum) in involution. The characteristic multiplier +1 of a periodic solution of such a system has multiplicity 4, see [7,11,17]. $\square$

3. Lagrangian triangular configuration

Consider the three-body problem with general masses normalized by $m_1+m_2+m_3 = 1$. The Hamiltonian of the three-body problem with coordinates $q_j, p_j \in \mathbb{R}^2$ $j = 1, 2, 3$ is

$$H_3 = \sum_{j=1}^{3} \frac{\|p_j\|^2}{2m_j} - S_3(q), \quad S_3(q) = \sum_{1 \leqslant j<k \leqslant 3} \frac{m_i m_j}{\|q_j - q_k\|}.$$

An equilateral triangular central configuration is given by

$$a_1 = (1, 0) - cm, \quad a_2 = (-1/2, \sqrt{3}/2) - cm, \quad a_3 = (-1/2, -\sqrt{3}/2) - cm,$$

where the vector

$$cm = (1/2)(2m_1 - m_2 - m_3, \sqrt{3}m_2 - \sqrt{3}m_3)$$

is chosen so that the center of mass is at the origin. It is possible to scale all distances by dividing by a common factor. If we choose the factor to be $\sqrt{3}\beta$ with

$$\beta^2 = m_1 m_2 + m_2 m_3 + m_1 m_3$$

then $m_1\|a_1\|^2 + m_2\|a_2\|^2 + m_3\|a_3\|^2 = 1$ and the position coordinates for the triangular configuration are given by

$$a_1 = \left(\frac{\sqrt{3}(m_2 + m_3)}{2\beta}, \frac{m_3 - m_2}{2\beta} \right), \quad a_2 = \left(-\frac{\sqrt{3}m_1}{2\beta}, \frac{m_1 + 2m_3}{2\beta} \right),$$

$$a_3 = \left(-\frac{\sqrt{3}m_1}{2\beta}, \frac{-m_1 - 2m_2}{2\beta} \right).$$

The three masses have to rotate with angular velocity $\omega = \beta^3$ around the origin, that is the center of mass, so that their configuration remains at a relative equilibrium.

Let $Q = (q_1, q_2, q_3)$, $P = (p_1, p_2, p_3) \in \mathbb{R}^6$ and correspondingly $X = (g, z, w)$, $Y = (G, Z, W) \in \mathbb{R}^6$ (all considered as column vectors). Make the symplectic change of coordinates of the form

$$Q = AX, \quad P = A^{-T} Y,$$

where A is a 6×6 matrix. This was the first example we constructed, but not by the general Gram–Schmidt procedure given in Theorem 2.1. If you were given two orthonormal vectors in $\mathbb{R}^3$ and asked to find a third to form an orthonormal triple and hence an orthogonal matrix you would simply take the cross product. This example was constructed by analogy to the above. Think of A as a block matrix made up of 2×2 matrices, i.e.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where each A_{ij} is a 2×2 matrix. The first two columns are

$$A_{11} = A_{21} = A_{31} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$A_{i2} = [a_i, \ -Ja_i], \quad i = 1, 2, 3$$

that is, the first column of A_{i2} is a_i and the second column is $-Ja_i$. Lastly,

$$A_{13} = \sqrt{\frac{m_2 m_3}{m_1}} \left(A_{21} A_{32} - A_{31} A_{22} \right)^T, \quad A_{23} = -\sqrt{\frac{m_1 m_3}{m_2}} \left(A_{11} A_{32} - A_{31} A_{12} \right)^T,$$

$$A_{33} = \sqrt{\frac{m_1 m_2}{m_3}} \left(A_{11} A_{22} - A_{21} A_{12} \right)^T.$$

These last three definitions are given in symmetric form and are inspired by the cross product formulas. They can be simplified by remembering that A_{i1} is the identity matrix.

Specifically,

$$A = \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}(m_2+m_3)}{2\beta} & \frac{m_2-m_3}{2\beta} & 0 & \frac{-\sqrt{m_2m_3}}{\beta\sqrt{m_1}} \\ 0 & 1 & -\frac{m_2-m_3}{2\beta} & \frac{\sqrt{3}(m_2+m_3)}{2\beta} & \frac{\sqrt{m_2m_3}}{\beta\sqrt{m_1}} & 0 \\ 1 & 0 & -\frac{\sqrt{3}m_1}{2\beta} & -\frac{m_1+2m_3}{2\beta} & \frac{\sqrt{3}\sqrt{m_1m_3}}{2\beta\sqrt{m_2}} & \frac{\sqrt{m_1m_3}}{2\beta\sqrt{m_2}} \\ 0 & 1 & \frac{m_1+2m_3}{2\beta} & -\frac{\sqrt{3}m_1}{2\beta} & -\frac{\sqrt{m_1m_3}}{2\beta\sqrt{m_2}} & \frac{\sqrt{3}\sqrt{m_1m_3}}{2\beta\sqrt{m_2}} \\ 1 & 0 & -\frac{\sqrt{3}m_1}{2\beta} & \frac{2m_2+m_1}{2\beta} & -\frac{\sqrt{3}\sqrt{m_1m_2}}{2\beta\sqrt{m_3}} & \frac{\sqrt{m_1m_2}}{2\beta\sqrt{m_3}} \\ 0 & 1 & -\frac{2m_2+m_1}{2\beta} & -\frac{\sqrt{3}m_1}{2\beta} & -\frac{\sqrt{m_1m_2}}{2\beta\sqrt{m_3}} & -\frac{\sqrt{3}\sqrt{m_1m_2}}{2\beta\sqrt{m_3}} \end{bmatrix}.$$

One can verify directly that $A^T MA = \mathbb{I}$ and $A^{-1}\mathbb{J}A = \mathbb{J}$, and so total angular momentum is preserved. Kinetic energy in these coordinates is

$$K = \tfrac{1}{2}(\|G\|^2 + \|Z\|^2 + \|W\|^2),$$

angular momentum is

$$C = g^T JG + z^T JZ + w^T JW$$

and the self-potential is independent of g, and is

$$S(z,w) = \sum_{1 \leqslant i<j \leqslant 3} \frac{m_i m_j}{d_{ij}},$$

where

$$\beta^2 d_{12}^2 = z_1^2 + z_2^2 + \frac{m_3(m_1^2 + m_1m_2 + m_2^2)}{m_1m_2}(w_1^2 + w_2^2) + \frac{\sqrt{3m_2m_3}}{\sqrt{m_1}}(z_2w_1 - z_1w_2)$$
$$-\frac{\sqrt{m_3}(2m_1 + m_2)}{\sqrt{m_1m_2}}(z_1w_1 + z_2w_2),$$

$$\beta^2 d_{23}^2 = z_1^2 + z_2^2 + \frac{m_1(m_2^2 + m_2m_3 + m_3^2)}{m_2m_3}(w_1^2 + w_2^2) - \frac{\sqrt{3m_1}(m_2 + m_3)}{\sqrt{m_2m_3}}(z_2w_1 - z_1w_2)$$
$$+\frac{\sqrt{m_1}(m_2 - m_3)}{\sqrt{m_2m_3}}(z_1w_1 + z_2w_2),$$

$$\beta^2 d_{31}^2 = z_1^2 + z_2^2 + \frac{m_2(m_1^2 + m_1 m_3 + m_3^2)}{m_1 m_3}(w_1^2 + w_2^2) + \frac{\sqrt{3m_2 m_3}}{\sqrt{m_1}}(z_2 w_1 - z_1 w_2)$$
$$+ \frac{\sqrt{m_2}(2m_1 + m_3)}{\sqrt{m_1 m_3}}(z_1 w_1 + z_2 w_2).$$

3.1. Kepler's problem

The invariant subspace $\mathcal{B}$ is given by $g = w = G = W = 0$. The Hamiltonian in the variables z and Z is then the Kepler problem with $\lambda = \beta^3$

$$H_{\mathrm{K}} = \frac{1}{2}\|Z\|^2 - \frac{\lambda}{\|z\|}.$$

Among the solutions of the corresponding differential equations

$$\ddot{z} + \frac{\lambda z}{\|z\|^3} = 0,$$

we consider elliptic orbits with semi-major axis a and eccentricity e, and whose perigee lies on the positive z_1-axis. In terms of the true anomaly f these solutions are given by

$$z_1 = r\cos f, \quad z_2 = r\sin f,$$

where

$$r = \|z\| = \frac{p}{1 + e\cos f} \tag{14}$$

and $p = a(1-e^2)$ is the latus rectum. The mean motion is $n = \sqrt{\lambda/a^3}$. The Lagrangian triangular configuration rotates with angular velocity ω. Since the two mean motions have to be the same we can find from $\omega = n$ the value of the semi-major axis $a = 1/\beta$.

In what follows we will use the true anomaly as the new independent variable and we go to pulsating coordinates. Thus, we do not use the above result directly, but use instead the following relationships for the Kepler problem

$$r^2\dot{f} = \sqrt{\lambda p} = \sqrt{\lambda a(1-e^2)}, \quad \ddot{r} = \lambda\left(\frac{p}{r^3} - \frac{1}{r^2}\right). \tag{15}$$

3.2. Equations in rotating and pulsating coordinates

Setting $g = G = 0$ fixes the center of mass at the origin and we can restrict ourselves to consider the Hamiltonian

$$H = \frac{1}{2}(Z_1^2 + Z_2^2 + W_1^2 + W_2^2) - S(z, w).$$

The three bodies move on elliptic orbits around the origin in such a way that they always form a central configuration. This motion takes place in the invariant subspace $\mathcal{B}$ and it was described in the previous section. We now change to nonuniformly rotating and pulsating coordinates, so that the configuration appears to be stationary.

