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# Doubly-Symmetric Periodic Solutions of the Spatial Restricted Three-Body Problem[1]

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The existence of two new families of periodic solutions to the spatial restricted three-body problem is shown. These solutions are independent of the mass ratio of the primaries, have large inclinations and are symmetric with respect to two coordinate planes. In one family the infinitesimal particle is very far from the primaries and in the other family the infinitesimal is very close to one of the primaries. © 2000 Academic Press

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## 1. INTRODUCTION

This paper establishes the existence of two new families of periodic solutions to the spatial restricted three-body problem by Poincaré's continuation method. These families exist for all values of the mass ratio parameter $\mu$ and have large inclinations. In one family the infinitesimal particle is far from the primaries in which case it will be called the *comet* and in the other case the infinitesimal is very close to a primary in which case it will be called the *moon*. These periodic solutions are perturbations of circular solutions of the Kepler problem. By the Kepler problem we mean the spatial central force problem with the inverse square law of attraction.

The small parameter $\varepsilon$ will be introduced as a scale parameter in both cases. In the comet problem $\varepsilon$ small means the infinitesimal is near infinity and for the lunar problem $\varepsilon$ small means the infinitesimal is close to one of the primaries. The perturbation problem is very degenerate. First of all, even to the second approximation the characteristic multipliers are all $+1$. Second, the periodic solutions that we establish are undefined when $\varepsilon = 0$, and third, for the comet problem the period of the solutions goes to infinity

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as $\varepsilon \to 0$. These difficulties are overcome by exploiting the symmetries of the problem and using the implicit function theorem of Arenstorf [1].

In 1965 Jeffreys [5] showed that there exist doubly symmetric, periodic solutions to the three dimensional restricted three-body problem. The method of the proof used in this case depends heavily on a symmetry argument, together with a standard perturbation method applied to the mass ratio $\mu$. Since that time, various treatments of the problem have appeared (see, for example [2, 3]), all involving either a perturbation of the mass ratio parameter $\mu$, or a perturbation of a solution with a special inclination.

Very few nondegenerate periodic solutions of the spatial restricted problem have been established rigorously, but there are interesting families of periodic solutions which have been established using symmetry arguments. Jefferys [5] used two time-reversing symmetries of the spatial restricted problem to establish the existence of periodic solutions which are symmetric with respect to two planes in phase space—hence the name doubly-symmetric periodic solutions.

## 2. SYMMETRIES AND SPECIAL COORDINATES

The Hamiltonian of the three-dimensional circular restricted three-body problem in rotating coordinates is

$$H = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - x_1 y_2 + x_2 y_1 - \left\{ \frac{\mu}{[(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2]^{1/2}} + \frac{1-\mu}{[(x_1 + \mu)^2 + x_2^2 + x_3^2]^{1/2}} \right\}, \tag{1}$$

(see [6]). This Hamiltonian is invariant under the two anti-symplectic reflections:

$$\begin{aligned} \mathscr{R}_1 &: (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_1, -x_2, -x_3, -y_1, y_2, y_3), \\ \mathscr{R}_2 &: (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_1, -x_2, x_3, -y_1, y_2, -y_3). \end{aligned} \tag{2}$$

These are time-reversing symmetries, so if $(x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t))$ is a solution, then so are $(x_1(-t), -x_2(-t), \pm x_3(-t), -y_1(-t), y_2(-t), \mp y_3(-t))$. The fixed set of these two symmetries are Lagrangian subplanes, i.e.

$$\mathscr{L}_1 = \{(x_1, 0, 0, 0, y_2, y_3)\}, \qquad \mathscr{L}_2 = \{(x_1, 0, x_3, 0, y_2, 0)\},$$

are fixed by the symmetries $\mathscr{R}_1, \mathscr{R}_2$. If a solution starts in one of these Lagrangian planes at time $t=0$ and hits the other at a later time $t=T$ then the solution is $4T$-periodic and the orbit of this solution is carried into itself by both symmetries. We shall call such a periodic solution *doubly-symmetric.* Geometrically, an orbit intersects $\mathscr{L}_1$ if it hits the $x_1$-axis perpendicularly and it intersects $\mathscr{L}_2$ if it hits the $x_1, x_3$-plane perpendicularly.

To be more specific, let

$$\begin{aligned}(X_1(t, \alpha, \beta, \gamma), X_2(t, \alpha, \beta, \gamma), X_3(t, \alpha, \beta, \gamma),\\ Y_1(t, \alpha, \beta, \gamma), Y_2(t, \alpha, \beta, \gamma), Y_3(t, \alpha, \beta, \gamma)),\end{aligned} \tag{3}$$

be a solution which starts at $(\alpha, 0, 0, 0, \beta, \gamma) \in \mathscr{L}_1$ when $t=0$, i.e.

$$\begin{aligned} X_1(0, \alpha, \beta, \gamma) &= \alpha, & X_2(0, \alpha, \beta, \gamma) &= 0, & X_3(0, \alpha, \beta, \gamma) &= 0,\\ Y_1(0, \alpha, \beta, \gamma) &= 0, & Y_2(0, \alpha, \beta, \gamma) &= \beta, & Y_3(0, \alpha, \beta, \gamma) &= \gamma.\end{aligned} \tag{4}$$

The solution with $\alpha=\alpha_0$, $\beta=\beta_0$, $\gamma=\gamma_0$ will be doubly-symmetric periodic with period $4T$ if it hits the $\mathscr{L}_2$ plane after a time $T$, i.e.

$$X_2(T, \alpha_0, \beta_0, \gamma_0)=0, \qquad Y_1(T, \alpha_0, \beta_0, \gamma_0)=0, \qquad Y_3(T, \alpha_0, \beta_0, \gamma_0)=0. \tag{5}$$

This solution will be a *nondegenerate doubly-symmetric periodic solution* if the Jacobian

$$\frac{\partial(X_2, Y_1, Y_3)}{\partial(t, \alpha, \beta, \gamma)}(T, \alpha_0, \beta_0, \gamma_0) \tag{6}$$

has rank three.

It follows from the Implicit Function Theorem that nondegenerate doubly-symmetric periodic solutions can be continued in the parameter $\mu$. In general, a nondegenerate doubly-symmetric periodic solution may not be nondegenerate in the classical sense, i.e. a nondegenerate doubly-symmetric periodic solution may have all its multipliers equal to one.

Jefferys [5] proved the existence of nondegenerate doubly-symmetric periodic solutions of the spatial restricted three-body problem by first setting the mass ratio parameter $\mu$ equal to zero to get the Kepler problem in rotating coordinates. He then showed that some of the circular solutions of the Kepler problem where nondegenerate doubly symmetric periodic solutions. Thus, by the above remarks these solutions can be continued into the restricted problem for small $\mu$. The solutions we seek will exist for all $\mu$, but will not be nondegenerate in the above sense. This makes the analysis much more delicate.

We follow Jefferys by using a variation of the Poincaré–Delaunay elements. First, the Delaunay elements $(\ell, g, k, L, G, K)$ are a coordinates on the elliptic domain of the Kepler problem. The elliptic domain is the open set in $\mathbb{R}^6$ which is filled with the elliptic solutions of the Kepler problem. The elements are: $\ell$ the mean anomaly measured from perigee, $g$ the argument of the perigee measured from the ascending node, $k$ the longitude of the ascending node measured from the $x_1$ axis, $L$ semi-major axis of the ellipse, $G$ total angular momentum, $K$ the component of angular momentum about the $x_3$-axis. $\ell$, $g$, and $k$ are angular variables defined modulo $2\pi$, and $L$, $G$ and $K$ are radial variables. If $i$ is the inclination of the orbital plane to the $x_1, x_2$ reference plane, then $K = \pm G \cos i$, and so an orbit is in the $x_1, x_2$-plane when $K = G$. (Often, $k$ and $K$ are denoted by $h$ and $H$, but we are Hamiltonophiles.)

An orbit hits $\mathscr{L}_1$ at time $t = 0$ if it is perpendicular to the $x_1$-axis. So its orbital plane must be through the $x_1$-axis or $k \equiv 0 \bmod \pi$, its perigee must be on the $x_1$-axis or $g \equiv 0 \bmod \pi$, and it must be at perigee (apogee) or $\ell \equiv 0 \bmod \pi$. Thus, $\mathscr{L}_1$ in Delaunay elements is defined by $\ell \equiv g \equiv k \equiv 0 \bmod \pi$.

