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Limit Periodic Functions, Adding Machines, and Solenoids[1]

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We prove that a stable adding machine invariant set for a homeomorphism of the plane is the limit of periodic points and also that a stable solenoid minimal invariant set for a three dimensional flow is the limit of periodic orbits. We give an example to show that a similar result is false in higher dimensions.

KEY WORDS: Dynamical systems; almost periodic; solenoid; adding machine.
SUBJECT CLASSIFICATIONS: 34C35, 34C27, 54H20.

1. INTRODUCTION

Consider a flow depending on a parameter μ. Imagine that as the parameter μ varies a sequence of bifurcations of periodic solutions occurs at the parameter values $\mu_1, \mu_2,...$. At the parameter value μ_i a new periodic solution bifurcates from the old with period p_i times the old period ($p_i \in \mathbb{Z}$, $p_i \geqslant 2$). It may happen that as $\mu_i \rightarrow \mu_\infty < \infty$ this sequence of periodic solutions converges uniformly to an almost periodic function. Such an almost periodic solution is called a limit periodic function and its hull is a solenoid minimal set. In a dissipative system usually $p_i = 2$ for all i so only one type of solenoid occurs, but for a Hamiltonian system the p_i are unrestricted in general, so an uncountable number of possible solenoids may occur. Indeed, in Markus and Meyer (1980), it was shown that generically a C^∞ Hamiltonian on a compact symplectic manifold has solenoid minimal sets of every possible type. Solenoid minimal sets admit a cross section which

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is a Cantor set and the section map is a generalized adding machine. So adding machines occur naturally in Hamiltonian systems.

Professor George Sell asked if an adding machine and solenoid minimal set are always the limit of periodic orbits. In low dimensions the answer is yes. In this note we will show that a stable adding machine minimal set for a homeomorphism of the plane is the limit of periodic points and also a flow on a three-dimensional manifold which has a stable solenoid minimal set is the limit of periodic solutions. Buescu and Stewart (preprint) show that for continuous maps of the interval, stable adding machines are limits of periodic orbits. Finally, we give a homeomorphism of $\mathbb{R}^3$ which has a stable adding machines as an invariant set, but which has no periodic points.

2. LIMIT PERIODIC FUNCTIONS, SOLENOIDS AND ADDING MACHINES

In this section we will recall some basic definitions and their interconnections. A continuous function $f\colon \mathbb{R}^1 \to \mathbb{R}^m$ is called *limit periodic* if it is the uniform limit of continuous periodic functions, but not periodic itself. It is a special kind of almost periodic function. We use the term "almost periodic" in the sense of the founder of the subject Bohr (1951). As an example consider

$$F(t) = \sum_0^\infty a_k \cos 2\pi t/2^k$$

where the $a_k \in \mathbb{R}^m$, $a_k \neq 0$ and are chosen to that the series converges uniformly. The nth partial sum is periodic with period 2^n but the limit is clearly not periodic.

Let $\mathscr{C} = \mathscr{C}(\mathbb{R}, \mathbb{R}^m)$ be the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^m$ with the topology of uniform convergence on compact sets. Given $f \in \mathscr{C}$ and $\tau \in \mathbb{R}$ define the *translate of f by τ* to be f_τ where $f_\tau(t) = f(t+\tau)$. The map $\Pi\colon \mathbb{R} \times \mathscr{C} \to \mathscr{C}\colon (\tau, f) \to f_\tau$ defines a flow on $\mathscr{C}$. For any $f \in \mathscr{C}$ the *hull of f* is $H(f) = Cl\{f_\tau\colon \tau \in \mathbb{R}\}$ i.e. the hull is the orbit closure of the trajectory through f. For example the hull of $F(t)$ is all the functions of the form

$$G(t) = \sum_0^\infty a_k \cos 2\pi(t - \phi_k)/2^k$$

where ϕ_k is an angle defined modulo 2^k and $\phi_k \equiv \phi_{k+1} \bmod 2^k$. If f is an almost periodic function, then the restriction $\Pi | H(f)$ is a compact minimal set, each element of $H(f)$ is almost periodic, $\Pi | H(f)$ is equicontinuous, and $H(f)$ can be given a compact, connected, Abelian group structure. See Nemytskii and Stepanov (1960). If f is limit periodic then $H(f)$ is a solenoid minimal set, Pontryagin (1966).