Lemma 3.1. *There exist symplectic coordinates $\zeta = (\bar{z}_1, \bar{z}_2, \bar{w}_1, \bar{w}_2, \bar{Z}_1, \bar{Z}_2, \bar{W}_1, \bar{W}_2)$ such that the variational equations for the Lagrangian triangular configuration depend on the true anomaly f and are given by $d\zeta/df = \Phi\zeta$ with*

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \frac{2-\cos f}{1+e\cos f} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{8m_1-m_2-m_3-4e\cos f}{4(1+\cos f)} & \frac{3\sqrt{3}(m_2-m_3)}{4(1+e\cos f)} & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3\sqrt{3}(m_2-m_3)}{4(1+e\cos f)} & \frac{-4m_1+5(m_2+m_3)-4e\cos f}{4(1+e\cos f)} & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Proof. We first change to coordinates $\tilde{z}$ and $\tilde{w}$ which rotate with the speed of the true anomaly. The transformation matrix is given by

$$A = \begin{bmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{bmatrix}.$$

The transformation can be generated by the function

$$F(Z, W, \tilde{z}, \tilde{w}) = -Z^T A\tilde{z} - W^T A\tilde{w}$$

and is given by

$$z = -\frac{\partial F}{\partial Z} = A\tilde{z}, \qquad \tilde{Z} = -\frac{\partial F}{\partial \tilde{z}} = A^T Z,$$

$$w = -\frac{\partial F}{\partial W} = A\tilde{w}, \qquad \tilde{W} = -\frac{\partial F}{\partial \tilde{w}} = A^T W.$$

Since the transformation is time dependent we have to add to the transformed Hamiltonian

$$\frac{\partial F}{\partial t} = -Z^T \dot{A}\tilde{z} - W^T \dot{A}\tilde{w} = -\tilde{Z}^T A^T \dot{A}\tilde{z} - \tilde{W}^T A^T \dot{A}\tilde{w},$$

so that the Hamiltonian in the coordinates rotating with the true anomaly is given by

$$H = \tfrac{1}{2}(\tilde{Z}_1^2 + \tilde{Z}_2^2 + \tilde{W}_1^2 + \tilde{W}_2^2) + (\tilde{z}_2\tilde{Z}_1 - \tilde{z}_1\tilde{Z}_2 + \tilde{w}_2\tilde{W}_1 - \tilde{w}_1\tilde{W}_2)\dot{f} - S(\tilde{z}, \tilde{w}).$$

The next step is to introduce coordinates $\hat{z}$ and $\hat{w}$ which pulsate with r as given in (14). The position coordinates are transformed by

$$\tilde{z} = r\hat{z}, \quad \tilde{w} = r\hat{w}.$$

It would suffice to scale the momenta by $1/r$ in order to make the transformation symplectic but it turns out that the resulting Hamiltonian is simpler if instead the transformation

$$\tilde{Z} = \frac{1}{r}\hat{Z} + \dot{r}\hat{z}, \quad \tilde{W} = \frac{1}{r}\hat{W} + \dot{r}\hat{w}$$

is used. This transformation can be generated by the function

$$F(\tilde{z}, \tilde{w}, \hat{Z}, \hat{W}) = \frac{1}{r}(\tilde{z}_1\hat{Z}_1 + \tilde{z}_2\hat{Z}_2 + \tilde{w}_1\hat{W}_1 + \tilde{w}_2\hat{W}_2) + \frac{\dot{r}}{2r}(\tilde{z}_1^2 + \tilde{z}_2^2 + \tilde{w}_1^2 + \tilde{w}_2^2)$$

via

$$\hat{z} = \frac{\partial F}{\partial \hat{Z}}, \quad \tilde{Z} = \frac{\partial F}{\partial \tilde{z}}, \quad \hat{w} = \frac{\partial F}{\partial \hat{W}}, \quad \tilde{W} = \frac{\partial F}{\partial \tilde{w}}.$$

With

$$\frac{\partial F}{\partial t} = -\frac{\dot{r}}{r}(\hat{z}_1\hat{Z}_1 + \hat{z}_2\hat{Z}_2 + \hat{w}_1\hat{W}_1 + \hat{w}_2\hat{W}_2) + \frac{\ddot{r}r - \dot{r}^2}{2}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2)$$

the transformed Hamiltonian is

$$\begin{aligned} H = {} & \frac{1}{2r^2}(\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) + (\hat{z}_2\hat{Z}_1 - \hat{z}_1\hat{Z}_2 + \hat{w}_2\hat{W}_1 - \hat{w}_1\hat{W}_2)\dot{f} \\ & + \frac{r\ddot{r}}{2}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2) - \frac{1}{r}S(\hat{z}, \hat{w}). \end{aligned}$$

The third step is to introduce the true anomaly as the new independent variable. We achieve this by dividing the Hamiltonian by $\dot{f}$. We also use the relationships (15) and arrive at

$$H = \frac{1}{2\sqrt{p\lambda}}(\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) + \hat{z}_2\hat{Z}_1 - \hat{z}_1\hat{Z}_2 + \hat{w}_2\hat{W}_1 - \hat{w}_1\hat{W}_2 + \frac{\lambda(p-r)}{2\sqrt{p\lambda}}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2) - \frac{r}{\sqrt{p\lambda}}S(\hat{z}, \hat{w}).$$

The last step uses the scaling $\hat{Z} \to \sqrt{p\lambda}\hat{Z}$ and $\hat{W} \to \sqrt{p\lambda}\hat{W}$. This is a symplectic transformation with multiplier $1/\sqrt{p\lambda}$, so that finally the Hamiltonian is given by

$$H = \frac{1}{2}(\hat{Z}_1^2 + \hat{Z}_2^2 + \hat{W}_1^2 + \hat{W}_2^2) + \hat{z}_2\hat{Z}_1 - \hat{z}_1\hat{Z}_2 + \hat{w}_2\hat{W}_1 - \hat{w}_1\hat{W}_2 + \frac{p-r}{2p}(\hat{z}_1^2 + \hat{z}_2^2 + \hat{w}_1^2 + \hat{w}_2^2) - \frac{r}{p\lambda}S(\hat{z}, \hat{w}).$$

With $\lambda = \beta^3$ the corresponding system of differential equations has a stationary solution for $\hat{z}_1 = \hat{Z}_2 = 1$, $\hat{z}_2 = \hat{Z}_1 = \hat{w}_1 = \hat{w}_2 = \hat{W}_1 = \hat{W}_2 = 0$. In order to study what happens to solutions nearby we look at the variational equations. We set

$$\hat{z}_1 = 1 + \bar{z}_1, \quad \hat{z}_2 = \bar{z}_2, \quad \hat{Z}_1 = \bar{Z}_1, \quad \hat{Z}_2 = 1 + \bar{Z}_2,$$

$$\hat{w}_1 = \bar{w}_1, \quad \hat{w}_2 = \bar{w}_2, \quad \hat{W}_1 = \bar{W}_1, \quad \hat{W}_2 = \bar{W}_2,$$

and determine the second-order terms of the Hamiltonian H:

$$\begin{aligned} H_2 = {} & \frac{1}{2}(\bar{Z}_1^2 + \bar{Z}_2^2) + \bar{Z}_1\bar{z}_2 - \bar{Z}_2\bar{z}_1 - \frac{2 - e\cos f}{2(1 + e\cos f)}\bar{z}_1^2 + \frac{1}{2}\bar{z}_2^2 \\ & + \frac{1}{2}(\bar{W}_1^2 + \bar{W}_2^2) + \bar{W}_1\bar{w}_2 - \bar{W}_2\bar{w}_1 \\ & + \frac{1}{8(1 + e\cos f)}((-8 + 9m_2 + 9m_3 + 4e\cos f)\bar{w}_1^2 \\ & - 6\sqrt{3}(m_2 - m_3)w_1w_2 + (4 - 9m_2 - 9m_3 + 4e\cos f)\bar{w}_2^2). \end{aligned}$$

The matrix Φ for the variational equations follows from this Hamiltonian. □

As expected the variational equations split into two components. The part corresponding to the first line in H_2 gives the variational equations for elliptic orbits in the two body problem. The four characteristic multipliers in this case are all 1. Thus we can concentrate on the second part in H_2. With the vector $\bar{w} = (\bar{w}_1, \bar{w}_2, \bar{W}_1, \bar{W}_2)$ the variational equations in matrix form are

$$\frac{d\bar{w}}{dt} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{8-9m_2-9m_3-4e\cos f}{4(1+e\cos f)} & \frac{3\sqrt{3}(m_2-m_3)}{4(1+e\cos f)} & 0 & 1 \\ \frac{3\sqrt{3}(m_2-m_3)}{4(1+e\cos f)} & \frac{-4+9m_2+9m_3-4e\cos f}{4(1+e\cos f)} & -1 & 0 \end{bmatrix} \bar{w}. \tag{16}$$

It is advisable to bring the 2×2 submatrix in the lower left corner of the matrix in (16) into diagonal form, as then the matrix will only depend on the parameter β. This diagonalization can be accomplished by a rotation in the position and in the momenta space.

Since the sum of the three masses is equal to 1, given β we can then determine the masses m_1 and m_2 in terms of m_3 and β by

$$m_1 = \tfrac{1}{2}(1 - m_3 - \sqrt{(1 - m_3)(1 + 3m_3) - 4\beta^2}),$$

$$m_2 = \tfrac{1}{2}(1 - m_3 + \sqrt{(1 - m_3)(1 + 3m_3) - 4\beta^2}).$$

Since the above square root will appear frequently in the matrix for the rotation we abbreviate by $S = \sqrt{(1 - m_3)(1 + 3m_3) - 4\beta^2}$. The rotation matrix for the position variables and also for the momenta is given by the orthonormal matrix

$$T = \begin{bmatrix} \frac{-1+3m_3+3S-4\sqrt{1-3\beta^2}}{2\sqrt{2}\sqrt{4(1-3\beta^2)+(1-3m_3-3S)\sqrt{1-3\beta^2}}} & \frac{-1+3m_3+3S+4\sqrt{1-3\beta^2}}{2\sqrt{2}\sqrt{4(1-3\beta^2)-(1-3m_3-3S)\sqrt{1-3\beta^2}}} \\ \frac{\sqrt{3}(-1+3m_3-S)}{2\sqrt{2}\sqrt{4(1-3\beta^2)+(1-3m_3-3S)\sqrt{1-3\beta^2}}} & \frac{\sqrt{3}(-1+3m_3+S)}{2\sqrt{2}\sqrt{4(1-3\beta^2)-(1-3m_3-3S)\sqrt{1-3\beta^2}}} \end{bmatrix}.$$

We thus have the following result, which can be verified by carrying out the transformation.