An orbit hits $\mathscr{L}_2$ at time $t = T$ if it is perpendicular to the $x_1, x_3$-plane. So its orbital plane must be perpendicular to the $x_1, x_3$-plane or $k \equiv \pi/2 \bmod \pi$, its perigee must be in the $x_1, x_3$-plane or $g \equiv \pi/2 \bmod \pi$, and it must be at perigee (apogee) or $\ell \equiv 0 \bmod \pi$. Thus, $\mathscr{L}_2$ in Delaunay elements is defined by $\ell \equiv 0$, $g \equiv k \equiv \pi/2 \bmod \pi$.

Since these coordinates are not valid in a neighborhood of the circular orbits of the Kepler problem, we change to Poincaré elements as follows: first make the symplectic linear change of variables

$$\begin{aligned} q_1 &= l + g + k, & p_1 &= L - G + K, \\ q_2 &= -k - g, & p_2 &= L - G, \\ q_3 &= l + g, & p_3 &= G - K, \end{aligned}$$

and now apply the symplectic change of variables defined by the generating function

$$W(q, P) = q_1 P_1 + \frac{P_2^2}{2} \tan q_2 + P_3 q_3$$

so that $P_2 = \sqrt{2p_2} \cos q_2$ and $Q_2 = \sqrt{2p_2} \sin q_2$. This combination of variable changes gives the new variables:

$$
\begin{aligned}
Q_1 &= q_1 = l + g + k, & P_1 &= p_1 = L - G + K,\\
Q_2 &= -\sqrt{2(L-G)}\,\sin(k+g), & P_2 &= \sqrt{2(L-G)}\,\cos(k+g),\\
Q_3 &= q_3 = l + g, & P_3 &= p_3 = G - K.
\end{aligned} \tag{7}
$$

These variables are valid on circular orbits which occur at $L=G$ (see [4, 8]). The circular orbits with $L=G$ correspond to $Q_2 = P_2 = 0$.

Thus, $\mathscr{L}_1$ in Poincaré elements is defined by $Q_2=0$, $Q_1 \equiv Q_3 \equiv 0 \bmod \pi$, and $\mathscr{L}_2$ in Poincaré elements is defined by $Q_2 = 0$, $Q_1 \equiv 0 \bmod \pi$, $Q_3 \equiv \pi/2 \bmod \pi$.

## 3. APPROXIMATE SOLUTIONS TO THE COMET PROBLEM

To consider orbits close to infinity, scale the variables by $x \to \varepsilon^{-2}x$, $y \to \varepsilon y$, which is symplectic with multiplier $\varepsilon$. Thus, with $H \to \varepsilon H$ the Hamiltonian (1) becomes

$$H = \varepsilon^3 \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2) - x_1 y_2 + x_2 y_1 - U \tag{8}$$

with potential

$$U = \varepsilon^3 \left\{ \frac{\mu}{[(x_1 - \varepsilon^2(1-\mu))^2 + x_2^2 + x_3^2]^{1/2}} + \frac{1-\mu}{[(x_1 + \varepsilon^2\mu)^2 + x_2^2 + x_3^2]^{1/2}} \right\}. \tag{9}$$

By expanding $U$ in terms of $\varepsilon^2$, we can write

$$H = \varepsilon^3 \left( \frac{|y|^2}{2} - \frac{1}{|x|} \right) - x_1\, y_2 + x_2\, y_1 + \varepsilon^7 H^\dagger(x, \varepsilon, \mu) \tag{10}$$

where $H^\dagger$ is order 1 in $\varepsilon$ and meromorphic in $x$. In Delaunay elements, this becomes

$$H = \frac{-\varepsilon^3}{2L^2} - K + \varepsilon^7 H^\dagger(\ell, g, k, L, G, K, \varepsilon, \mu). \tag{11}$$

Since these coordinates are not valid in a neighborhood of the circular orbits, we change to Poincaré elements (7) and the Hamiltonian becomes

$$H = \frac{-\varepsilon^3}{2(P_1+P_3)^2} - P_1 + \frac{1}{2}(P_2^2 + Q_2^2) + \varepsilon^7 H^\dagger(Q_1, Q_2, Q_3, P_1, P_2, P_3, \varepsilon, \mu) \tag{12}$$

and the equations of motion are

$$
\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^3}{(P_1+P_3)^3} - 1 + \varepsilon^7 f_1, & \dot{P}_1 &= 0 + \varepsilon^7 f_4, \\
\dot{Q}_2 &= P_2 + \varepsilon^7 f_2, & \dot{P}_2 &= -Q_2 + \varepsilon^7 f_5, \\
\dot{Q}_3 &= \frac{\varepsilon^3}{(P_1+P_3)^3} + \varepsilon^7 f_3, & \dot{P}_3 &= 0 + \varepsilon^7 f_6.
\end{aligned}
\tag{13}
$$

where the $f_i$ are the appropriate partials of $H^\dagger$.

The problem is essentially independent of the mass ratio $\mu$, so we can consider it a fixed parameter for what follows. The parameter $\varepsilon$ is inversely proportional to the square root of the distance of the third body from the primaries. Thus, as $\varepsilon \to 0$, this distance goes to infinity and the form of the differential equation (13) degenerates. We cannot, therefore, use perturbation methods which rely on solving the differential equation when $\varepsilon = 0$. Instead, we need to obtain solutions for $\varepsilon$ in a deleted neighborhood of $\varepsilon = 0$, and to do this we need approximate solutions to this system of differential equations and good estimates. Also, since we are looking for periodic solutions far from the primaries and therefore of long period, we need these approximate solutions for large values of $t$ and small values of $\varepsilon$.

For now let us define the equations of the first approximation by dropping the $\varepsilon^7$ terms, i.e. consider the equations

$$
\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^3}{(P_1+P_3)^3} - 1, & \dot{P}_1 &= 0, \\
\dot{Q}_2 &= P_2, & \dot{P}_2 &= -Q_2, \\
\dot{Q}_3 &= \frac{\varepsilon^3}{(P_1+P_3)^3}, & \dot{P}_3 &= 0.
\end{aligned}
\tag{14}
$$

These are of course, the equations of motion for the Kepler problem in the scaled, rotating Poincaré elements.

Consider a solution of (14) which starts on $\mathscr{L}_1$ at $t = 0$ with initial conditions $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$, $P_1 = p_1$, $P_2 = p_2$, $P_3 = p_3$, where $p_1, p_2, p_3$ are constants to be determined and $i$ and $j$ can be either 0 or 1. This solution is of the form

$$
\begin{aligned}
Q_1(t) &= \left( \frac{\varepsilon^3}{(p_1+p_3)^3} - 1 \right) t + i\pi, \\
Q_2(t) &= p_2 \sin t, \\
Q_3(t) &= \left( \frac{\varepsilon^3}{(p_1+p_3)^3} \right) t + j\pi.
\end{aligned}
\tag{15}
$$

To satisfy the conditions that at time $t = T$ this solution is in $\mathscr{L}_2$ it is sufficient to solve the set of three equations in four unknowns:

$$\begin{aligned} Q_1(T) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} - 1 \right) T + i\pi = (i + k)\,\pi, \\ Q_2(T) &= p_2 \sin T = 0, \\ Q_3(T) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} \right) T + j\pi = (j + m + 1/2)\,\pi. \end{aligned} \tag{16}$$

The second equation is easy to solve by taking $p_2 = 0$, thus selecting a circular orbit of the Kepler problem. The difference between the first and third equation has a solution with $T = (m - k + 1/2)\,\pi$ and arbitrary $p_1$ and $p_3$. It remains only to solve either the first or the third equations, say the third. With this choice of $T$ the third equation becomes

$$\left( \frac{\varepsilon^3}{(p_1 + p_3)^3} - 1 \right) \left( m - k + \frac{1}{2} \right) \pi = \left( m + \frac{1}{2} \right) \pi$$

or (17)

$$(p_1 + p_3)^3 = \frac{\varepsilon^3 \left( m - k + \frac{1}{2} \right)}{\left( m + \frac{1}{2} \right)}.$$

Recall that $P_1 + P_3 = L$ which is the semi-major axis in the Kepler problem. Thus we seek a solution which is order 1 in $p_1 + p_3$. (If we take $p_1 + p_3$ of order $\varepsilon^3$ then in the original unscaled variables the solutions are order 1 which would just give us back Jefferys' solutions.) In order to solve Eq. (17) with $p_1 + p_3 = 1$ we will fix $m$ and make $k$ a large integer by choosing

$$k = -\frac{\left( m + \frac{1}{2} \right)}{\varepsilon^3} + m + \frac{1}{2}. \tag{18}$$

With the above choices $p_3$ is arbitrary. Recall, that $P_3 = G - K$, $K = \pm G \cos i$ where $G$ is the total angular momentum, $K$ is the $x_3$-component of angular momentum, and $i$ is the inclination. So, with this selection the inclination $i$ is arbitrary.