Topologically a solenoid is the inverse limit system

$$\mathbf{S}: S^1 \xleftarrow{z^{p_1}} S^1 \xleftarrow{z^{p_2}} S^1 \xleftarrow{z^{p_3}} \cdots$$

where S^1 is the unit circle in the complex plane, the $p_k \geqslant 2$ are integers, and $S^1 \xleftarrow{z^{p_k}} S^1$ denotes the mapping of the circle into itself by $z \to z^{p_k}$. A point $\mathbf{z} \in S$ is of the form $\mathbf{z} = (z_0, z_1, z_2, \ldots)$ where $z_{k-1} = z_k^{p_k}$. In the case when $p_k \equiv 2$ an element of S is of the form

$$(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}, \ldots)$$

where $\theta_k = 2\pi\phi_k/2^k$ and ϕ_k is an angle defined modulo 2^k and $\phi_k \equiv \phi_{k+1} \bmod 2^k$. This is the point in the inverse limit system corresponding to $G \in H(F)$ given earlier. This corresponds to the usual torus inside a torus description of a solenoid as shown in Fig. 1.

The solenoid minimal flow, $\Pi_t: \mathbf{S} \to \mathbf{S}$, is defined by

$$\Pi_t: (\ldots, z_k, \ldots) \to (\ldots, z_k e^{(i2\pi t/q_k)}, \ldots)$$

where $q_0 = 1$, and $q_k = p_1 \cdots p_k$ for $k \geqslant 1$. An elementary discussion of the case where $p_k \equiv 2$ and additional references can be found in Meyer and Seller (1989) [pp. 68–74]. This flow admits a cross section $\mathbf{C} = \{z \in \mathbf{S}: z_0 = 1\}$ with first return time $T = 1$ and Poincaré map $\mathbf{P} = \Pi_1$.

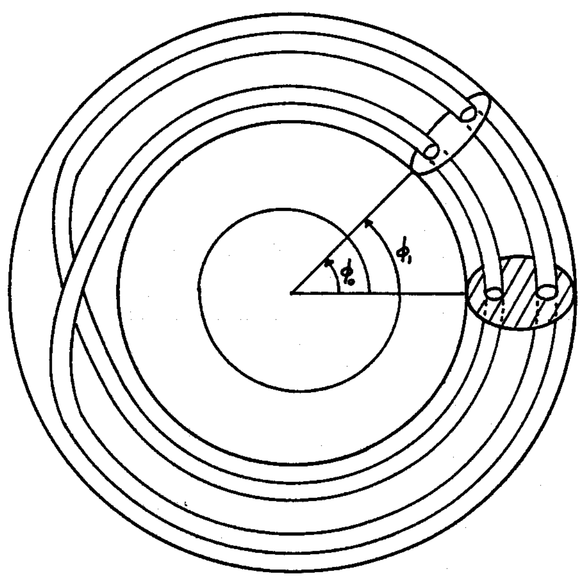

Fig. 1. A solenoid in 3-space.

A point $z \in \mathbf{C}$ is of the form $z = (1, z_1, z_2, \ldots)$ where z_1 is a p_1 root of unity, z_2 is a p_2 root of p_1 etc.

Since all the nth roots of unity are of the form $\exp(i2\pi\alpha/n)$ where $\alpha \in \mathbb{Z}_n$ we see that $\mathbf{C}$ is equivalent to the inverse limit system

$$\mathbf{D}: \mathbb{Z}_{q_1} \xleftarrow{\imath} \mathbb{Z}_{q_2} \xleftarrow{\imath} \mathbb{Z}_{q_3} \xleftarrow{\imath} \cdots$$

where $\imath$ is the map $\imath: \mathbb{Z}_{q_k} \to \mathbb{Z}_{q_{k-1}}: \alpha \to \alpha \bmod q_{k-1}$. So if $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbf{D}$ then $\alpha_{k-1} \equiv \alpha_k \bmod q_{k-1}$. The map $\mathbf{P}$ on $\mathbf{D}$ is

$$\mathbf{P}: \mathbf{D} \to \mathbf{D}: (\ldots, \alpha_k, \ldots) \to (\ldots, \alpha_k + 1, \ldots)$$