Proposition 3.1. *The transformed variational equations only depend on β or on a related parameter*

$$\sigma = \sqrt{27}\beta$$

and the variational equations are given by

$$\frac{dw}{dt} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{1+\sqrt{9-\sigma^2}-2e\cos f}{2(1+e\cos f)} & 0 & 0 & 1 \\ 0 & \frac{1-\sqrt{9-\sigma^2}-2e\cos f}{2(1+e\cos f)} & -1 & 0 \end{bmatrix} w. \tag{17}$$

For the coordinates in the rotated frame we have reused the vector $w = (w_1, w_2, W_1, W_2)$. *The variational equations can also represented by the Hamiltonian*

$$\begin{aligned} H &= \frac{(1-2e\cos f)(w_1^2 + w_2^2) - \sqrt{9-\sigma^2}(w_1^2 - w_2^2)}{4(1+e\cos f)} \\ &\quad + \frac{1}{2}(W_1^2 + W_2^2) + w_2 W_1 - w_1 W_2. \end{aligned} \tag{18}$$

3.3. Versal normal form

The change of coordinates and the resulting equations are very similar to those of the elliptic restricted three-body problem near the Lagrangian point L_4. Thus, it is not surprising, that the methods used in [22] are also applicable here. It appears impossible to integrate the Eq. (17) in closed form. Instead we treat e as a small parameter and work with series expansion in e. We can bring these series into normal form with the help of the Lie transformation of Deprit [12]. Before this can be done the matrix (17) has to be put into normal form when $e = 0$. The matrix in (17) is then

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{1}{2}(1+\sqrt{9-\sigma^2}) & 0 & 0 & 1 \\ 0 & \frac{1}{2}(1-\sqrt{9-\sigma^2}) & -1 & 0 \end{bmatrix}. \tag{19}$$

It has the eigenvalues $\pm\sqrt{(-1 \pm \sqrt{1-\sigma^2})/2} = \pm \mathrm{i}(\sqrt{1+\sigma} \pm \sqrt{1-\sigma})/2$. Since the stability changes at $\sigma = 1$ (or $\beta = 1/\sqrt{27}$) we will use a versal normal form near $\sigma = 1$. With complex coordinates the versal for (17) is given by

$$\Lambda = \begin{bmatrix} \frac{\mathrm{i}}{2}\sqrt{1+\sigma} & 0 & 0 & \frac{1-\sigma}{4} \\ 0 & -\frac{\mathrm{i}}{2}\sqrt{1+\sigma} & \frac{1-\sigma}{4} & 0 \\ 0 & -1 & -\frac{\mathrm{i}}{2}\sqrt{1+\sigma} & 0 \\ -1 & 0 & 0 & \frac{\mathrm{i}}{2}\sqrt{1+\sigma} \end{bmatrix}. \tag{20}$$

The transformation matrix $T = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ has complex valued column vectors with $\alpha_2 = \overline{\alpha}_1$ and $\alpha_3 = \overline{\alpha}_4$. The condition $T\Lambda = AT$ results in the set of equations

$$(A - \mathrm{i}\sqrt{1-\sigma^2}\ \mathbb{I})\alpha_1 = \alpha_4, \tag{21}$$

$$(A - \mathrm{i}\sqrt{1-\sigma^2}\ \mathbb{I})\alpha_4 = \frac{1-\sigma}{4}\alpha_1. \tag{22}$$

By substituting (22) into (21) we obtain the system of equations

$$\left((A - \mathrm{i}\sqrt{1-\sigma^2}\ \mathbb{I})^2 - \frac{1-\sigma}{4}\ \mathbb{I}\right)\alpha_1 = 0$$

for α_1. The system of equations has two linearly independent solutions, given here with the two arbitrary parameters r_1 and r_2

$$\alpha_1 = r_1 \begin{bmatrix} -2(3+\sigma) \\ -\mathrm{i}\sqrt{1+\sigma}(3+\sigma+\sqrt{9-\sigma^2}) \\ 0 \\ (2+\sigma)\sqrt{9-\sigma^2} \end{bmatrix} + r_2 \begin{bmatrix} \mathrm{i}\sqrt{1+\sigma}(3+\sigma-\sqrt{9-\sigma^2}) \\ -2(3+\sigma) \\ (2+\sigma)\sqrt{9-\sigma^2} \\ 0 \end{bmatrix}.$$

The vector α_4 is found from (21) to be

$$\alpha_4 = r_1 \begin{bmatrix} \mathrm{i}\sqrt{1+\sigma}\sqrt{9-\sigma^2} \\ -\frac{1}{2}(3+\sigma)(3-\sigma+\sqrt{9-\sigma^2}) \\ 3+\sigma+\sqrt{9-\sigma^2} \\ \frac{\mathrm{i}}{2}\sqrt{1+\sigma}(\sigma+3)(\sigma-2) \end{bmatrix} + r_2 \begin{bmatrix} -\frac{1}{2}(3+\sigma)(-3+\sigma+\sqrt{9-\sigma^2}) \\ -\mathrm{i}\sqrt{1+\sigma}\sqrt{9-\sigma^2} \\ -\frac{\mathrm{i}}{2}\sqrt{1+\sigma}(3+\sigma)(\sigma-2) \\ 3+\sigma-\sqrt{9-\sigma^2} \end{bmatrix}.$$

The parameters r_1 and r_2 can now be chosen so that the transformation is symplectic. This results in the two equations

$$0 = (3+\sigma+\sqrt{9-\sigma^2})r_1^2 - (3+\sigma-\sqrt{9-\sigma^2})r_2^2,$$

$$\frac{1}{(\sigma-2)(\sigma+3)(\sigma+1)} = (3+\sigma+\sqrt{9-\sigma^2})r_1^2 + (3+\sigma-\sqrt{9-\sigma^2})r_2^2,$$

which corresponds to the intersection of two lines through the origin with an ellipse. A solution is given by

$$r_1 = \frac{1}{\sqrt{2(\sigma+1)(\sigma+3)(\sigma-2)(3+\sigma+\sqrt{9-\sigma^2})}},$$

$$r_2 = \frac{1}{\sqrt{2(\sigma+1)(\sigma+3)(\sigma-2)(3+\sigma-\sqrt{9-\sigma^2})}}.$$

We have thus shown how to construct a symplectic transformation to the versal normal form, which is summarized in the following lemma.

Lemma 3.2. *Let $\zeta = (\xi_1, \xi_2, \eta_1, \eta_2)^T$ be the coordinates for the complex normal form (20), that is $\zeta = Tw$, where ξ_1 and ξ_2 are the position coordinates and η_1 and η_2 are the momenta. Real solutions are given when $\bar{\xi}_1 = \xi_2$ and $\bar{\eta}_1 = \eta_2$. The Hamiltonian corresponding to (20) is*

$$K_{0,0} = \mathrm{i}\omega(\xi_1\eta_1 - \xi_2\eta_2) + \xi_1\xi_2 + \varepsilon\eta_1\eta_2$$

with $\varepsilon = (1-\sigma)/4$ and $\omega = \frac{1}{2}\sqrt{1+\sigma}$. Values of interest are those of σ near 1.

Remark. If the real versal normal form of (19) is desired then it is given by

$$\begin{bmatrix} 0 & -\frac{\sqrt{1+\sigma}}{2} & \frac{1-\sigma}{4} & 0 \\ \frac{\sqrt{1+\sigma}}{2} & 0 & 0 & \frac{1-\sigma}{4} \\ -1 & 0 & 0 & -\frac{\sqrt{1+\sigma}}{2} \\ 0 & -1 & \frac{\sqrt{1+\sigma}}{2} & 0 \end{bmatrix}$$

and the transformation to it is given by the matrix $\sqrt{2}(\mathrm{Re}(\alpha_1), \mathrm{Im}(\alpha_1), -\mathrm{Re}(\alpha_4), -\mathrm{Im}(\alpha_4))$.

3.4. Lie transformation to normal form when $e \neq 0$

Lemma 3.3. *For $e \neq 0$ the normal form of the Hamiltonian for the variational problem near $\sigma = 1$ is given by*

$$\begin{aligned} K^* &= \mathrm{i}\omega(\xi_1\eta_1 - \xi_2\eta_2) + \xi_1\xi_2 + \varepsilon\eta_1\eta_2 \\ &\quad + \sum_{n=1} \frac{e^{2n}}{(2n)!}(\mathrm{i}p_{2n}(\xi_1\xi_2 - \eta_1\eta_2) + q_{2n}\eta_1\eta_2), \end{aligned} \tag{23}$$

where p_{2n} and q_{2n} are numerical constants, which can be considered to be corrections to ω and ε when $e \neq 0$ and $\sigma \neq 1$.