We shall show in Section 5 that these solutions of the approximate equations (14) are actually approximations of actual doubly-symmetric periodic solutions of the true equations (13). Thus, our first theorem is

THEOREM 3.1. *There exist doubly-symmetric periodic solutions of the spatial restricted three-body problems for all values of the mass ratio parameter* $\mu$ *with large inclination which are arbitrarily far away from the primaries.*

The reader can now see the complexity of the problem. The period of the solutions is of order $\varepsilon^{-3}$ and the solutions are undefined when $\varepsilon = 0$. The details of the estimates and the complete proof will be give in Section 5.

## 4. APPROXIMATE SOLUTIONS TO THE LUNAR PROBLEM

To consider lunar problem, move one of the primaries to the origin by making the change of variable $x_1 \to x_1 - \mu$, $y_2 \to y_2 - \mu$ in the Hamiltonian (1). Now to move the third mass close to the origin, scale the variables by $x \to \varepsilon^2 x(1-\mu)^{1/3}$, $y \to \varepsilon^{-1}(1-\mu)^{1/3} y$, which is symplectic with multiplier $\varepsilon^{-1}(1-\mu)^{-2/3}$. Letting $H \to \varepsilon^{-1}(1-\mu)^{-2/3} H$, expanding the potential in $\varepsilon$, and by dropping the constant terms, the Hamiltonian becomes

$$H = \varepsilon^{-3} \left\{ \frac{|y|^2}{2} - \frac{1}{|x|} \right\} - (x_1 y_2 - x_2 y_1) + \mathcal{O}(\varepsilon^3). \tag{19}$$

As in the comet problem use the Poincaré elements (7) so that

$$H = \frac{-\varepsilon^{-3}}{2(P_1 + P_3)^2} - P_1 + \frac{1}{2}(P_2^2 + Q_2^2) + \varepsilon^3 H^\dagger(Q_1, Q_2, Q_3, P_1, P_2, P_3, \varepsilon, \mu), \tag{20}$$

where $H^\dagger$ is order 1 in $\varepsilon$. Thus the equations of motion are $\dot{Q} = H_P$, $\dot{P} = -H_Q$ or

$$\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^{-3}}{(P_1 + P_3)^3} - 1 + \varepsilon^3 f_1, & \dot{P}_1 &= 0 + \varepsilon^3 f_4, \\
\dot{Q}_2 &= P_2 + \varepsilon^3 f_2, & \dot{P}_2 &= -Q_2 + \varepsilon^3 f_5, \\
\dot{Q}_3 &= \frac{\varepsilon^{-3}}{(P_1 + P_3)^3} + \varepsilon^3 f_3, & \dot{P}_3 &= 0 + \varepsilon^3 f_6,
\end{aligned} \tag{21}$$

where the $f_i$ are the appropriate partials of $H^\dagger$.

As in the previous section let us consider the approximate equations first in order to find the correct approximate periodic solutions. Consider the approximate equations

$$\begin{aligned} \dot{Q}_1 &= \frac{\varepsilon^{-3}}{(P_1+P_3)^3} - 1, & \dot{P}_1 &= 0, \\ \dot{Q}_2 &= P_2, & \dot{P}_2 &= -Q_2, \\ \dot{Q}_3 &= \frac{\varepsilon^{-3}}{(P_1+P_3)^3}, & \dot{P}_3 &= 0. \end{aligned} \tag{22}$$

The solution of Eq. (22) are

$$\begin{aligned} Q_1(t) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1\right) t + q_1, & P_1(t) &= p_1, \\ Q_2(t) &= q_2 \cos t + p_2 \sin t, & P_2(t) &= -q_2 \sin t + p_2 \cos t, \\ Q_3(t) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\right) t + q_3, & P_3(t) &= p_3, \end{aligned} \tag{23}$$

for initial conditions $(q_1, q_2, q_3, p_1, p_2, p_3)$ at $t=0$.

The periodicity conditions are the same as those in the Section 2. That is, at $t=0$; $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$ and at $t=T$; $Q_1 = (i+k)\,\pi$, $Q_2 = 0$, $Q_3 = (j+m+1/2)\,\pi$ where $i$ and $j$ are 0 or 1, and $k$, and $m$ are arbitrary integers. To satisfy these symmetry condition at $t=0$ and at $t=T$ we have so solve the equations

$$\begin{aligned} Q_1(T) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1\right) T + i\pi = (i+k)\,\pi, \\ Q_2(T) &= p_2 \sin T = 0, \\ Q_3(T) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\right) T + j\pi = (j+m+1/2)\,\pi, \end{aligned} \tag{24}$$

The second equation is solved by taking $p_2 = 0$, thus selecting a circular orbit of the Kepler problem. The difference between the first and third equation has a solution with $T = (m-k+1/2)\,\pi$. It remains to solve the third equation. With this choice of $T$ it becomes

$$(p_1+p_3)^3 = \frac{\varepsilon^{-3}\left(m-k+\frac{1}{2}\right)}{\left(m+\frac{1}{2}\right)}. \tag{25}$$

Again with the above choices $p_3$ is arbitrary and so as with the comet problem the inclination is arbitrary.

Again we seek solutions which are order 1 in the sum $p_1 + p_3$. Let $n$ be a fixed small integer. To solve (25) set $m + 1/2 = \varepsilon^{-3}$, $k = m - n$, and $(p_1 + p_3)^3 = (n + 1/2)$. With this choice the period becomes $T = n + 1/2$

We shall show in Section 6 that these solutions of the approximate equations (22) are actually approximations of actual doubly-symmetric periodic solutions of the true equations (21). Thus, our second theorem is

THEOREM 4.1. *There exist doubly-symmetric periodic solutions of the spatial restricted three-body problems for all values of the mass ratio parameter $\mu$ with large inclination which are arbitrarily close to one of the primaries.*

## 5. PROOF FOR THE COMET PROBLEM

In order to prove Theorem 3.1 we need good long term estimates on the solutions of Eq. (13). From time to time, we shall write Eq. (13) in the form

$$\dot{z} = F(z, \varepsilon) + \varepsilon^7 \vec{f}(z, \varepsilon) \tag{26}$$

where $z = (Q_1, Q_2, Q_3, P_1, P_2, P_3)$, $F = (\varepsilon^3/(P_1 + P_3)^3 - 1,\ P_2,\ \varepsilon^3/(P_1 + P_3)^3,\ 0,\ -Q_2,\ 0)$ and $\vec{f} = (f_1, f_2, f_3, f_4, f_5, f_6)$.

LEMMA 5.1. *Let $(q_1, q_2, q_3, p_1, p_2, p_3)$ be initial conditions such that for the equations of the first approximation (14) the solutions remain bounded and bounded away from singularities. Let $\vec{\varphi}(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t), \varphi_6(t))$ be the solution to (13) with $\varepsilon \neq 0$, and $(\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(0), \varphi_5(0), \varphi_6(0)) = (q_1, q_2, q_3, p_1, p_2, p_3)$. Then this solution is of the form*

$$\begin{aligned}
\varphi_1(t) &= \left(\frac{\varepsilon^3}{(p_1 + p_3)^3} - 1\right) t + q_1 + \varepsilon^4 g_1, & \varphi_4(t) &= p_1 + \varepsilon^4 g_4,\\
\varphi_2(t) &= q_2 \cos t + p_2 \sin t + \varepsilon^4 g_2, & \varphi_5(t) &= -q_2 \sin t + p_2 \cos t + \varepsilon^4 g_5,\\
\varphi_3(t) &= \left(\frac{\varepsilon^3}{(p_1 + p_3)^3}\right) t + q_3 + \varepsilon^4 g_3, & \varphi_6(t) &= p_3 + \varepsilon^4 g_6,
\end{aligned} \tag{27}$$

*for $0 \leqslant t \leqslant \gamma\varepsilon^{-3}$, where $\gamma$ is a constant independent of $\varepsilon$, and where the $g_i = g_i(t, q_1, q_2, q_3, p_1, p_2, p_3, \varepsilon)$ are uniformly bounded as $\varepsilon$ approaches zero for $t \in [0, \gamma\varepsilon^{-3}]$.*

*Proof.* To show this, we will compare the solution of Eqs. (13) and (14). Let $\vec{\psi}(t) = (\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t), \psi_6(t))$ be the solution to (14), with $\vec{\psi}(0) = (q_1, q_2, q_3, p_1, p_2, p_3)$ and let $\mathscr{C} \subseteq \mathbb{R}^6$ be a compact neighborhood of this solution. In particular $\vec{\psi}(t)$ is the solution in (27) with $g_i \equiv 0$.