Clearly $\mathbf{D}$ is a Cantor set and $\mathbf{P}$ is a homeomorphism. This is not the usual definition of the generalized adding machine, but it is conjugate to the usual adding machine. The usual definition is defined on the set

$$\mathbf{E} = \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$$

with the product topology and the adding machine operator, $A: \mathbf{E} \to \mathbf{E}$, is defined by $\mathbf{A}: (\ldots, \alpha_k, \ldots) \to (\ldots, \gamma_k, \ldots)$ where $\gamma_k = 0$ for all k if $\alpha_k = p_k - 1$ for all k or if the first index where $\alpha_k < p_k - 1$ is r then $\gamma_k = 0$ for $1 \leqslant k < r$, $\gamma_r = \alpha_r + 1$; and $\gamma_k = \alpha_k$ for $k > r$.

The conjugacy map is defined by $\mathbf{H}: \mathbf{D} \to \mathbf{E}: (\ldots, \alpha_k, \ldots) \to (\ldots, \beta_k, \ldots)$ where $\alpha_k = \beta_k q_{k-1} + \alpha_{k-1}$. It is easy to see that $\mathbf{H}$ is a homeomorphism and the map $\mathbf{P}$ becomes $\mathbf{A} = \mathbf{H} \circ \mathbf{P} \circ \mathbf{H}^{-1}$ which is the generalized addition operator defined before.

This addition algorithm will be familiar to anyone who knew the old British coinage: four farthings makes a penny, twelve pence makes a shilling, and twenty shillings makes a pound. (How much is 2/19/11 3/4 and 1/4?, answer: £3.) One can see a picture of a turn of the century adding machine, a *comptometer*, [Horsburgh (1914) p. xi] which was built especially for the British market. On the far right the is a column of three keys marked, 1/4, 1/2, 3/4 for farthings; to its left is a column of eleven keys marked 1, 2,..., 11 for pence; to the left are two column of keys the first marked 1,..., 9 and the second column with one key marked 10 for shillings; and to the far left there are several columns of keys marked 1,..., 9 for pounds.

To realize the adding machine with $p_k \equiv 2$ as an invariant set of a plane homeomorphism we follow Buescu and Stewart (preprint), who treat the general case. Consider Fig. 2 which illustrates a flow on a closed disk with boundary C_1 and three fixed points, a saddle and two sinks, which are the ω-limit sets of all the trajectories. Figure 2 is symmetric about the

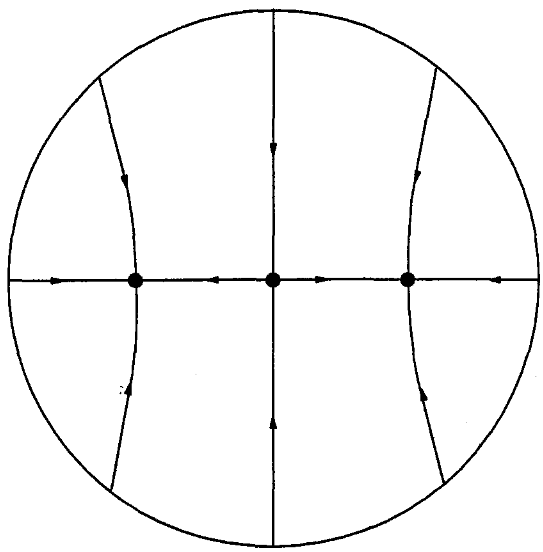

Fig. 2. A flow on the disk.

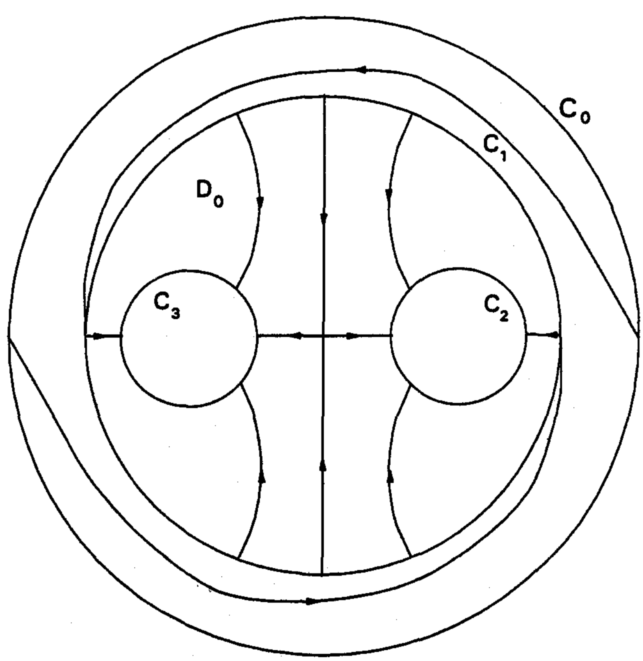

Fig. 3. The homeomorphism of the disk with two disks excised.