Proof. When the eccentricity e is not zero the Hamiltonian for the variational equations has the form

$$K = K_{0,0} + \sum_{n=1} \frac{e^n}{n!} K_{0,n}.$$

The terms in $K_{0,n}$ have the form

$$\alpha e^{ikf}\xi_1^{i_1}\xi_2^{i_2}\eta_1^{j_1}\eta_2^{j_2} \quad \text{with} \quad i_1+i_2+j_1+j_2=2 \quad \text{and} \quad |k|\leqslant n.$$

The Lie transformation uses a near identity transformation, which is generated by

$$W=\sum_{n=0}\frac{e^n}{n!}W_{n+1},$$

where the functions W_{n+1} are constructed order-by-order so that the transformed Hamiltonian

$$K^*=K_{0,0}+\sum_{n=1}\frac{e^n}{n!}K_{n,0}$$

is as simple as possible. Since our Hamiltonian is nonautonomous the nth order terms are determined by

$$K_{n,0}=P+\left\{K_{0,0};\,W_n\right\}-\frac{\partial W_n}{\partial f},$$

where P depends only on known terms and $\{\,\cdot\,;\,\cdot\,\}$ denotes the usual Poisson bracket. All terms in the range of the operator

$$\left\{K_{0,0};\,\cdot\,\right\}-\frac{\partial}{\partial f} \tag{24}$$

can be eliminated. Since there are only 10 quadratic monomials in the four variables of ζ we can determine the kernel of the adjoint operator of (24) directly. Representation theory of sl$(2, R)$ shows that the space of quadratic monomials splits into four invariant subspaces for (24), see [4]. Bases for these subspaces are

$$\{\eta_1^2, 2\xi_2\eta_1, 2\xi_2^2\},\ \{\eta_1\eta_2, \xi_1\eta_1+\xi_2\eta_2, 2\xi_1\xi_2\},\ \{\eta_2^2, 2\xi_1\eta_2, 2\xi_1^2\},\ \{\mathrm{i}(\xi_1\eta_1-\xi_2\eta_2)\}. \tag{25}$$

Consider the first subspace given in (25) and let

$$\alpha_1(f)\eta_1^2+2\beta_1(f)\xi_2\eta_1+2\gamma_1(f)\xi_2^2$$

be the term in P which lies in this subspace. We try to find the coefficients $a_1(f)$, $b_1(f)$ and $c_1(f)$ for the corresponding terms in W_n so that the terms in $K_{n,0}$ are zero. In matrix form this condition reads

$$\frac{d}{df}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} - \begin{bmatrix} 2\mathrm{i}\omega & -2\varepsilon & 0 \\ 1 & 2\mathrm{i}\omega & -2\varepsilon \\ 0 & 1 & 2\mathrm{i}\omega \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}.$$

The right-hand side contains various powers of $\mathrm{e}^{\mathrm{i}f}$, but with ω near $\sqrt{2}/2$ there is no problem with resonances and all these terms in $K_{n,0}$ can be eliminated by selecting the appropriate functions $a_1(f)$, $b_1(f)$ and $c_1(f)$.

For the second subspace in (25) we obtain the condition

$$\frac{d}{df}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 & -2\varepsilon & 0 \\ 1 & 0 & -2\varepsilon \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix}.$$

Again for a periodic right-hand side a solution for $a_2(f)$, $b_2(f)$ and $c_2(f)$ exists provided that $\alpha_2(f)$ does not contain a constant term. It means that all terms in $K_{0,n}$ can be eliminated with the exception of $q_n\eta_1\eta_2$ with q_n being a constant term.

For the third subspace in (25) we obtain the condition

$$\frac{d}{df}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} - \begin{bmatrix} -2\mathrm{i}\omega & -2\varepsilon & 0 \\ 1 & -2\mathrm{i}\omega & -2\varepsilon \\ 0 & 1 & -2\mathrm{i}\omega \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{bmatrix}.$$

For all possible right-hand sides there exists a solution for $a_3(f)$, $b_3(f)$ and $c_3(f)$ so that all corresponding terms in $K_{0,n}$ can be eliminated. Finally for the last, one-dimensional subspace in (25) the normalizing condition reads

$$\frac{da_4}{df} = \alpha_4(f)$$

so that all terms except $\mathrm{i}p_n(\xi_1\xi_2 - \eta_1\eta_2)$ with p_n constant can be eliminated. Due to the way in which the true anomaly enters into the Hamiltonian, p_n and q_n will be zero for odd n and real for even n. Thus the normal form of the Hamiltonian is

$$\begin{aligned} K^* &= \mathrm{i}\omega(\xi_1\eta_1 - \xi_2\eta_2) + \xi_1\xi_2 + \varepsilon\eta_1\eta_2 \\ &\quad + \sum_{n=1} \frac{e^{2n}}{(2n)!}(\mathrm{i}p_{2n}(\xi_1\xi_2 - \eta_1\eta_2) + q_{2n}\eta_1\eta_2). \qquad \square \end{aligned}$$

Since the higher-order terms are thus simply corrections to ε and ω, we set

$$w = \omega + \sum_{n=1} \frac{e^{2n}}{(2n)!} p_{2n} \quad \text{and} \quad \varepsilon = \varepsilon + \sum_{n=1} \frac{e^{2n}}{(2n)!} q_{2n}$$

and get

$$K^* = \mathrm{i}w(\xi_1\eta_1 - \xi_2\eta_2) + \xi_1\xi_2 + \varepsilon\eta_1\eta_2.$$

From this we derive the differential equations

$$\begin{aligned} \xi_1' &= \mathrm{i}w\xi_1 + \varepsilon\eta_2, \\ \xi_2' &= -\mathrm{i}w\xi_2 + \varepsilon\eta_1, \\ \eta_1' &= -\mathrm{i}w\eta_1 - \xi_2, \\ \eta_2' &= \mathrm{i}w\eta_2 - \xi_1. \end{aligned}$$

Due to the reality conditions we only have to consider the first and last of these equations. They are easily solved by setting

$$\xi_1 = \mathrm{e}^{\mathrm{i}wf} x \quad \text{and} \quad \eta_2 = \mathrm{e}^{\mathrm{i}wf} y$$

which gives

$$x' = \varepsilon y \quad \text{and} \quad y' = -x.$$

Therefore, the change in stability occurs when $\varepsilon = 0$ that is for

$$\varepsilon = -\sum_{n=1} \frac{e^{2n}}{(2n)!} q_{2n}.$$

3.5. *Direct computation of the stability boundary*

Unfortunately, the computational effort is substantial when the method of Lie transformation is used. Nevertheless, the knowledge obtained in the previous section allows us to determine the stability boundary more easily by finding it directly from (16) or from the equivalent system of second-order differential equations

$$\begin{aligned} Q_1'' - 2Q_2' - g(1+h)Q_1 &= 0, \\ Q_2'' + 2Q_1' - g(1-h)Q_2 &= 0 \end{aligned}$$

with

$$g = \frac{3}{2(1 + e\cos f)} \quad \text{and} \quad h = \sqrt{1 - \sigma^2/9}.$$

The system can be written with complex position coordinates $z = Q_1 + \mathrm{i}Q_2$ as

$$z'' + 2\mathrm{i}z' - gz - gh\bar{z} = 0. \tag{26}$$

We will look for solutions of the form

$$z = u\mathrm{e}^{\mathrm{i}wf} + v\mathrm{e}^{-\mathrm{i}wf}.$$

The resulting differential equations are then

$$\begin{aligned} u'' + 2\mathrm{i}(w+1)u' - (w^2 + 2w + g)u - gh\bar{v} &= 0, \\ v'' - 2\mathrm{i}(w-1)v' - (w^2 - 2w + g)v - gh\bar{u} &= 0. \end{aligned}$$

The solution will be found as a series in e order-by-order, that is, we set

$$\begin{aligned} u &= \sum_{n=0} u_n e^n, \\ v &= \sum_{n=0} v_n e^n, \\ w &= \frac{\sqrt{2}}{2} + \sum_{n=1} w_n e^n, \\ \varepsilon &= \frac{1}{4}\left(1 - \sigma + \sum_{n=1} \sigma_n e^n\right). \end{aligned}$$

Since we want to determine the stability boundary and we know that it occurs for $\varepsilon = 0$ the series to be used for σ is

$$\sigma = 1 + \sum_{n=1} \sigma_n e^n.$$

The function g is given by

$$g = \frac{3r}{2p} = \frac{3}{2} + \sum_{n=1} g_n e^n \quad \text{with} \quad g_n = \frac{3(-1)^n}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \mathrm{e}^{\mathrm{i}(n-2k)f}. \tag{27}$$

Since

$$h = \sqrt{1-\sigma^2/9} = \frac{2\sqrt{2}}{3} + \sum_{n=1} h_n e^n$$

it can also be found as a series in e.

The differential equations for u_0 and v_0 read

$$\begin{aligned} u_0'' + \mathrm{i}(\sqrt{2}+2)u_0' - (\sqrt{2}+2)u_0 - \sqrt{2}\,\overline{v}_0 &= 0, \\ v_0'' - \mathrm{i}(\sqrt{2}-2)v_0' + (\sqrt{2}-2)v_0 - \sqrt{2}\,\overline{u}_0 &= 0. \end{aligned}$$

These equations have the constant solution

$$u_0 = 1 - \sqrt{2}/2, \quad v_0 = -\sqrt{2}/2$$

and a linear solution in f, which is of no interest at the moment. The differential equations for u_n and v_n have the form

$$u_n'' + \mathrm{i}(\sqrt{2}+2)u_n' - (\sqrt{2}+2)u_n - \sqrt{2}\,\overline{v}_n = \alpha, \tag{28}$$

$$v_n'' - \mathrm{i}(\sqrt{2}-2)v_n' + (\sqrt{2}-2)v_n - \sqrt{2}\,\overline{u}_n = \beta. \tag{29}$$

Due to (27) the functions α and β have the form

$$\alpha = \sum_{k=-n}^{n} \alpha_k \mathrm{e}^{\mathrm{i}kf}, \quad \beta = \sum_{k=-n}^{n} \beta_k \mathrm{e}^{\mathrm{i}kf},$$

where all coefficients depend on terms which have already been calculated. There is one exception, since α_0 and β_0 contain w_n and σ_n in a linear manner. At each order $n > 0$ we set

$$u_n = \sum_{k=-n}^{n} A_k \mathrm{e}^{\mathrm{i}kf}, \quad v_n = \sum_{k=-n}^{n} B_k \mathrm{e}^{\mathrm{i}kf}.$$