Let $M > 0$ be such that the solution to (13) exist in $\mathscr{C}$ for all $0 \leqslant t \leqslant M$ and all $0 \leqslant \varepsilon \leqslant 1$. We first show that $\|\vec{\varphi} - \vec{\psi}\|$ is $\mathscr{O}(\varepsilon^7)$, for $0 \leqslant t \leqslant M$. In order to do this we must group the equations into two sets (the equations for $(Q_1, Q_3, P_1, P_3)$ and the equations for $(Q_2, P_2)$), so we define $\bar{\varphi} = (\varphi_1, \varphi_3, \varphi_4, \varphi_6)$, $\tilde{\varphi} = (\varphi_2, \varphi_5)$. Define $\bar{\psi}$, $\tilde{\psi}$ etc. in a similar manner.

Consider the non-homogeneous system given by

$$\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^3}{(P_1 + P_3)^3} - 1 + \varepsilon^7 f_1, \qquad & \dot{P}_1 &= 0 + \varepsilon^7 f_4, \\
\dot{Q}_3 &= \frac{\varepsilon^3}{(P_1 + P_3)^3} + \varepsilon^7 f_3, \qquad & \dot{P}_3 &= 0 + \varepsilon^7 f_6,
\end{aligned} \tag{28}$$

with $(Q_1(0), Q_3(0), P_1(0), P_3(0)) = (q_1, q_3, p_1, p_3)$, and the $f_i$ are as in (13), except that $f_i = f_i(Q_1, \varphi_2(t), Q_3, P_1, \varphi_5(t), P_3)$. Thus, $\bar{\varphi}(t)$ is a solution of (28) for $t \in [0, M]$.

Now

$$\dot{\bar{\varphi}} = \bar{F}(\varphi_4(t), \varphi_6(t), \varepsilon) + \varepsilon^7 \bar{f}(\vec{\varphi}(t), \varepsilon),$$

$$\dot{\bar{\psi}} = \bar{F}(\psi_4(t), \psi_6(t), \varepsilon).$$

Let $c$ be a constant such that $\|\vec{f}\| \leqslant c$ on $\mathscr{C}$ for $0 \leqslant \varepsilon \leqslant 1$. From the form of the equations it is clear that $\bar{F}$ has Lipschitz constant $\varepsilon^3 \beta$, for some real $\beta > 0$ on $\mathscr{C}$ for $0 \leqslant \varepsilon \leqslant 1$. Then

$$\begin{aligned}
\|\bar{\varphi}(t) - \bar{\psi}(t)\| &\leqslant \int_0^t \|\bar{F}(\varphi_4(s), \varphi_6(s), \varepsilon) - \bar{F}(\psi_4(s), \psi_6(s), \varepsilon)\| + \varepsilon^7 \|\bar{f}(\vec{\varphi}(s), \varepsilon)\| \, ds \\
&\leqslant \int_0^t \varepsilon^3 \beta \|\bar{\varphi}(s) - \bar{\psi}(s)\| + (\varepsilon^7 c) \, ds \\
&= \varepsilon^7 c t + \int_0^t \varepsilon^3 \beta \|\bar{\varphi}(s) - \bar{\psi}(s)\| \, ds.
\end{aligned}$$

So by Gronwall's inequality, we have

$$\begin{aligned}\|\bar{\varphi}(t)-\bar{\psi}(t)\| &\leqslant \varepsilon^7 ct + c\varepsilon^{10}\beta\int_0^t s e^{\beta\varepsilon^3(t-s)}\,ds\\ &= \varepsilon^7 ct - \frac{c\varepsilon^4}{\beta}\left(\beta\varepsilon^3 t + 1 - e^{\beta\varepsilon^3 t}\right)\\ &= \frac{c\varepsilon^4}{\beta}\left(e^{\beta\varepsilon^3 t}-1\right),\end{aligned}$$

for $t\in[0, M]$.

The remaining solution components are bounded in similar way. Note that $(\varphi_2(t), \varphi_5(t))$ is the solution to the non-homogeneous system

$$\begin{aligned}\dot{Q}_2 &= P_2 + \varepsilon^7 f_2(\varphi_1(t), Q_2,, \varphi_3(t), \varphi_4(t), P_2, \varphi_6(t), \varepsilon),\\ \dot{P}_2 &= -Q_2 + \varepsilon^7 f_5(\varphi_1(t), Q_2,, \varphi_3(t), \varphi_4(t), P_2, \varphi_6(t), \varepsilon),\end{aligned}$$

with $(\varphi_2(0), \varphi_5(0)) = (q_2, p_2)$. Then by variation of parameters formula we have

$$\tilde{\varphi}(t) = \tilde{\phi}(t) + \tilde{h}(t)$$

where

$$\tilde{h}(t) = \begin{pmatrix} h_2(t)\\ h_5(t)\end{pmatrix} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\int_0^t \begin{pmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{pmatrix}\begin{pmatrix}\varepsilon^7 f_2(\vec{\varphi}(s), \varepsilon)\\ \varepsilon^7 f_5(\vec{\varphi}(s), \varepsilon)\end{pmatrix} ds$$

Thus, $\|\tilde{h}(t)\| = \|\tilde{\varphi}(t) - \tilde{\psi}(t)\| \leqslant c\varepsilon^7 t$ for $0\leqslant t\leqslant M$.

Combining the two results above, we have $\|\vec{\varphi}(t) - \vec{\psi}(t)\| = \mathcal{O}(\varepsilon^7)$ as long as the solutions remain in $\mathcal{O}$. Actually we have $\|\vec{\varphi}(t) - \vec{\psi}(t)\| \leqslant k\varepsilon^4$ for $0\leqslant t \leqslant \gamma\varepsilon^{-3}$ where $k$ is a constant as long as the solutions remain in $\mathscr{C}$. But, since $\mathscr{C}$ is compact this estimate insures that the solution remains in $\mathscr{C}$ for $0\leqslant t\leqslant \gamma\varepsilon^{-3}$ provided $\varepsilon$ is sufficiently small. ▮

LEMMA 5.2. *Let the $\varphi_i(t; q_1, q_2, q_3, p_1, p_2, p_3, \varepsilon)$ and $g_i$ be as in Lemma 5.1. Then for any fixed $t\in[0, \gamma\varepsilon^{-3}]$, the $\partial g_i/\partial p_j$ are uniformly bounded as $\varepsilon$ approaches zero.*

*Proof.* As in Lemma 5.1, we let $\vec{\psi}(t; \vec{p}, \vec{q})$ be the solution to (26) with $\varepsilon = 0$ and let $\vec{\varphi}(t; \vec{p}, \vec{q})$ be the solution to (26) for $\varepsilon \neq 0$. Thus, from Lemma 5.1, we know that $\varepsilon^4\vec{g}(t; \vec{p}, \vec{q}) = \vec{\varphi}(t; \vec{p}, \vec{q}) - \vec{\psi}(t; \vec{p}, \vec{q})$, where $\vec{g} = (g_1, g_2, g_3, g_4, g_5, g_6)$. Omitting the vector notation, the variational equation for $g$ is:

$$\begin{aligned}
\frac{d}{dt}\left[\varepsilon^4 \frac{\partial}{\partial p} g(t; p, q)\right] &= \frac{\partial}{\partial p}(\dot{\varphi}(t; p, q) - \dot{\psi}(t; p, q)) \\
&= \frac{\partial}{\partial p}[F(\varphi(t; p, q)) - F(\psi(t; p, q)) + \varepsilon^7 f(\varphi(t; p, q))] \\
&= DF(\varphi)\frac{\partial \varphi}{\partial p} - DF(\psi)\frac{\partial \psi}{\partial p} + \varepsilon^7\, Df(\varphi)\frac{\partial \varphi}{\partial p} \\
&= DF(\varphi)\frac{\partial(\psi + \varepsilon^4 g)}{\partial p} - DF(\psi)\frac{\partial \psi}{\partial p} + \varepsilon^7\, Df(\varphi)\frac{\partial(\psi + \varepsilon^4 g)}{\partial p} \\
&= [DF(\varphi) - DF(\psi)]\frac{\partial \psi}{\partial p} + \varepsilon^7\, Df(\varphi)\frac{\partial \psi}{\partial p} \\
&\quad + DF(\varphi)\,\varepsilon^4\frac{\partial g}{\partial p} + \varepsilon^7\, Df(\varphi)\,\varepsilon^4\frac{\partial g}{\partial p}.
\end{aligned}$$