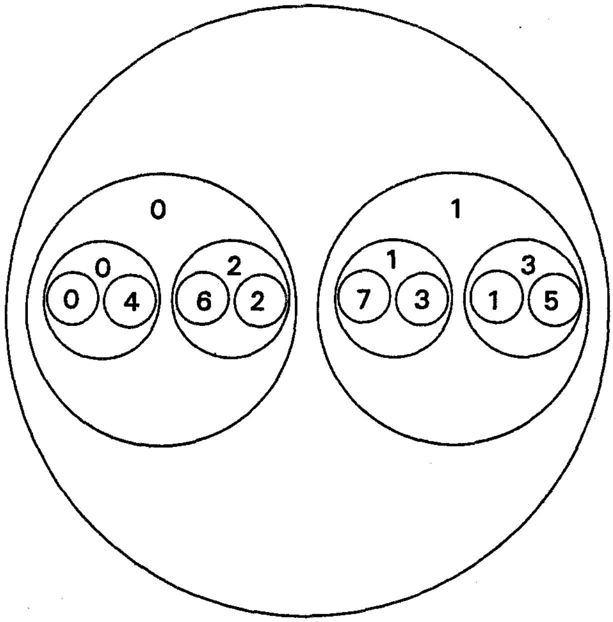

Fig. 4. The pattern of how the disks are labeled and mapped.

origin. Excise from this disk two equal small disks about the two sinks with boundary circles C_2 and C_3. Slow the flow down so that the three circles C_1, C_2, C_3 are fixed under the flow and let H_1 be the time one map of this flow composed with a rotation by 180°. Thus H_1 maps C_1 into itself and interchanges C_2 and C_3. Now enclose C_1 in a slightly larger circle C_0 and extend H_1 out to C_0 by mapping concentric circles into themselves and adjusting the rotation on these circles so that H_1 is the identity on C_0. Let this disk with two holes be D_0. See Fig. 3.

Paste two copies of D_0, called $D_{0,0}$ and $D_{0,1}$ onto D_0 one in each of the excised circles as shown in Fig. 4 giving a disk with four circles excised. Paste four copies of D_0 onto these boundary circles denoting the disk $D_{0,0,0}$, $D_{0,0,2}$, $D_{0,1,1}$, $D_{0,1,3}$ etc. Extend H_1 at each step and repeat the process until a homeomorphism of the plane is obtained.

In Fig. 4 only the last digit in the numbering of the disks is shown, so the disk marked 5 is inside the disk marked 3 which is inside 1 which is inside 0 and so is disk $D_{0,1,3,5}$. The smallest disks marked 0–7 are mapped into each other by adding 1 mod 8. Thus the resulting homeomorphism maps the disk into itself and has an adding machine as an invariant set—note that this adding machine is stable and is the limit of hyperbolic periodic points. It can be extended to all of $\mathbb{R}^2$ by making it the identity outside C_0.

The suspension of this homeomorphism is a flow on $S^1 \times S^3$ which has a solenoid minimal set.

3. PERIODIC POINTS AND ADDING MACHINES

The basic references for the material of this section are Bell (1976, 1977) and the references therein. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism.

Definitions:

(1) A *continuum* is a nonempty, compact, connected set.

(2) Let $A \subset \mathbb{R}^2$ then the *topological hull* of A, $T(A)$, is the union of A and all of its bounded complementary domains.

(3) Let $A \subset \mathbb{R}^2$ then the ω-limit set of A (under f) is

$$\omega(A) = \bigcap_{j=1}^{\infty} cl\left(\bigcup_{k=j}^{\infty} f^k(A)\right)$$

where cl is the closure operator.