By substituting into (28) and (29) and comparing coefficients we obtain for $k \neq 0$

$$\begin{aligned} -(k^2 + (\sqrt{2}+2)k + 2 + \sqrt{2})A_k - \sqrt{2}B_{-k} &= \alpha_k, \\ -\sqrt{2}A_k - (k^2 - (2-\sqrt{2})k + 2 - \sqrt{2})B_{-k} &= \beta_{-k}. \end{aligned}$$

The system of equations has the solution

$$A_k = \frac{(-k^2 + (2-\sqrt{2})(k-1))\alpha_k + \sqrt{2}\beta_{-k}}{k^2(k+\sqrt{2})^2},$$
$$B_{-k} = \frac{\sqrt{2}\alpha_k - (k^2 + (2+\sqrt{2})(k+1))\beta_{-k}}{k^2(k+\sqrt{2})^2}.$$

For $k = 0$ the equations from (28) and (29) are

$$-(\sqrt{2}+2)A_0 - \sqrt{2}B_0 = \alpha_0,$$
$$-\sqrt{2}A_0 + (\sqrt{2}-2)B_0 = \beta_0.$$

At all orders $n > 0$ we are free to choose the initial conditions for the solution and we can select $A_0 = B_0 = 0$. We then solve $\alpha_0 = 0$ and $\beta_0 = 0$ for σ_n and w_n. The boundary curve for the change in stability is then $\sigma = 1 + \sum_{n=1} \sigma_n e^n$. By performing these computations we obtain the following result:

Theorem 3.1. *The curve where two nontrivial multipliers of the variational equations are* 1, *is given by*

$$\sigma = 1 + e^2 - \frac{17}{16}e^4 + \frac{803}{256}e^6 - \frac{2\,416\,719}{229\,376}e^8 + \frac{20\,166\,411\,233}{51\,380\,224}e^{10} \pm \cdots. \tag{30}$$

To the left of the curve the multipliers are on the unit circle, but to the right they are on the real axis, resulting in linear instability.

3.6. Normal form for the short and long period families

For the moment consider zero eccentricity, that is $e = 0$ in (17). Then for $0 \leqslant \sigma < 1$, that is $0 < \beta < 1/\sqrt{27}$, the eigenvalues of the matrix in (17) are purely imaginary. They are of the form $\pm \mathrm{i}\omega_\mathrm{s}$, $\pm \mathrm{i}\omega_\mathrm{l}$ with $0 < \omega_\mathrm{l} < \frac{1}{\sqrt{2}} < \omega_\mathrm{s} < 1$. The frequencies are given by

$$\omega_\mathrm{l} = \sqrt{\frac{1}{2}(1 - \sqrt{1-\sigma^2})}, \quad \omega_\mathrm{s} = \sqrt{\frac{1}{2}(1 + \sqrt{1-\sigma^2})}.$$

The indices were chosen to remind of the families of long and short periodic orbits, as the situation is very similar to the one near the Lagrangian point L_4 in the restricted problem of three bodies.

Lemma 3.4. *With action angle variables* $(\rho_1, \rho_2, \varphi_1, \varphi_2)$ *the Hamiltonian of the variational equations* (17) *has the normal form*

$$H = \omega_{\mathrm{s}}\rho_1 - \omega_{\mathrm{l}}\rho_2$$

for $0 < \sigma < 1$ *and* $e = 0$.

Proof. For $e = 0$ the matrix in (17) can be brought into real normal form by the transformation $w = Tx$, with the column vector as used previously and $x = (x_1, x_2, X_1, X_2)^T$. The transformation matrix is

$$T = \begin{bmatrix} \frac{-4-\sqrt{1-\sigma^2}+\sqrt{9-\sigma^2}}{c_1} & \frac{4-\sqrt{1-\sigma^2}-\sqrt{9-\sigma^2}}{c_2} & 0 & 0 \\ 0 & 0 & -\frac{4\omega_{\mathrm{s}}}{c_1} & -\frac{4\omega_{\mathrm{l}}}{c_2} \\ 0 & 0 & (\frac{\sqrt{9-\sigma^2}-\sqrt{1-\sigma^2})\omega_{\mathrm{s}}}{c_1} & (\frac{\sqrt{9-\sigma^2}+\sqrt{1-\sigma^2})\omega_{\mathrm{l}}}{c_2} \\ \frac{-2+\sqrt{1-\sigma^2}+\sqrt{9-\sigma^2}}{c_1} & \frac{2+\sqrt{1-\sigma^2}-\sqrt{9-\sigma^2}}{c_2} & 0 & 0 \end{bmatrix}.$$

The factors c_1 and c_2 are chosen such that T is symplectic and they are given by

$$c_1^2 = \sqrt{1-\sigma^2}(4 + \sqrt{1-\sigma^2} - \sqrt{9-\sigma^2})\omega_{\mathrm{s}}/2,$$
$$c_2^2 = \sqrt{1-\sigma^2}(4 - \sqrt{1-\sigma^2} - \sqrt{9-\sigma^2})\omega_{\mathrm{l}}/2.$$

In Cartesian coordinates the Hamiltonian is therefore

$$H = \frac{\omega_{\mathrm{s}}}{2}(x_1^2 + X_1^2) - \frac{\omega_{\mathrm{l}}}{2}(x_2^2 + X_2^2)$$

from which the normal form in action angle variables follows. □

For $e \neq 0$ the above transformation produces many additional terms. As before we expand the transformed Hamiltonian (18) into a series in e and obtain

$$H = \omega_{\mathrm{s}}\rho_1 - \omega_{\mathrm{l}}\rho_2 + \sum_{n \geqslant 1} \sum A_{i_0 i_1 i_2} \cos(i_1\varphi_1 + i_2\varphi_2 + i_0 f). \tag{31}$$

The indices for the inner sum are restricted by

$$|i_1| + |i_2| \leqslant 2 \quad \text{and} \quad |i_0| \leqslant n, \tag{32}$$

and the coefficients $A_{i_0 i_1 i_2}$ are homogeneous polynomials of degree 2 in $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$.

Via a Lie transformation all terms in (3.6) can be eliminated except those that are in the kernel of the operator

$$\omega_s \frac{\partial}{\partial \varphi_1} - \omega_l \frac{\partial}{\partial \varphi_2} - \frac{\partial}{\partial f}. \tag{33}$$

Due to the restriction (32) on the indices, resonances between ω_l and ω_s do not play a role at this order, but when $\omega_l = \frac{1}{2}$ a resonance with the forcing function is possible. This happens when $\sigma = \frac{\sqrt{3}}{2}$. The characteristic multipliers are at -1 then and can leave the unit circle.

Lemma 3.5. *For $0 < \sigma < 1$ but $\sigma \neq \sqrt{3}/2$ the normal form of the Hamiltonian* (3.6) *is that of two harmonic oscillators. For $\sigma = \sqrt{3}/2 + \sum e^n \sigma_n$ the normal form is*

$$H = \omega_1 \rho_1 - \omega_2 \rho_2 + eA\rho_2 \cos(2\varphi_2 - f) \tag{34}$$

where $\omega_1 = \sqrt{3}/2 + e\tilde{\omega}_1$, $\omega_2 = \frac{1}{2} + e\tilde{\omega}_2$. The functions $\tilde{\omega}_1$, $\tilde{\omega}_2$ and A depend on e and on σ_n, $n = 1, 2, \ldots$.

Proof. Roberts has already shown in [18] the existence of this exceptional value and that it gives rise to an interval of instability when $e \neq 0$, a feature which was missed by Danby in [6]. That (34) is the normal form of (3.6) follows from the kernel (33). □

Lemma 3.6. *For $|\tilde{\omega}_2| > |A|$ the origin in the Hamiltonian* (34) *is stable, and unstable when $|\tilde{\omega}_2| < |A|$.*

Proof. With the new variable $\psi_2 = \varphi_2 - \frac{1}{2} f$ and without the ignorable variable ρ_1 the Hamiltonian (34) becomes $H = e\rho_2(-\tilde{\omega}_2 + A\cos 2\psi_2)$. By looking at the level curves of this function the result follows. Therefore, change of stability occurs when $|\tilde{\omega}_2| = |A|$, that is at solutions with twice the period of the forcing function. □

3.7. *Direct computation of the period doubling curves*

The series expansion for the period doubling curve $\sigma = \sqrt{3}/2 + \sum_{n>0} \sigma_n e^n$ is found more easily by solving (26) directly. The function g is given in (27), but this time

$$h = \sqrt{1 - \frac{\sigma^2}{9}} = \frac{\sqrt{33}}{4} + \sum h_n e^n$$

and we are looking for solutions of the form

$$z = \sum_{n \geqslant 0} z_n e^n \quad \text{with} \quad z_n = \sum_{k=-2n-1}^{2n+1} z_{n,k} \mathrm{e}^{\mathrm{i}kf/2}.$$

For $n = 0$ a nonzero π-periodic solution of (26) is

$$z_0 = \sqrt{3}\mathrm{e}^{\mathrm{i}f/2} - \sqrt{11}\mathrm{e}^{-\mathrm{i}f/2}.$$

For $n > 0$ the differential equations (26) leads to the algebraic equations

$$\left(-\frac{k^2}{4} - k - \frac{3}{2}\right) z_{n,k} - \frac{\sqrt{33}}{4} z_{n,-k} = \alpha_{n,k}, \tag{35}$$

$$-\frac{\sqrt{33}}{4} z_{n,k} + \left(-\frac{k^2}{4} + k - \frac{3}{2}\right) z_{n,-k} = \alpha_{n,-k}, \tag{36}$$

with $k = 1, \ldots, 2n+1$ and where $\alpha_{n,k}$ and $\alpha_{n,-k}$ depend only on known terms from previous orders. One exception is $\alpha_{n,1}$ and $\alpha_{n,-1}$, which depend linearly on σ_n.