Then, letting $z = \varepsilon^4\, \partial g/\partial p$, we consider the system of differential equations

$$\dot{z} = DF(\varphi)\, z + \varepsilon^7\, Df(\varphi)\, z + [DF(\varphi) - DF(\psi)]\frac{\partial \psi}{\partial p} + \varepsilon^7\, Df(\varphi)\frac{\partial \psi}{\partial p}, \quad (29)$$

which we can also write as

$$\dot{z} = DF(\varphi)\, z + \varepsilon^7\, Df(\varphi)\, z + K(t)$$

since the last two terms of (29) do not depend on $z$. Applying the variation of constants formula to this equation, we get that

$$z(t) = Y(t)\left[Y^{-1}(0)\, z(0) + \int_0^t Y^{-1}(s)(\varepsilon^7\, Df(\varphi(s))\, z(s) + K(s))\, ds\right], \quad (30)$$

where $Y(t)$ is the fundamental solutions matrix for the linear equation $\dot{z} = DF(\varphi(t))\, z$.

Since $z(t) = \varepsilon^4\, \partial g(t)/\partial p = \partial\varphi(t)/\partial p - \partial\psi(t)/\partial p$, and since $\partial\varphi(0)/\partial p = \partial\psi(0)/\partial p = 1$, we note that $z(0) = 0$, and (30) becomes

$$z(t) = Y(t)\int_0^t Y^{-1}(s)(\varepsilon^7\, Df(\varphi(s))\, z(s) + K(s))\, ds. \quad (31)$$

Now, since $\varphi(t; p, q)$ is known from Lemma 5.1 and $DF$ can be calculated, we can find $Y(t; p, q)$ explicitly since it is the solution to the differential equation

$$\dot{z}_1 = \frac{-3\varepsilon^3}{(p_1+\varepsilon^4 g_4 + p_3 + \varepsilon^4 g_6)^4} \cdot (z_4 + z_6), \qquad \dot{z}_4 = 0,$$

$$\dot{z}_2 = z_5, \qquad \dot{z}_5 = -z_2,$$

$$\dot{z}_3 = \frac{-3\varepsilon^3}{(p_1+\varepsilon^4 g_4 + p_3 + \varepsilon^4 g_6)^4} \cdot (z_4 + z_6), \qquad \dot{z}_5 = 0.$$

That is, this equation decouples so, letting $v(t) = \varepsilon^3 \int_0^t -3(p_1 + \varepsilon^4 g_4(s) + p_3 + \varepsilon^4 g_6(s))^{-4}\, ds$, we can obtain by direct computation that $Y(t)$ is

$$\begin{bmatrix} 1 & 0 & 0 & v(t) & 0 & v(t) \\ 0 & \cos t & 0 & 0 & \sin t & 0 \\ 0 & 0 & 1 & v(t) & 0 & v(t) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\sin t & 0 & 0 & \cot t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $c_1 = \sup\{\|Df(p, q)\| : (p, q) \in \mathscr{C}\}$, so by Lemma 5.1 we know $\|Df(\varphi(t))\| \leqslant c_1$ for $t \in [0, M]$ where $M = \gamma\varepsilon^{-3}$.

Next, since $K = [DF(\varphi) - DF(\psi)]\, \partial\psi/\partial p + \varepsilon^7\, Df(\varphi)\, \partial\psi/\partial p$, we calculate the term $DF(\varphi) - DF(\psi)$ to be

$$\begin{bmatrix} 0 & 0 & 0 & w(t, q, p) & 0 & w(t, q, p) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w(t, q, p) & 0 & w(t, q, p) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$w(t, p, q) = \frac{3\varepsilon^3}{(p_1 + p_3)^4} - \frac{3\varepsilon^3}{(p_1 + \varepsilon^4 g_4 + p_3 + \varepsilon^4 g_6)^4}.$$

Since $3/x^4$ is Lipschitz for $x$ bounded away from zero, and since $|p_1 + p_3| \geqslant \alpha > 0$ in a solution bounded away from collision, we have $|w(t, p, q)| \leqslant \varepsilon^3 \beta\, |\varepsilon^4 g_4(t, p, q) + \varepsilon^4 g_6(t, p, q)|$ where $\beta$ is the appropriate Lipschitz constant. But of course, from Lemma 5.1 we know that the $g_i$ are uniformly bounded, and letting $c_2$ be this bound, we have that $|w(t, p, q)| \leqslant \varepsilon^7 \beta c_2$, and thus that $\|DF(\varphi) - DF(\psi)\| \leqslant \varepsilon^7 \beta c_2 c_3$ for some positive constant $c_3$. Then, by direct calculation, we find that $\|\partial\psi/\partial p\| \leqslant \max_{t\in[0,M]} c_4 \max\{3\varepsilon^3 t/(p_1+p_3)^4, 1\}$ for some positive constant $c_4$. We can assume, without loss of generality, that $\max\{3\varepsilon^3 t/(p_1+p_3)^4, 1\}=1$ for $t\in[0,M]$. Thus, all together we obtain that

$$\begin{aligned}\|K\| &= \|[DF(\varphi)-DF(\psi)]\,\partial\psi/\partial p + \varepsilon^7 Df(\varphi)\,\partial\psi/\partial p\|\\ &\leqslant \|DF(\varphi)-DF(\psi)\|\,\|\partial\psi/\partial p\| + \varepsilon^7\,\|Df(\varphi)\|\,\|\partial\psi/\partial p\|\\ &\leqslant \varepsilon^7\beta c_2c_3c_4+\varepsilon^7 c_1c_4\\ &=\varepsilon^7 c_4(\beta c_2 c_3 + c_1),\end{aligned}$$

for $t\in[0,M]$.

Since the $g_i$ are bounded by Lemma 5.1, we can let $b=\max_{t\in[0,M]}(3/(p_1+\varepsilon^4 g_4+p_3+\varepsilon^4 g_6)^4)$ so that $\|Y(t)^{-1}\| = \|Y(t)\| \leqslant c_5 \max_{t\in[0,M]}\{\varepsilon^3 bt, 1\}$. Again, without loss of generality, we can assume that $\max_{t\in[0,M]}\{\varepsilon^3 bt, 1\}=1$.