(4) An invariant set A for f is *stable* if for every neighborhood U of A there is a neighborhood V of A such that $f^k(V) \subset U$ for all $k \geqslant 0$. Note that by replacing V by $\bigcup_{k=0}^{\infty} f^k(V)$ if necessary one can assume that

$$f^k(V) \subset V \subset U$$

Lemma 1. *Let $A \in \mathbb{R}^2$. If A is a continuum then $T(A)$ is a nonseparating plane continuum. If A is connected, $A \cap f(A) \neq \varnothing$, and $f^k(A)$ is contained in a compact set for $k \geqslant 0$ then $\omega(A)$ is an invariant continuum and $T(\omega(A))$ is a nonseparating, invariant continuum.*

See Nemytskii and Stepanov (1960) and Bell (1977) for details.

Theorem 1. (The Bell-Cartwright-Littlewood Theorem) *If B is a nonseparating, invariant continuum for the homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$, then f has a fixed point in B.*

See Cartwright and Littlewood (1951); and Bell (1976, 1977).

Theorem 2. *If A is a stable invariant set for the homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$, such that $f|A$ is conjugate to the adding machine then A is the limit of periodic points of f.*

This theorem follows at once from the following theorem which is not surprising since Buescu and Stewart (preprint) show that stable Cantor sets are adding machines in general.

Theorem 3. *If Λ is a stable, totally disconnected, transitive, invariant set for the homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, then Λ is the limit of periodic points.*

Proof. Let $p \in \Lambda$ and P any neighborhood of p—without loss of generality we may take P to be a closed disk, so P is compact and convex. Λ is totally disconnected so there exist open sets A and B such that (i) $U = A \cup B$ is a neighborhood of Λ, (ii) $clA \cap clB = \emptyset$, (iii) $clA \subset P$.

Since Λ is stable there is a neighborhood V of Λ such that $f^k(V) \subset V \subset U$ for all $k \geqslant 0$. Let Q be the connected component of V which contains p, so $Q \subset A$.

Since f is transitive on Λ there is an $s > 0$ and an $r \in Q$ such that $q = f^s(r) \in Q$. Let $F = f^s$. Since Q is a component of V; $F(V) \subset V$; and $r, q = F(r) \in Q$ it follows that $F(Q) \subset A$ and $\Delta = \bigcup_{n \geqslant 0} F^n(Q) \subset A$. Since $Q \subset P$ and P is compact $\omega(Q)$ is a continuum in P. Then $T(\omega(Q)) \subset P$ since P is convex. $T(\omega(Q))$ is an F-invariant, nonseparating continuum and so by the Bell-Cartwright-Littlewood theorem F has a fixed point in $T(\omega(Q))$ and hence in P. This fixed point of F is a periodic point of f. □

4. PERIOD ORBITS AND SOLENOIDS

Let Φ_t be a smooth flow on a m-dimensional manifold M with distance function d, so $\Phi_t\colon M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$.

Definitions.

(1) An invariant set A for Φ_t is *stable* if for every neighborhood U of A there is a neighborhood V of A such that $\Phi_t(V) \subset U$ for all $t \geqslant 0$. Note that by replacing V by $\bigcup_{t \geqslant 0} \Phi_t(V)$ if necessary one can assume that $\Phi_t(V) \subset V \subset U$ for all $t \geqslant 0$.

(2) A *flow box* at $p \in M$ is a coordinate chart (W, α) at p where W is a neighborhood of p in M, $\alpha\colon W \to [-1, 1]^m \subset \mathbb{R}^m$ is a homeomorphism, $\alpha(p) = 0$, and the flow in this chart is $(t, (y_1, y_2, \ldots, y_m)) \to (y_1 + \omega t, y_2, \ldots, y_m)$ where $(y_1, \ldots, y_m)$ are coordinates in $\mathbb{R}^m$ and $\omega > 0$. A flow box exist at each noncritical point of a smooth flow Meyer and Hall (1992).

Theorem 2. *Let Φ_t be a smooth flow on a 3-manifold M. If $\Lambda \subset M$ is a stable invariant set conjugate to the solenoid minimal flow, then Λ is the limit of periodic orbits.*

Proof. As discussed earlier the solenoid minimal flow admits a cross section. Once we show that this cross section can be extended to a nice neighborhood and that the section map is continuous the proof follows the proof given here. This is true because the Bell-Cartwright-Littlewood Theorem does not require that the homeomorphism be defined in the whole plane but only in a simply connected neighborhood of the invariant continuum.