The determinant for (35,36) is $(k^2 - 1)(k^2 - 3)/16$, so that he coefficients $z_{n,k}$ can be found uniquely for $k \geqslant 2$. When $k = 1$, Eq. (35) can be used to determine also σ_n. In order to make the solution unique an additional condition like $z_{n,1} = z_{n,-1}$ can be imposed. Finally, setting $z_{n,0} = 0$ at all orders will satisfy (35) when $k = 0$.

Theorem 3.2. *A period doubling curve starts at $\sigma = \sqrt{3}/2$, that is at $\beta = \frac{1}{6}$. A series expansion for this period doubling curve is given by*

$$\sigma = \frac{\sqrt{3}}{2} + \frac{\sqrt{11}}{4} e - \frac{25}{64\sqrt{3}} e^2 - \frac{483}{1024\sqrt{11}} e^3 - \frac{2113}{49\,152\sqrt{3}} e^4 - \frac{565\,461}{2\,883\,584\sqrt{11}} e^5 - \cdots. \tag{37}$$

Proof. A computer algebra program was used to calculate the series by the method outlined above to an order much higher than what is listed. The perigee of an elliptic orbit can shift by 180 degrees when the eccentricity goes through zero. Negative values for the eccentricity can be interpreted in this way and therefore the series (37) is valid

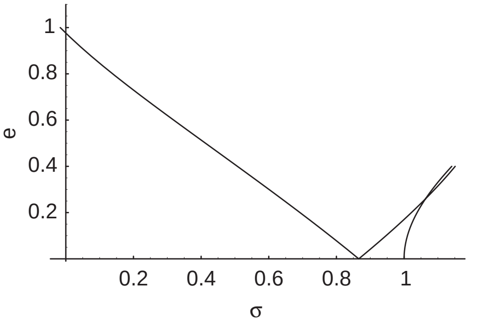

Fig. 1. Curves in the e–σ plane, where the stability Lagrangian triangular configuration changes. The curves are plotted from series expansions in e.

for positive and negative values of e. Nevertheless, the period doubling curve is drawn only in the first quadrant of the e–σ plane in Fig. 1 in order to visualize the region of instability starting on the σ-axis at $\sigma = \sqrt{3}/2$. The curve to the right of $\sigma = \sqrt{3}/2$ belongs to $e > 0$ and the one to the left to $e < 0$.

Since the coefficients in the series (37) appear to be bounded we can plot σ as a function of e for $|e| \leqslant 1$. On the other hand, the coefficients of the series in (30) increase in magnitude when higher-order terms are computed. When graphing this curve it becomes apparent that the series (30) converges for about $|e| < 0.4$.

Fig. 1 is in agreement with the numerical work of Roberts in [18]. He finds that the stability curve starting at $\sigma = 1$ and the one starting at $\sigma = \sqrt{3}/2$ become tangent to each other. Roberts also finds that the period doubling curve is tangent to the e-axis at $e = 1$. That it does not show up in our figure is not surprising, since we have computed series, which are only accurate for small values of e. $\square$

4. Regular polygon configurations

For regular polygon configurations with a central mass it is inconvenient to normalize the masses so that their sum is equal to one. Instead the transformation matrix A, which was constructed in Theorem 2.1, can be modified in order to accommodate this case. For arbitrary masses the first two columns of A, that is c_1 and c_2, have to be divided by $\sqrt{\sum_{i=1}^{n} m_i}$ and the next two columns by $\sqrt{\sum_{i=1}^{n} |a_i|^2 m_i}$.

For regular polygon configurations circulant matrices are useful. They are defined as follows:

Definition 4.1. Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$ be an nth root of unity. A circulant matrix has the following form:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^2 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}.$$

For properties of circulant matrices see for example [3]. Here, it suffices to state that each complex number ω^k corresponds to a 2×2 submatrix of A_{ij} of our transformation matrix since it has the special form as required by Theorem 2.1:

$$\begin{bmatrix} \cos 2\pi k/n & -\sin 2\pi k/n \\ \sin 2\pi k/n & \cos 2\pi k/n \end{bmatrix}.$$

When considering a configuration with n unit masses at the vertices of a regular polygon and a mass of size μ at the center of the polygon we can use circulant matrices to determine the first $2n$ columns of the transformation matrix. The column c_{2n+1} can be set up like c_1 except that the last nonzero component in c_{2n+1} will be given a different value, so that c_1 and c_{2n+1} are orthogonal to each other.

As the easiest example we consider first the regular triangular configuration with a central mass. The three unit masses will be unit distance away from the mass μ at the origin. When the three bodies move on circular orbits they appear to be at a relative equilibrium in a coordinate system, which rotates uniformly with angular velocity $\omega = \sqrt{\sqrt{3}/3 + \mu}$ around the origin. In order to consider the case where the three bodies move on elliptic orbits we first carry out the transformation as outlined above.

Proposition 4.1. *Let $m_1 = m_2 = m_3 = 1$ and $m_4 = \mu$ denote the masses in a four body problem. A solution of (3) is the central configuration $a_1 = (1, 0)$, $a_2 = (-1/2, \sqrt{3}/2)$, $a_3 = (-1/2, -\sqrt{3}/2)$, and $a_4 = (0, 0)$. It represents an equilateral triangular solution with equal masses and another mass at the origin. Let $X = (g, z, w_1, w_2) \in \mathbb{R}^8$ be the transformed position vector and $Q = (q_1, q_2, q_3, q_4)$ the original position vector. With the transformation $Q = AX$ for the position coordinates and $P = A^{-T}Y$ for the momenta the transformed Hamiltonian is given by*

$$H = \tfrac{1}{2}(\| G \|^2 + \| Z \|^2 + \| W_1 \|^2 + \| W_2 \|^2) - S(z, w_1, w_2, \mu).$$

In the subspace $\mathcal{B}$, where $g = w_1 = w_2 = G = W_1 = W_2 = 0$, the Hamiltonian represents the Kepler problem

$$H_{\mathrm{K}} = \frac{1}{2} \parallel Z \parallel^2 - \frac{3(1+\sqrt{3}\mu)}{\parallel z \parallel}.$$

Proof. The transformation matrix is constructed as outlined above and given by

$$A = \begin{bmatrix} \frac{1}{\sqrt{3+\mu}} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} & 0 \\ 0 & \frac{1}{\sqrt{3+\mu}} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} \\ \frac{1}{\sqrt{3+\mu}} & 0 & -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} & 0 \\ 0 & \frac{1}{\sqrt{3+\mu}} & \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{3}}{6} & 0 & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} \\ \frac{1}{\sqrt{3+\mu}} & 0 & -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} & 0 \\ 0 & \frac{1}{\sqrt{3+\mu}} & -\frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} & 0 & \frac{-\sqrt{\mu}}{\sqrt{3(3+\mu)}} \\ \frac{1}{\sqrt{3+\mu}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{\sqrt{\mu(3+\mu)}} & 0 \\ 0 & \frac{1}{\sqrt{3+\mu}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{\sqrt{\mu(3+\mu)}} \end{bmatrix}.$$

Since the Hamiltonian does not depend on g we will ignore it and its conjugate variable by setting $g = G = 0$. It should be noted that A becomes singular for $\mu = 0$. (The case $\mu = 0$ reduces to the equilateral triangle central configuration treated above.) Although negative masses are physically meaningless, we can consider them mathematically. The transformation matrix A requires that $\mu > -\sqrt{3}$, but $\lambda > 0$ in the Kepler problem of the proposition gives the more restrictive condition $\mu > -\sqrt{3}/3$. $\square$

In order to consider elliptic orbits of the Kepler problem, we again go to pulsating coordinates which rotate nonuniformly with the true anomaly of the Kepler problem. The change of coordinates are the same as those for Lemma 3.1 so that in these coordinates the Hamiltonian is given by

$$\begin{aligned} H = {} & \frac{1}{2}(\parallel \bar{Z} \parallel^2 + \parallel \bar{W}_1 \parallel^2 + \parallel \bar{W}_2 \parallel^2) - \bar{z}^T J \bar{Z} - \bar{w}_1^T J \bar{W}_1 - \bar{w}_2 J \bar{W}_2 \\ & + \frac{e \cos f}{2(1+e\cos f)}(\parallel \bar{z} \parallel^2 + \parallel \bar{w}_1 \parallel^2 + \parallel \bar{w}_2 \parallel^2) \\ & - \frac{S(\bar{z}, \bar{w}_1, \bar{w}_2, \mu)}{3(1+\sqrt{3}\mu)(1+e\cos f)}. \end{aligned} \tag{38}$$

Proposition 4.2. *The Hamiltonian* (38) *has a stationary point at* $z_1 = Z_2 = 1$, $z_2 = w_1 = w_2 = w_3 = w_4 = Z_1 = W_1 = W_2 = W_3 = W_4 = 0$. *The Hamiltonian for the variational equations in the subspace* $\mathcal{C}$ *is given by the quadratic terms of* (38). *For these terms we use the notation* $\bar{w}_1 = (w_1, w_2)$, $\bar{w}_2 = (w_3, w_4)$, $\bar{W}_1 = (W_1, W_2)$, *and* $\bar{W}_2 = (W_3, W_4)$ *and find*

$$\begin{aligned} H_2 = {} & \frac{1}{2}(W_1^2 + W_2^2 + W_3^2 + W_4^2) + w_2 W_1 - w_1 W_2 + w_4 W_3 - w_3 W_4 \\ & - \frac{1}{4(1 + e\cos f)} \left(w_1^2 + w_2^2 + \frac{\sqrt{3}(3 + \sqrt{\mu})}{1 + \sqrt{3}\mu}(w_3^2 + w_4^2) \right. \\ & \left. - \frac{6\sqrt{3\mu(3 + \mu)}}{1 + \sqrt{3}\mu}(w_1 w_3 - w_2 w_4) \right). \end{aligned}$$

Proof. Due to the rotational symmetry of elliptic relative equilibrium the entire unit circle in the z_1–z_2 plane consists of stationary points. We have selected a representative. □

We restricted our discussion to the case $e = 0$. In this case the variational equations are given by $dw/df = \Phi w$ with the column vector $w = (w_1, w_2, w_3, w_4, W_1, W_2, W_3, W_4)$. The matrix A is then independent of the true anomaly and is given by

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & -\frac{3\sqrt{3\mu(3+\mu)}}{2\sqrt{1+\sqrt{3}\mu}} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3\sqrt{3\mu(3+\mu)}}{2\sqrt{1+\sqrt{3}\mu}} & -1 & 0 & 0 & 0 \\ -\frac{3\sqrt{3\mu(3+\mu)}}{2\sqrt{1+\sqrt{3}\mu}} & 0 & \frac{\sqrt{3}(3+\mu)}{2(1+\sqrt{3}\mu)} & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{3\sqrt{3\mu(3+\mu)}}{2\sqrt{1+\sqrt{3}\mu}} & 0 & \frac{\sqrt{3}(3+\mu)}{2(1+\sqrt{3}\mu)} & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{39}$$

Proposition 4.3. *Consider an equilateral triangular configuration with three equal masses of 1 and a central body of mass* μ. *The bodies rotate on circles* ($e = 0$) *around the center of mass. This configuration is unstable for any mass* μ.