Finally, we can bound $z(t)$ as follows:

$$\begin{aligned}\|z(t)\| &= \left\| Y(t)\int_0^t Y^{-1}(s)(\varepsilon^7\,Df(\varphi(s))\,z(s)+K(s))\,ds\right\|\\ &\leqslant \varepsilon^7\,\|Y(t)\|\int_0^t \|Y^{-1}(s)\|\,\|Df(\varphi(s))\|\,\|z(s)\|\,ds\\ &\quad+\|Y(t)\|\int_0^t \|Y^{-1}(s)\|\,\|K(s)\|\,ds\\ &\leqslant \varepsilon^7 c_5\int_0^t c_5c_1\,\|z(s)\|\,ds + c_5\int_0^t c_5\varepsilon^7 c_4(\beta c_2c_3+c_1)\,ds\\ &=\varepsilon^7 c_5^2 c_1\int_0^t \|z(s)\|\,ds+\varepsilon^7 c_5^2 c_4(\beta c_2 c_3+c_1)\,t\\ &=\varepsilon^7 k_1\int_0^t \|z(s)\|\,ds+\varepsilon^7 k_2 t,\end{aligned}$$

where the last step is merely simplifying the constants. Then, by Gronwall's inequality we have:

$$\begin{aligned}\|z(t)\| &\leqslant \varepsilon^7 k_2 t+\int_0^t \varepsilon^7 k_2 s\varepsilon^7 k_1 e^{(t-s)\,\varepsilon^7 k_1}\,ds\\ &=\varepsilon^7 k_2 t - k_2\,\frac{t\varepsilon^7 k_1+1-e^{t\varepsilon^7 k_1}}{k_1}\\ &=\frac{k_2}{k_1}\,(e^{t\varepsilon^7 k_1}-1).\end{aligned}$$

Now, recalling that $z(t) = \varepsilon^4\, \partial g/\partial p$, we have proved that

$$\left\| \frac{\partial g}{\partial p} \right\| \leqslant k_2 \frac{e^{t\varepsilon^7 k_1} - 1}{\varepsilon^4 k_1}$$

for $t \in [0, M] = [0, \gamma\varepsilon^{-3}]$. That is, of course,

$$\left\| \frac{\partial g}{\partial p} \right\| \leqslant k_2 \frac{e^{t\varepsilon^7 k_1} - 1}{\varepsilon^4 k_1} \leqslant k_2 \gamma \frac{e^{\gamma\varepsilon^4 k_1} - 1}{\gamma\varepsilon^4 k_1},$$

and the right hand side is uniformly bounded as $\varepsilon$ approaches zero. ∎

Now return to the proof of Theorem 3.1. The conditions which prescribe the doubly-periodic trajectories are, in the Poincaré elements, $t = 0$; $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$ and at $t = T$; $Q_1 = (i + k)\,\pi$, $Q_2 = 0$, $Q_3 = (j + m + 1/2)\,\pi$.

Letting $(p_1, p_2, p_3)$ be initial conditions to be determined, the functions

$$\begin{aligned}
Q_1(t) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} - 1 \right) t + \varepsilon^4 g_1 + i\pi, \\
Q_2(t) &= p_2 \sin t + \varepsilon^4 g_2, \\
Q_3(t) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} \right) t + \varepsilon^4 g_3 + j\pi,
\end{aligned}$$

are of the correct form and satisfy the initial conditions at $t = 0$. Here the $g_i$, which are functions of the initial conditions, have been evaluated at the three initial conditions $q_1 = i\pi$, $q_2 = 0$, $q_3 = j\pi$, but the $p_i$ are still free. To satisfy the conditions at $t = T$, it is sufficient to solve the set of three equations in four unknowns:

$$\begin{aligned}
Q_1(t) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} - 1 \right) t + \varepsilon^4 g_1(t, p_1, p_2, p_3, \varepsilon) + i\pi = (i + k)\,\pi, \\
Q_2(t) &= p_2 \sin t + \varepsilon^4 g_2(t, p_1, p_2, p_3, \varepsilon) = 0, \\
Q_3(t) &= \left( \frac{\varepsilon^3}{(p_1 + p_3)^3} \right) t + \varepsilon^4 g_3(t, p_1, p_2, p_3, \varepsilon) + j\pi = \left( j + m + \frac{1}{2} \right) \pi.
\end{aligned}$$

This is done by applying the Implicit Function Theorem twice. First, fix $m = m_0 \geqslant 1$ and let

$$R(t, p_1, p_2, p_3) = Q_3(t) - (m_0 + \tfrac{1}{2} + j)\,\pi - Q_1(t) + (i + k)\,\pi$$

and consider the system of equations

$$0 = R = t - (m_0 + \tfrac{1}{2} - k)\,\pi + \varepsilon^4 g_3(t, p_1, p_2, p_3, \varepsilon) - \varepsilon^4 g_1(t, p_1, p_2, p_3, \varepsilon),$$

$$0 = Q_2(t) = p_2 \sin t + \varepsilon^4 g_2(t, p_1, p_2, p_3, \varepsilon).$$

Then for $\varepsilon = 0$, $(p_1, p_3)$ arbitrary, we have solution $t = (m_0 + 1/2 - k)\,\pi$, $p_2 = 0$.

Lemma 5.2 means that $\varepsilon^4\, \partial g_i/\partial p_i((m_0 + 1/2 - k)\,\pi, q_1, q_2, q_3, p_1, p_2, p_3, \varepsilon)|_{\varepsilon=0} = 0$ as long as $(m_0 + 1/2 - k)\,\pi \in [0, \gamma\varepsilon^{-3}]$. Thus

$$\frac{\partial(R, Q_2)}{\partial(t, p_2)} = \begin{vmatrix} 1 & 0 \\ 0 & \sin((m_0 + 1/2 - k)\,\pi) \end{vmatrix} = \pm 1 \neq 0.$$

Then by the Implicit Function Theorem, there exists a neighborhood $\mathcal{N}$ of 0 and functions $T(p_1, p_3, \varepsilon)$ near $(m_0 + 1/2 - k)\,\pi$ and $p_2(p_1, p_3, \varepsilon)$ near 0 such that

$$T(p_1, p_3, \varepsilon) - (m_0 + \tfrac{1}{2} - k)\,\pi + \varepsilon^4 g_3 - \varepsilon^4 g_1 = 0,$$

$$p_2(p_1, p_3, \varepsilon) \sin T(p_1, p_3, \varepsilon) + \varepsilon^4 g_2 = 0,$$

for $\varepsilon \in \mathcal{N}$ and $(p_1, p_3)$ arbitrary.

Now $k$ was arbitrary, so we can let $k = -2m_0/\varepsilon^3$. Then for each $\varepsilon \in \mathcal{N} - \{0\}$ such that $\varepsilon^3 = 1/n$, for any integer $n$, we have

$$T(p_1, p_3, \varepsilon) = \left(m_0 + \frac{1}{2} + \frac{2\pi m_0}{\varepsilon^3}\right)\pi - \varepsilon^4 g_3(T, p_1, p_2, p_3, \varepsilon)$$
$$+ \varepsilon^4 g_1(T, p_1, p_2, p_3, \varepsilon)$$

or

$$T(p_1, p_3, \varepsilon) = (m_0 + \tfrac{1}{2} + 2\pi m_0 n)\,\pi - \varepsilon^4 g_3 + \varepsilon^4 g_1.$$

Now substituting this $T$ into the original equation for $Q_3(T) = (j + m_0 + 1/2)\,\pi$ we need to solve:

$$\frac{\varepsilon^3}{(p_1 + p_3)^3}\left[\left(m_0 + \frac{1}{2} - k\right)\pi - \varepsilon^4 g_3(T, p_1, p_3, \varepsilon) + \varepsilon^4 g_1(T, p_1, p_3, \varepsilon)\right]$$
$$-\left(m_0 + \frac{1}{2}\right)\pi + \varepsilon^4 g_3(T, p_1, p_3, \varepsilon) = 0$$

for $\varepsilon \in \mathcal{N} - \{0\}$.

Since the solution will be in a neighborhood of $p_1 + p_3 = [4m_0/(2m_0 + 1)]^{1/3}$, we fix $p_3$ arbitrarily and, if necessary, take $\mathcal{N}$ smaller so that $\varepsilon^4 g_3(T, p_1, p_3, \varepsilon) \neq (m_0 + 1/2)\pi$ in some open neighborhood $\mathscr{V}$ of $p_1^* = [4m_0/(2m_0+1)]^{1/3} - p_3$. Now to solve:

$$(p_1+p_3)^3 = \frac{\varepsilon^3\left[\left(m_0+\frac{1}{2}-k\right)\pi - \varepsilon^4 g_3 + \varepsilon^4 g_1\right]}{\left(m_0+\frac{1}{2}\right)\pi - \varepsilon^4 g_3},$$

recall that on $\mathscr{N} - \{0\}$, $k = -2m_0/\varepsilon^3$ and let

$$S(p_1, \varepsilon) = \frac{\varepsilon^3\left[\left(m_0+\frac{1}{2}+\frac{2m_0}{\varepsilon^3}\right)\pi - \varepsilon^4 g_3 + \varepsilon^4 g_1\right]}{\left(m_0+\frac{1}{2}\right)\pi - \varepsilon^4 g_3} - (p_1+p_3)^3$$

$$= \varepsilon^3 + \frac{2m_0\pi + \varepsilon^4 g_1}{(m_0+1/2)\pi - \varepsilon^4 g_3} - (p_1+p_3)^3,$$

where $T$ is defined implicitly as above. It is important to note that $T$ is not defined at $\varepsilon = 0$, but only that the $g_i(T, p_1, p_3, \varepsilon)$ and $\partial g_i/\partial p_i(T, p_1, p_3, \varepsilon)$ are order 1 in $\varepsilon$ by Lemmas 5.1 and 5.2, so that we cannot use the Implicit Function Theorem here directly. Instead, we must use Arenstorf's Implicit Function Theorem, since we do have that $S(p_1, 0) = 4m_0/(2m_0+1) - (p_1+p_3)^3$. Arenstorf's Theorem [1] is as follows:

THEOREM 5.1. *Let $P$ and $Y$ be Banach spaces with elements $p$ and $y$. Let $f$ be a mapping from the product space $P \times Y$ into $P$, given by $(p, y) \to f(p, y) \subset P$, and defined for $p$ in a ball $B^*$ around some $p^* \in P$ and $y$ in a region $B$ of $Y$ containing $y = 0$, with $f(p^*, 0) = p^*$, and*

$$B^*\colon \|p - p^*\| \leqslant \alpha, \qquad \alpha > 0.$$

*If, for every $y$ in $B$, $f$ is differentiable with respect to $p$ in $B^*$ and*

$$\|f_p(p, y)\| \leqslant \eta \leqslant \tfrac{1}{2} \qquad \text{on } B^* \times B,$$

*(where $f_p$ denotes the partial derivative of $f$ with respect to $p$, and the norm is the sup norm of the linear operator from $P$ to itself) and if*

$$\|f(p^*, y) - p^*\| \leqslant \tfrac{1}{2}\alpha \qquad \text{on } B,$$

*then there exists a function $p(y)$ with*

$$f(p(y), y) = p(y),\ p(y) \subset B^* \qquad \text{for} \quad y \subset B, \quad p(0) = p^*.$$

Now we can show, with the help of Lemmas 5.1 and 5.2, that the function $S(p_1, \varepsilon) + p_1$, which has a fixed point at $p_1^*$ when we let $p_1^* = [4m_0/(2m_0+1)]^{1/3} - p_3$, satisfies the hypotheses of Arenstorf's Implicit Function Theorem.

LEMMA 5.3. *Let the function $S(p_1, \varepsilon)$ be given as above. Then the function $(1/5)\,S(p_1, \varepsilon) + p_1$ satisfies the hypotheses of Arenstorf's Implicit Function Theorem. Thus, there exists an $\varepsilon$-neighborhood of 0, $B$, and a function $p_1(\varepsilon)$ with $S(p_1(\varepsilon), \varepsilon) = 0$, $p_1(\varepsilon)$ contained in a neighborhood $\mathscr{P}$ of $[4m_0/(2m_0+1)]^{1/3} - p_3$, for all $\varepsilon \in B$.*

*Proof.* To prove this lemma, we first let $H(p_1, \varepsilon) = (1/5)\,S(p_1, \varepsilon) + p_1$. Thus,

$$H(p_1, \varepsilon) = \frac{1}{5}\left(\varepsilon^3 + \frac{2m_0\pi + \varepsilon^4 g_1}{(m_0+1/2)\,\pi - \varepsilon^4 g_3} - (p_1+p_3)^3\right) + p_1,$$

Then if we let $p_1^* = [2m_0/(m_0+1/2)]^{1/3} - p_3$, we have

$$H(p_1^*, 0) = \tfrac{1}{5}\,S(p_1^*, 0) + p_1^* = 0 + p_1^* = p_1^*.$$

since we can define the $\varepsilon^4 g_i(T, p_1, \varepsilon)$ to be continuous at $\varepsilon = 0$ by letting it take on the value of its limit, which is *zero* (from Lemma 5.1). In this way, $H$ is defined on $\mathscr{V} \times \mathscr{N}$.

Now, for $m_0 \geqslant 1$, we have that $p_1^* + p_3 \leqslant 2^{1/3}$, so that the expression $|1 - (3/5)(p_1^* + p_3)^2| < 1/8$. Since this is a function continuous in $p_1^*$, let $\beta > 0$ be chosen so that $|p_1 - p_1^*| < \beta$ implies that $||1 - (3/5)(p_1 + p_3)^2| - |1 - (3/5)(p_1^* + p_3)^2|| < 1/8$. Choose $0 < \alpha < \beta$ and let

$$B^* = \{p_1 \colon |p_1 - p_1^*| \leqslant \alpha < \beta\} \cap \mathscr{V}$$

Each term of $\varepsilon^4 g_i(T, p_1, \varepsilon)$ is differentiable at $\varepsilon = 0$ if we let the derivative be the zero operator for any $p_1$. While the derivative is not continuous, the partial $\partial(\varepsilon^4 g_i)/\partial p_1$ exists and is equal to zero by Lemma 5.2, and this is sufficient for fulfilling the hypotheses of the theorem. That is, we have that $H$ is differentiable with respect to $p_1$ in $B^*$ for every $\varepsilon \in \mathscr{N}$. ∎

For $\varepsilon \neq 0$, we calculate that

$$\frac{\partial H}{\partial p_1} = \frac{1}{5}\,[\varepsilon^4 J - 3(p_1+p_3)^2] + 1$$

where $J$ is uniformly bounded as $\varepsilon$ approaches zero by Lemmas 5.1 and 5.2. Thus, we can choose neighborhood $\mathcal{N}^*\mathcal{N}$ of zero such that $(1/5)\,\varepsilon^4\,|J| \leqslant 1/4$. Then $p_1 \in B^*$ implies that

$$\begin{aligned}&|1-(3/5)(p_1+p_3)^2| - |1-(3/5)(p_1^*+p_3)^2|\\ &\quad \leqslant ||1-(3/5)(p_1+p_3)^2| - |1-(3/5)(p_1^*+p_3)^2|| < 1/8\end{aligned}$$

so that

$$|1-(3/5)(p_1+p_3)^2| < 1/8 + |1-(3/5)(p_1^*+p_3)^2| \leqslant 1/4.$$

Thus,

$$\begin{aligned}\left|\frac{\partial H}{\partial p_1}\right| &= \left|\frac{1}{5}\,[\varepsilon^4 J - 3(p_1+p_3)^2] + 1\right|\\ &\leqslant \left|\frac{1}{5}\,(\varepsilon^4 J)\right| + |1-(3/5)(p_1+p_3)^2|\\ &\leqslant \frac{1}{2}\end{aligned}$$

on the neighborhood $B^* \times \mathcal{N}^*$.

Next,

$$\begin{aligned}H(p_1^*, \varepsilon) - p_1^* &= \frac{1}{5}\left(\varepsilon^3 + \frac{2m_0\pi + \varepsilon^4 g_1}{(m_0+1/2)\,\pi - \varepsilon^4 g_3} - (p_1^* + p_3)^3\right)\\ &= \frac{1}{5}\left(\varepsilon^3 + \frac{2m_0\pi + \varepsilon^4 g_1}{(m_0+1/2)\,\pi - \varepsilon^4 g_3} - \frac{2m_0}{m_0+1/2}\right),\end{aligned}$$

and since the function $(2m_0\pi + \varepsilon^4 g_1)/((m_0+1/2)\,\pi - \varepsilon^4 g_3)$ is continuous in $\varepsilon$ at $\varepsilon = 0$, we can find a neighborhood $\tilde{B}$ of zero such that, for all $\varepsilon \in \tilde{B}$ we have that

$$|H(p_1^*, \varepsilon) - p_1^*| \leqslant \frac{1}{5}\,\varepsilon^3 + \frac{1}{5}\left|\frac{2m_0\pi + \varepsilon^4 g_1}{(m_0+1/2)\,\pi - \varepsilon^4 g_3} - \frac{2m_0}{m_0+1/2}\right| < \frac{\alpha}{2}.$$

Letting $B = \tilde{B} \cap \mathcal{N}^*$, we have both of the last two inequalities on the neighborhood $B$.