Pick any point $p \in \Lambda$ and an arbitrarily small neighborhood of p which we may take as a flow box (W, α) at p. Then $\Gamma = \alpha^{-1}(\{y_1 = 0\}) \subset M$ is a cross section to the flow on M at p. Since a solenoid is locally the product of an interval and a Cantor set the intersection $\Lambda' = \Lambda \cap \Gamma$ is totally disconnected. Let A be an open neighborhood of p in Γ totally contained in Γ such that $\partial A \cap \Lambda' = \emptyset$. Let $\Delta = A \cap \Lambda'$ and $\varepsilon = d(\Delta, \partial A) > 0$. Make ε smaller if necessary so that an ε-neighborhood of Δ is contained in W.

Since the flow on Λ is almost periodic there is a $\tau > 0$ such that $d(\Phi_\tau(q), q) < \varepsilon$ for all $q \in \Delta$. But this implies that $\Phi_\tau(q) \in W$. Since any trajectory which hits W must cross Γ there is a small change in τ to $\tau(q)$ such that $\Phi_{\tau(q)}(q) \in \Gamma$. But $\Phi_{\tau(q)}(q) \in \Lambda$ so $\Phi_{\tau(q)}(q) \in \Delta$. Clearly $\tau(q)$ is continuous. Thus the section map

$$P(q) = \Phi_{\tau(q)}(q) \colon \Delta \to \Delta$$

is continuous. Since the flow on the solenoid is minimal every orbit is dense, this implies that the map $P | \Delta$ is transitive.

The flow is smooth, so for each $q \in \Delta$ the section map P can be continuously extended to an open neighborhood of q and so the section map P can be extended to an open neighborhood X of Δ. Since Δ is totally disconnected we may decrease X if necessary so that X is simply connected—see Lemma 2. The solenoid is stable so there is an open neighborhood V of Δ such that $P \colon V \to V \subset X$. Since P is defined on X and a homeomorphism where defined, P can be extended to $T(V)$ so by replacing V by $T(V)$ if necessary we may assume that V is simply connected. Let Z be a component of V so Z is connected and simply connected. Let $z \in Z$. There is a $k > 0$ such that $P^k(z) \in Z$ so $P^k \colon Z \to Z$. Now follow the argument given before, but applied to P^k restricted to Z. □

Lemma 2. *Let Δ be nonseparating, compact set in the plane and X any neighborhood of Δ. Then there is a simply connected neighborhood X' of Δ with $X' \subseteq X$.*

Proof. For each positive integer n let $\Delta_n = \Delta \cup \{[c, d] \colon c, d \in \Lambda$ and $\rho(c, d) < 1/n\}$ where $[c, d]$ denotes the line segment joining c and d and ρ is the distance function. Since Δ is compact, Δ_n is compact also.

Since Δ does not separate, for each $x \notin \Delta$ there is an arc C joining x to infinity which does not meet Δ. Since Δ is compact, $\rho(\Delta, C)$ is positive and thus there is an n such that $1/n < \rho(\Delta, C)$ or $\rho(\Delta_n, C) > 0$. Thus x is in the unbounded domain of the complement of Δ_n, i.e. $x \notin T(\Delta_n)$. This proves that $\Delta = \bigcap_1^\infty T(\Delta_n)$.

Choose one point of Δ in each component of Δ_n. The distance between two such points must be greater than $1/n$, and so the set of such points has no limit point. But all these points lie in the compact set Δ and so this set must be finite. Thus Δ_n has only a finite number of components. Each component of $T(\Delta_n)$ contains a component of Δ_n, so $T(\Delta_n)$ has only a finite number of components.

We may assume that X is an open neighborhood of Δ. Since $\{T(\Delta_n)\}$ is a decreasing sequence of compact sets whose intersection is Δ, there is a k such that $T(\Delta_k) \subset X$.

$T(\Delta_k)$ is a finite union of nonseparating plane continua. A nonseparating plane continuum, P, can be written as $P = \bigcap_1^\infty B_n$ where each B_n is a closed topological ball, *int*$(B_n) \supset P$, and *int* $B_n \supset B_{n+1}$, see Bell (1977). For each component of $T(\Delta_k)$ choose one of these closed balls which contains the component in its interior and is interior to X and such that the balls are pairwise disjoint. The union of these balls is the desired neighborhood X'. □

This lemma is false in three space as Antoine's necklace shows [Hocking and Young (1961)].