More precisely, for $0 < \mu < 0.274423$ *all eight eigenvalues of the matrix* A *given in* (39) *are in the complex plane. For* $\mu > 0.274423$ *four eigenvalues will be on the imaginary axis, but the other four will remain in the complex plane. At* $\mu = (81 + 64\sqrt{3})/249$ *two eigenvalues are zero, when bifurcations to new configurations occur. The two eigenvalues of* A *will remain on the imaginary axis when* μ *increases*

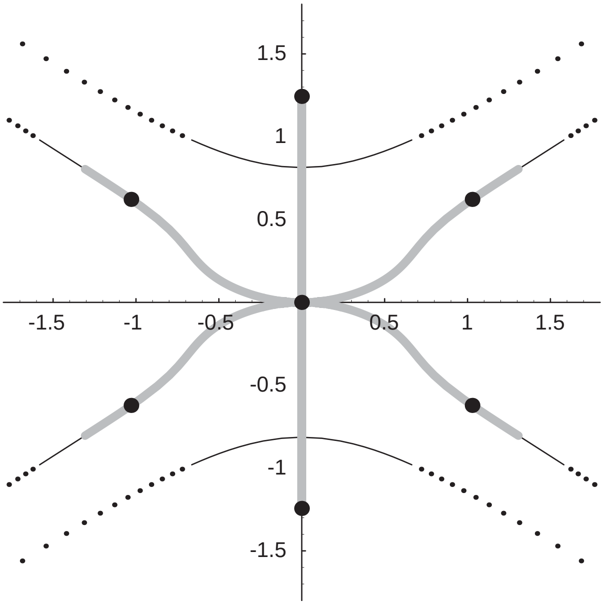

Fig. 2. Location of the eight eigenvalues of the matrix in (39) as μ varies. For $-\sqrt{3}/3 < \mu < 0$ the eigenvalues come from ∞ and are indicated by individual dots. For $0 < \mu < 0.274423$ all eigenvalues are complex and they lie on the thin solid lines. For $0.274423 < \mu$ the location of the eigenvalues is indicated by thicker gray level lines. Four eigenvalues are on the imaginary axis. The darker dots indicate the locations of the eigenvalues when $\mu = (81 + 64\sqrt{3})/249$ and thus two eigenvalues are zero.

further. When μ becomes large the eigenvalues on the imaginary axis tend to $\pm\sqrt{-1}$, whereas those in the complex plane tend towards zero as shown in Fig. 2.

Proof. The characteristic polynomial of A can be written in a special form, see [16,21],

$$(\lambda^4 + \alpha\lambda^2 + \beta)^2 + \gamma\lambda^2 = 0 \tag{40}$$

with

$$\alpha = \frac{3 - 3\sqrt{3} + 2\sqrt{3}\mu}{2(1 + \sqrt{3}\mu)},$$

$$\beta = \frac{3(2 + 3\sqrt{3} + (-18 + 5\sqrt{3})\mu)}{4(1 + \sqrt{3}\mu)^2},$$

$$\gamma = \frac{2(14 - 3\sqrt{3})}{(1 + \sqrt{3}\mu)^2}.$$

Although a configuration with a negative mass at the origin has no physical significance the eigenvalues of A can be computed as soon as the rate of rotation $\omega = \sqrt{\sqrt{3}/3 + \mu}$ is real. For $-\sqrt{3}/3 < \mu < 0$ the solutions of (40) are in the complex plane and come from complex infinity.

With $\gamma > 0$ the discriminant

$$\Delta = 16\beta(\alpha^2 - 4\beta)^2 + 4\alpha(4\alpha^2 - 36\beta)\gamma - 27\gamma^2 = 0 \tag{41}$$

can be used to determine when there are repeated roots of (40), see [16,21]. It results in a fourth-order polynomial equation for μ. Unfortunately, the solutions of this polynomial are too complicated in closed form. Two solutions are complex valued, one is negative and another one is positive. The numerical value for the positive solution is $\mu = 0.274423$. At this value two pairs of eigenvalues of A move onto the imaginary axis at 0.813956i and −0.813956i, respectively. Of these four eigenvalues two will move towards the origin and the other two away from it, as μ increases. When $\beta = 0$, that is for

$$\mu = (81 + 64\sqrt{3})/249 \approx 0.770487,$$

two eigenvalues on the imaginary axis will pass each other at the origin. The other two eigenvalues on the imaginary axis are at ±1.24362i and appear to have reached their maximum distance from the origin. The above value of μ is consistent with what is found in [14], as it allows bifurcations to new configurations.

As μ increases further all four eigenvalues on the imaginary axis will tend to ±i. The other four eigenvalues stay for all μ in the complex plane and they tend to zero as $\mu \to \infty$. □

When investigating the stability of the configuration for $e > 0$ the values of μ will be of interest where (40) has repeated solutions, that is at $\mu = 0.274423$ and at $\mu = (81 + 64\sqrt{3})/249$.

As the final example we consider the five-body problem, where four bodies of unit mass are at the corners of a square and a body of mass μ is at the center.

Proposition 4.4. *Let $m_1 = m_2 = m_3 = m_4 = 1$ and $m_5 = \mu$. A central configuration is given by $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (-1, 0)$, $a_4 = (0, -1)$, and $a_5 = (0, 0)$. Let $Q = (q_1, q_2, q_3, q_4, q_5) \in \mathbb{R}^{10}$ be the old position coordinates for the five bodies and $X = (g, z, w_1, w_2, w_3)$ the new position. The transformation $Q = AX$ and the related one for the momenta gives the Hamiltonian for the five-body problem in the form*

$$\begin{aligned} H \;=\; & \tfrac{1}{2}(\| G \|^2 + \| Z \|^2 + \| W_1 \|^2 + \| W_2 \|^2 + \| W_3 \|^2) \\ & -S(z, w_1, w_2, w_3, \mu). \end{aligned} \tag{42}$$

In the subspace $\mathcal{B}$, the Hamiltonian represents the Kepler problem

$$H_{\mathrm{K}} = \frac{1}{2} \parallel Z \parallel^2 - \frac{2+4\sqrt{2}+8\mu}{\parallel z \parallel}.$$

Proof. The transformation matrix is again constructed with the help of circulant matrices and given by

$$A = \begin{bmatrix} \frac{1}{\sqrt{4+\mu}} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} & 0 \\ 0 & \frac{1}{\sqrt{4+\mu}} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} \\ \frac{1}{\sqrt{4+\mu}} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} & 0 \\ 0 & \frac{1}{\sqrt{4+\mu}} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} \\ \frac{1}{\sqrt{4+\mu}} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} & 0 \\ 0 & \frac{1}{\sqrt{4+\mu}} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} \\ \frac{1}{\sqrt{4+\mu}} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} & 0 \\ 0 & \frac{1}{\sqrt{4+\mu}} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{-\sqrt{\mu}}{2\sqrt{4+\mu}} \\ \frac{1}{\sqrt{4+\mu}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{2}{\sqrt{\mu(4+\mu)}} & 0 \\ 0 & \frac{1}{\sqrt{4+\mu}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{2}{\sqrt{\mu(4+\mu)}} \end{bmatrix}. \qquad \square$$

Proposition 4.5. *In a nonuniformly rotating and pulsating coordinate system the Hamiltonian* (42) *becomes*

$$\begin{aligned} H \;=\; & \frac{1}{2}(\parallel \bar{Z} \parallel^2 + \parallel \bar{W}_1 \parallel^2 + \parallel \bar{W}_2 \parallel^2 + \parallel \bar{W}_3 \parallel^2) - \bar{z}^T J\bar{Z} - \bar{w}_1^T J\bar{W}_1 - \bar{w}_2 J\bar{W}_2 \\ & - \bar{w}_3 J\bar{W}_3 + \frac{e\cos f}{2(1+e\cos f)}(\parallel \bar{z} \parallel^2 + \parallel \bar{w}_1 \parallel^2 + \parallel \bar{w}_2 \parallel^2 + \parallel \bar{w}_3 \parallel^2) \\ & - \frac{S(\bar{z}, \bar{w}_1, \bar{w}_2, \bar{w}_3, \mu)}{(2+4\sqrt{2}+8\mu)(1+e\cos f)}. \end{aligned}$$