Finally, what remains is only to note that if $\mathcal{W} = \{\varepsilon \in (B - \{0\}) : \varepsilon^3 = 1/n,\ \text{for } n \in \mathbb{Z}^*\}$, then for $\varepsilon \in \mathcal{W}$ and $p_3$ arbitrary we can find functions $\tilde{T}(\varepsilon, p_3) \cong 2\pi(2m_0+1) + 8m_0\pi/\varepsilon^3$, $p_2(\varepsilon, p_3) \cong 0$, and $p_1(\varepsilon)$ such that $(p_1+p_3)^3 \cong 4m_0/(2m_0+1)$ and such that

$$\left(\frac{\varepsilon^3}{(p_1(\varepsilon)+p_3)^3}-1\right)\tilde{T}(\varepsilon, p_3)-\frac{2m_0\pi}{\varepsilon^3}+\varepsilon^4 g_1=0,$$

$$p_2(\varepsilon, p_3)\sin\tilde{T}(\varepsilon, p_3)+\varepsilon 4 g_2=0,$$

$$\left(\frac{\varepsilon^3}{(p_1(\varepsilon)+p_3)^3}\right)\tilde{T}(\varepsilon, p_3)-\left(m_0+\frac{1}{2}\right)\pi+\varepsilon^4 g_3=0.$$

So for each $\varepsilon \in \mathscr{W}$, since $\varepsilon^3=1/n$ so that $2m_0\pi/\varepsilon^3=k\pi$ for $k$ an integer, we obtain the necessary period and initial conditions.

Since $Q_2$ is close to 0 for $0 \leqslant t \leqslant \tilde{T}$ and thus for all time by the symmetry of the orbit, and $P_2 \cong p_2$ which is also close to 0, these periodic solutions are nearly circular (in the fixed frame of reference). Choosing the free initial condition $p_3$ is like choosing angular momentum, or total energy for the system. This completes the proof of Theorem 3.1.

## 6. PROOF FOR THE LUNAR PROBLEM

Consider the equations (21). The periodicity conditions remain: at $t=0$; $Q_1=i\pi$, $Q_2=0$, $Q_3=j\pi$ and at $t=T$; $Q_1=(i+k)\pi$, $Q_2=0$, $Q_3=(j+m+\frac{1}{2})\pi$ where $i$, $j$, are 0 or 1 and $k$, and $m$ are arbitrary integers. By an argument similar to the proof of Lemma 5.1 of the previous section, the solution to this system of differential equations (21) is of the form:

$$Q_1(t)=\left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}-1\right)t+q_1+\varepsilon^3 g_1, \qquad P_1(t)=p_1+\varepsilon^3 g_4,$$

$$Q_2(t)=q_2\cos t+p_2\sin t+\varepsilon^3 g_2, \qquad P_2(t)=p_2\cos t-q_2\sin t+\varepsilon^3 g_5,$$

$$Q_3(t)=\left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\right)t+q_3+\varepsilon^3 g_3, \qquad P_3(t)=p_3+\varepsilon^3 g_6,$$

for initial conditions $(q_1, q_2, q_3, p_1, p_2, p_3)$ and for time $t\varepsilon[0, \gamma\varepsilon]$, where $g_i=g_i(t, q_1, q_2, q_3, p_1, p_2, p_3)$.

To satisfy the symmetry condition at $t=0$ we have the solution

$$Q_1(t)=\left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}-1\right)t+i\pi+\varepsilon^3 g_1,$$

$$Q_2(t)=p_2\sin t+\varepsilon^3 g_2,$$

$$Q_3(t)=\left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\right)t+j\pi+\varepsilon^3 g_3.$$

Next we need to solve this for the symmetry condition at $t = T$ which are now:

$$Q_1(t) = \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1\right) t + i\pi + \varepsilon^3 g_1 = (i+k)\,\pi,$$

$$Q_2(t) = p_2 \sin t + \varepsilon^3 g_2 = 0,$$

$$Q_3(t) = \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\right) t + j\pi + \varepsilon^3 g_3 = \left(j + m + \frac{1}{2}\right)\pi,$$

or

$$\left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1\right) t - k\pi + \varepsilon^3 g_1 = 0,$$

$$p_2 \sin t + \varepsilon^3 g_2 = 0,$$

$$\left(\frac{\varepsilon^{-3}}{(P_1+P_3)^3} - 1\right) t - \left(m + \frac{1}{2}\right)\pi + \varepsilon^3 g_3 = 0.$$

This is done, as in Section 5, by applying the Implicit Function Theorem twice. First we consider the difference of the first and third equation, together with the second equation. This is the system of equations

$$t + (k - m - \tfrac{1}{2})\,\pi + \varepsilon^3 g_3 - \varepsilon^3 g_1 = 0,$$

$$p_2 \sin t + \varepsilon^3 g_2 = 0.$$

This has solution $t = (m + 1/2 - k)\,\pi$, $p_2 = 0$ at $\varepsilon = 0$. Along this solution, the determinant of the derivative of the system with respect to $t$ and $p_2$ is given by

$$\begin{vmatrix} 1 & 0 \\ 0 & \sin(m + 1/2 - k)\,\pi) \end{vmatrix} = \pm 1 \neq 0.$$

Thus by the Implicit Function Theorem, there exists a neighborhood $\mathcal{N}$ of 0 and functions $T(p_1, p_3, \varepsilon)$ near $(m + 1/2 - k)\,\pi$ and $p_2(p_1, p_3, \varepsilon)$ near 0 for $\varepsilon \in \mathcal{N}$ and $(p_1, p_3)$ arbitrary such that

$$T(p_1, p_3, \varepsilon) - (m + \tfrac{1}{2} - k)\,\pi + \varepsilon^3 g_3 - \varepsilon^3 g_1 = 0,$$

$$p_2(p_1, p_3, \varepsilon) \sin T(p_1, p_3, \varepsilon) + \varepsilon^3 g_2 = 0.$$

Putting this solution for $T$ into the third equation, we need to solve $Q_3(T)-(m+\frac{1}{2})\,\pi=0$, or

$$\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\left[\left(m+\frac{1}{2}-k\right)\pi-\varepsilon^3 g_3+\varepsilon^3 g_1\right]-\left(m+\frac{1}{2}\right)\pi+\varepsilon^3 g_3=0,$$

which is equivalent to solving

$$\left(m+\frac{1}{2}-k\right)\pi-\varepsilon^3 g_3+\varepsilon^3 g_1-\left(\varepsilon^3\left(m+\frac{1}{2}\right)\pi-\varepsilon^6 g_3\right)(p_1+p_3)^3=0$$

for $(p_1+p_3)^3$ whenever $\varepsilon\in\mathcal{N}-\{0\}$. Now the solution for $T$ left both $m$ and $k$ arbitrary, so for the moment regard $m$ and $k$ as free variables. Then letting $m+\frac{1}{2}=\varepsilon^{-3}$ and letting $k=m-n$ for $n$ a small integer, we seek to solve

$$R=(n+\tfrac{1}{2})\,\pi-\varepsilon^3 g_3+\varepsilon^3 g_1-(\pi-\varepsilon^6 g_3)(p_1+p_3)^3=0.$$

By this choice of $m$ and $k$, $T$ becomes $T(p_1, p_3, \varepsilon)=(n+1/2)\,\pi+\varepsilon^3 g_3-\varepsilon^3 g_1$ which is uniformly bounded as $\varepsilon$ approaches zero and, by part 2 of the Implicit Function Theorem, $\partial T/\partial p_1$ along solutions at $\varepsilon=0$ is given by $-4\,\partial(\varepsilon^3 g_6-\varepsilon^3 g_2)/\partial p_1$ where the partials of the $g_i$ are evaluated along solutions; $t=(n+\frac{1}{2})\,\pi$, $p_2=0$, $\varepsilon=0$, $(p_1, p_3)$ arbitrary. By the argument in the Section 5, the partials of the $g_i$ with respect to initial conditions are also uniformly bounded as $\varepsilon$ approaches zero. Thus we can differentiate $R$ along the solution $(p_1+p_3)^3=n+1/2$, $\varepsilon=0$ to get $\partial R/\partial p_1=-3\pi(n+1/2)^{2/3}\neq 0$.

Thus we have shown that there exists a deleted neighborhood $\mathcal{N}-\{0\}$ of 0 such whenever $\varepsilon\in\mathcal{N}-\{0\}$ and $\varepsilon$ is of the form $(m+1/2)^{-1/3}$ for $m$ an integer, the system has a periodic solution with period near $(4n+2)\,\pi$ for $n$ a small integer. These solutions are doubly symmetric, and approximately small circular orbits of the Kepler problem.

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