5. A COUNTER EXAMPLE IN $\mathbb{R}^3$

In this section we show that *there is a homeomorphism of* $\mathbb{R}^3$ *into itself which admits a stable adding machine invariant set, but which has no periodic points.*

Consider a standardly embedded solid torus, $S = S_0 \subset \mathbb{R}^3$, as illustrated in Fig. 5 with toral coordinates r, radius, θ, meridian, and ϕ, longitude. Let the angles θ, ϕ be defined mod2π, and let r be measured from the central longitude with $0 \leqslant r \leqslant 2$. The desired homeomorphism will be defined in steps. The first homeomorphism, F: $(r, \theta, \phi) \to (r', \theta', \phi')$, is

$$\begin{aligned} r' &= r\{1 + \varepsilon(r-1)(r-2)\} \\ \theta' &= \theta \\ \phi' &= \phi + \alpha\{1 - r(2-r)\} + \{\pi + \varepsilon \sin^2 \phi\}\, r(2-r) \end{aligned} \tag{1}$$

where ε is a small positive number and α is a small positive number such that α/π is irrational. Note that F does not depend on θ.

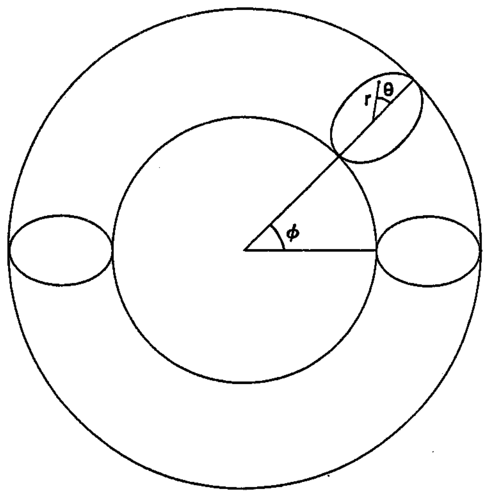

Fig. 5. Coordinates on the solid torus.

Denote the boundary torus where $r=2$ by T_2, the internal torus where $r=1$ by T_1 and the central circle where $r=0$ by T_0. Denote the two meridian circles on T_1 where $r=1$, $\phi=0$, π by C_0, C_1 respectively. See Fig. 6.

Note that the boundary torus, T_2, the internal torus T_1, and the central circle, T_0 are all mapped into themselves because $r'=r$ when $r=2$, 1, 0. On the boundary torus, T_2, and on the central circle, T_0, the map is

$$\theta'=\theta, \qquad \phi'=\phi+\alpha$$

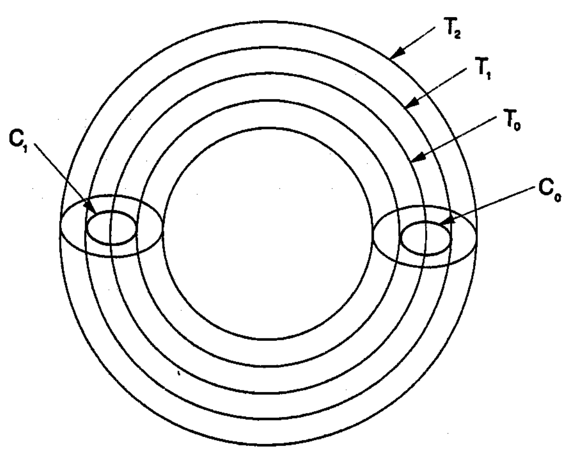

Fig. 6. Set labels.

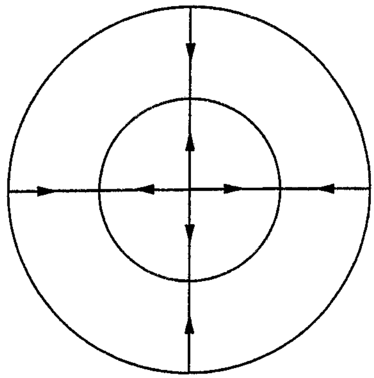

Fig. 7. Mapping of the r, θ disk.

which is periodic point free since α/π is irrational. On the internal torus, T_1 the maps is

$$\theta' = \theta, \qquad \phi' = \phi + \pi + \varepsilon \sin^2 \phi$$

Thus T_1 is rotated by π plus a small correction. The correction is positive except at $\phi = 0, \pi$. Thus the meridian circles C_0, C_1 are made up periodic points of period two. All other points on T_1 tend asymptotically to C_0, C_1 under iteration by F. Between T_2, T_1, T_0 points are mapped radially toward T_1 since

$$r' = r\{1 + \varepsilon(r-1)(r-2)\}$$

See Fig. 7.