The Hamiltonian has a stationary point at $\bar{z} = (1, 0)$, $\bar{Z} = (0, 1)$ with all other two-dimensional vectors set to zero. Use $\bar{w}_i = (w_{2i-1}, w_{2i})$ for $i = 1, 2, 3$ and use a similar notation for the momenta. Then the Hamiltonian for the nontrivial part of the variational problem can be written as the sum of two separate Hamiltonian functions, *that is,*

$$H_2 = H_2^{(1)} + H_2^{(2)}$$

with

$$\begin{aligned}
H_2^{(1)} = {} & \frac{1}{2}(W_1^2 + W_2^2 + W_5^2 + W_6^2) + W_1 w_2 - W_2 w_1 + W_5 w_6 - W_5 w_6 \\
& - \frac{1}{(1+2\sqrt{2}+4\mu)(1+e\cos f)}\{(\sqrt{2}+\mu)(w_1^2 + w_2^2) \\
& -6\sqrt{\mu(4+\mu)}(w_1 w_5 - w_2 w_6) \\
& +(4+\mu)(w_5^2 + w_6^2)\}
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
H_2^{(2)} = {} & \frac{1}{2}(W_3^2 + W_4^2) + W_3 w_4 - W_4 w_3 - \frac{1}{(1+2\sqrt{2}+4\mu)(1+e\cos f)} \\
& \times \left\{(1-\sqrt{2}+4\mu)w_3^2 - \left(\frac{1}{2} - 2\sqrt{2} + 4\mu\right) w_4^2\right\}.
\end{aligned} \tag{44}$$

Proof. The coordinate transformation and the change to the true anomaly as new independent variable has been carried out as described in Lemma 3.1. With it stationary solutions are always found at the same place that is for $\| \bar{z} \| = 1$. Using circulant matrices decomposes the variational equations into two parts one with variables from the set of indices {1,2,5,6} the other one with indices from {3,4}. This has been observed previously, see [13], and it is known that for any $N+1$-gon configuration moving on circular orbits, the variational equations can be considered in eight-dimensional subspaces, and when N is even in an additional four-dimensional subspace. The proposition shows that the same holds true for elliptic central configurations.

Let $w = (w_1, w_2, w_5, w_6, W_1, W_2, W_5, W_6)$ be an eight-dimensional column vector, or the four-dimensional column vector $w = (w_3, w_4, W_3, W_4)$. Then the variational equation in the appropriate subspace is

$$\frac{dw}{df} = \Phi w \quad \text{with} \quad \Phi = \begin{bmatrix} \mathbb{J} & \mathbb{I} \\ \Psi & \mathbb{J} \end{bmatrix} \tag{45}$$

with $\mathbb{J}$ a standard four- or two-dimensional symplectic matrix, $\mathbb{I}$ a four or two-dimensional identity matrix and Ψ the Hessian with respect to the position variables. For other $N+1$-gon configurations the structure of the variational equations (45) will remain. $\square$

We finish our discussion by analyzing the variational equations, which follow from (43) and (44), when $e=0$. For (43) the matrix Ψ in (45) is

$$\Psi = \frac{2}{1+2\sqrt{2}+4\mu}\begin{bmatrix} \sqrt{2}+\mu & 0 & -3\sqrt{\mu(4+\mu)} & 0 \\ 0 & \sqrt{2}+\mu & 0 & 3\sqrt{\mu(4+\mu)} \\ -3\sqrt{\mu(4+\mu)} & 0 & 4+\mu & 0 \\ 0 & 3\sqrt{\mu(4+\mu)} & 0 & 4+\mu \end{bmatrix}.$$

The characteristic polynomial for Φ is again of the form (40) with

$$\alpha = -\frac{-6+2\sqrt{2}+4\mu}{1+2\sqrt{2}+4\mu},$$

$$\beta = \frac{25+38\sqrt{2}-(84-36\sqrt{2})\mu}{(1+2\sqrt{2}+4\mu)^2},$$

$$\gamma = \frac{32(9-4\sqrt{2})}{(1+2\sqrt{2}+4\mu)^2}.$$

The location of the eight eigenvalues is indicated in Fig. 3. The value of μ where two pairs of eigenvalues meet on the imaginary axis is found by solving the discriminant (41) numerically. The value where two eigenvalues cross each other at the origin comes from $\beta=0$ and gives $\mu=(13+11\sqrt{2})/12$. It is the value where bifurcations to new configurations is possible, see [14]. As μ is increased further the eigenvalues on the imaginary axis tend to $\pm\mathrm{i}$, whereas the remaining four eigenvalues tend to zero.

For (44) the matrix Ψ in (45) is

$$\Psi = \frac{}{1+2\sqrt{2}+4\mu}\begin{bmatrix} 2(1-\sqrt{2}+4\mu) & 0 \\ 0 & -(1-4\sqrt{2}+4\mu) \end{bmatrix}$$

and the characteristic polynomial for Φ is

$$\lambda^4+\lambda^2+\frac{18\sqrt{2}(1+4\mu)}{(1+2\sqrt{2}+4\mu)^2}=0.$$

It has the solutions

$$\lambda^2 = -\frac{1}{2}\pm\frac{\sqrt{16\mu^2+(8-272\sqrt{2})\mu+9-68\sqrt{2}}}{2(1+2\sqrt{2}+4\mu)}.$$

The terms under the square root are negative for $\frac{1}{4}(-49+34\sqrt{2})<\mu<\frac{1}{4}(47+34\sqrt{2})$. For this range, which is $-0.229<\mu<23.77$, the eigenvalues λ are in the complex

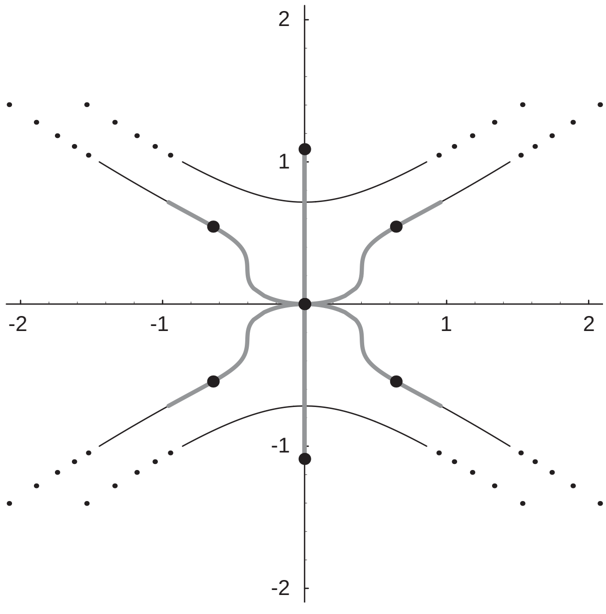

Fig. 3. Location of the eight eigenvalues of the variational matrix for (43). For $-(1+2\sqrt{2})/4 < \mu < 0$ the eigenvalues come from ∞ and are indicated by individual dots. For $0 < \mu < 1.00716$ all eigenvalues are complex and they lie on the thin solid lines. For $1.00716 < \mu$ the location of the eigenvalues is indicated by thicker gray lines. Four eigenvalues are on the imaginary axis. The darker dots indicate the locations of the eigenvalues when $\mu = (13+11\sqrt{2})/12$ and thus two eigenvalues are zero.

plane. In the plane for λ^2 it is obvious that the solutions lie on a line segment parallel to the imaginary axis with real value $-\frac{1}{2}$. When μ is at the left or right endpoint of the above interval then $\lambda = \pm i\sqrt{2}/2$. With this comment the location of the eigenvalues in Fig. 4 should become clear. It should be noted that two eigenvalues are at the origin when $\mu = -\frac{1}{4}$. Also the eigenvalues move into the complex plane when μ is still negative. The location of these eigenvalues when μ is negative are hidden by the gray curves.

5. Conclusion

The change of coordinates introduced in Section 2 decouples the phase space into three components. The first component is for the center of mass and the second one for the Keplerian motion of the bodies. These two spaces yield the eight trivial multipliers of +1 arising from the integrals of the problem. The third space, which is the compliment of the first two, gives the nontrivial multipliers of the problem.

We have shown that our approach reproduces the results of [18] for the Lagrangian triangular configuration in a systematic fashion. We have started the analysis for the $N+1$ regular polygon configuration and we have presented some details for $N = 3$

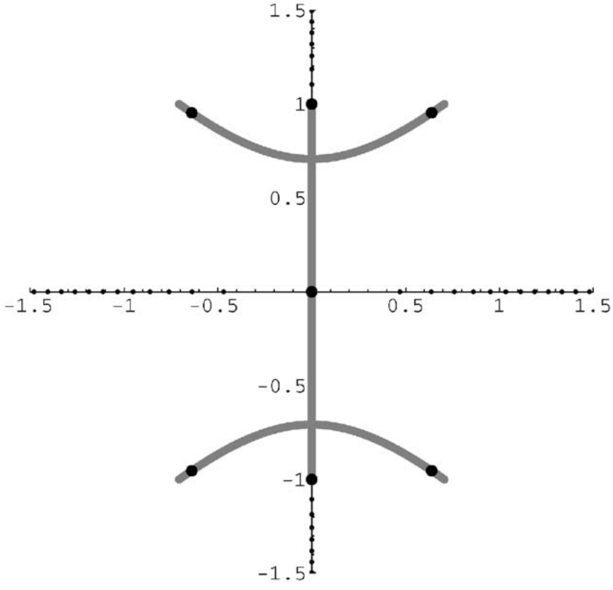

Fig. 4. Location of the four eigenvalues of the variational matrix for (44). For $-(1+2\sqrt{2})/4 < \mu$ the eigenvalues come from ∞ along the four axis and are indicated by individual dots. For $\mu > 0$ the location of the eigenvalues is indicated by thicker gray lines. As μ becomes large two eigenvalues tend to zero and the other two to $\pm$i. The darker dots on the imaginary axis indicate the locations of the four eigenvalues when $\mu = -\frac{1}{4}$. The darker dots in the complex plane are those for $\mu = 0$.

and 4. For $N \leqslant 6$, we find that the relative equilibrium is unstable for all values μ of the central mass in the circular case, $e = 0$. This was already pointed out in [15] and studied further in [21]. Unfortunately, for $e \neq 0$ the variational equations are nonautonomous and are complicated. Therefore, we have not yet analyzed them, and have no conjecture, if the stability of a relative equilibria changes, as the eccentricity e is varied.

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