Thus, *the only periodic points of F are on the two meridian circles C_0, C_1 which are all periodic points of period two.*

In the first step the homeomorphism F did not depend on θ. Now we will modify the map in a neighborhood of the meridian circles C_0, C_1 to eliminate all the periodic points. Choose toral coordinates (ρ_0, ψ_0, θ) in a neighborhood of C_0 as illustrated in Fig. 8. Here $0 \leqslant \rho_0 \leqslant 1$, and the angles ψ_0, θ are defined mod2π. Note that θ is the same angle as before, but in this new coordinate system it is the longitude whereas before it was the meridian. In a like manner choose toral coordinates (ρ_1, ψ_1, θ) in a neighborhood of C_1. The map F takes a neighborhood, N, where $0 \leqslant \rho_1 < 2/3$ of C_1 into a neighborhood of C_0, in these coordinates let it be

$$F: (\rho_1, \psi_1, \theta) \to (R_0, \Psi_0, \theta)$$

(recall that F leaves θ fixed). Let b: $\mathbb{R} \to \mathbb{R}$ be a continuous bump function which is 1 for $0 \leqslant x \leqslant 1/3$, positive for $1/3 < x < 2/3$ and 0 for $x \geqslant 2/3$.

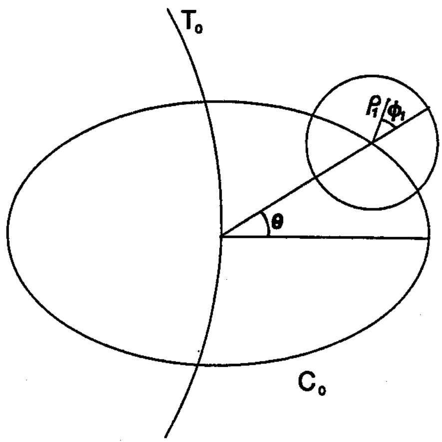

Fig. 8. Coordinates about C_0.

Denote the second map of the solid torus by H_0 where $H_0 = F$ outside the neighborhood N and on N it is

$$H_0: (\rho_1, \psi_1, \theta) \to (R_0, \Psi_0, \theta + b(\rho_1)\alpha)$$

Thus H_0 is just like F except the θ variable is augmented near C_1. H_0 maps C_0 onto C_1 as before (identity) but maps C_1 onto C_0 with the extra twist α. Thus H_0^2: $C_0 \to C_0 : \theta \to \theta + \alpha$ and similarly on C_1. Since α/π was irrational H_0 has no periodic points on C_0 or C_1.

In summary, H_0 *is a homeomorphism of the solid torus* S_0 *into itself which is periodic point free. On the boundary torus,* T_2, H_0 *maps the longitude* $\phi \to \phi + \alpha$ *and on the two meridian circles* C_0, C_1, $H_0^2 : \theta \to \theta + \alpha$.

Now proceed as with the planar example to define the homeomorphism iteratively. Blow up the circles C_0, C_1 into two boundary tori, such that H_0^2 still maps these boundary tori by $\theta \to \theta + \alpha$. Fill in the two tori so that they are solid tori, denoted by S_{01}, S_{11} and extend let H_1 be the extension of H_0 to the interior by mapping the solid tori S_{00} to S_{01} by the identity and S_{01} to S_{00} by the old H_0.

Now H_1 is a homeomorphism of $S = S_0$ which is periodic point free except on four circles each point of which is a point of period four. Blow up the circles to boundary tori, extend H_1 to their interior etc. In the limit, we have a homeomorphism H: $S \to S$ which is periodic point free. But H has a stable invariant Cantor set with a dense orbit, which is an adding machine, see Buescu and Stewart (1994).

It is easy to extend H to the exterior of S in $\mathbb{R}^3$ so that it is still periodic point free, or since S^3 is the union of two solid torus H can be extended to a periodic point free homeomorphism of S^3 with two stable adding machines.

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