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Apollonius Coordinates, the N-Body Problem, and Continuation of Periodic Solutions

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This paper treats the N-body problem and its relation to various restricted problems. For each solution of the Kepler problem a generalization of the pulsating coordinates used to express the Hamiltonian of the elliptic restricted three-body problem is given. These coordinates are called Apollonius coordinates. The method of symplectic scaling is used to give a precise derivation of the elliptic restricted problem showing the precise asymptotic relationship between the restricted problem and the full three-body problem. This derivation obviates the proof of the fact that a nondegenerate periodic solution of the elliptic restricted three-body problem can be continued into the full three-body problem under mild nonresonance assumptions. Also, the method of symplectic scaling is used to give a precise derivation of the elliptic Hill lunar equation showing the precise relationship between the elliptic Hill lunar equation and the full three-body problem. A similar continuation theorem is established.

KEY WORDS: N-body problem; elliptic restricted problem; continuation of periodic solutions; Hill's lunar problem.
AMS SUBJECT CLASSIFICATION: 34C25, 70F07, 70F10, 70H05.

I. INTRODUCTION

This paper deals with the planar N-body problem of classical celestial mechanics, its relation to various restricted problems which are defined, a special coordinate system, and the continuation of periodic solutions.

For each solution of the Kepler problem a generalization of the rotating-pulsating coordinates used to express the Hamiltonian of the elliptic restricted three-body problem is given. The Kepler problem is the central force problem with an attractive inverse square law force at the origin; see

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Eq. (I.1) below. In particular, any solution of the Kepler problem (be it circular, elliptic, parabolic, or hyperbolic) will give rise to a coordinate system in which the Hamiltonian of the full planar, N-body problem is relatively simple. If the solution of the Kepler problem is circular, then this coordinate system is the standard rotating coordinates, and if the solution of the Kepler problem is elliptic, this coordinate system is the rotating-pulsating coordinates used in the elliptic restricted three-body problem. The derivation given below stresses the role of the Kepler problem, and so avoids some of the tedious trigonometry of the standard derivation. It would be tempting to call these coordinates "Kepler coordinates" but that name already has a well-established meaning in celestial mechanics, so these coordinates are called Apollonius coordinates after Apollonius of Perga (ca. 262–200 B.C.), who wrote the definitive book on conic sections. The origins of rotating-pulsating coordinates and the elliptic restricted three-body problem goes back to the work of Scheibner (1866) and was rediscovered by Nechvile (1926) and others. The rotating-pulsating coordinates were used to put the three-body problem in a simple form by Waldvogel (1973) for a different goal. The notes of Szebehely (1967) have more information on the historical works.

A central configuration of the N-body problem is an equilibrium point in these coordinates, so it is called a relative equilibrium also. Given any central configuration of the N-body problem and any solution of the Kepler problem, then there is a restricted $(N+1)$-body problem where N of the bodies move on the solution of the Kepler problem while maintaining their relative position similar to the central configuration and there is an infinitesimal body moving under their gravitational attraction. For example, there is a restricted four-body problem where three bodies of arbitrary mass move on hyperbolic orbits of the Kepler problem such that at each instant they are at the vertices of an equilateral triangle and a fourth infinitesimal body moves under the gravitational attraction of the other three but does not in turn influence the motion of the three finite bodies. To my knowledge the only reference to something other than the circular or elliptic restricted problems is by Faintich (1972), who considered the hyperbolic restricted three-body problem.

The method of symplectic scaling is used to give a precise derivation of such a restricted problem showing the precise asymptotic relationship between the restricted problem and the full $(N+1)$-body problem. This derivation obviates the proof of the fact that a nondegenerate periodic solution of the elliptic restricted $(N+1)$-body problem can be continued into the full $(N+1)$-body problem under mild nonresonance assumptions. A similar theorem was proved for the circular restricted $(N+1)$-body problem by Meyer (1981, 1984).

Jacobi coordinates are extremely useful coordinates in celestial mechanics and Apollonius coordinates work well with them. The Hamiltonian of the N-body problem is particularly nice in Jacobi–Apollonius coordinates.

The method of symplectic scaling is used to give a precise derivation of an elliptic Hill lunar equation showing the precise relationship between the elliptic Hill lunar equation and the full three-body problem. This derivation obviates the proof of the fact that a nondegenerate periodic solution of the elliptic Hill's lunar equation can be continued into the full three-body problem. A similar theorem was proved for the (circular) Hill's lunar equation by Meyer and Schmidt (1982).

II. APOLLONIUS COORDINATES

First, recall some basic formulas from the Kepler problem and its solution. Let $\phi = (\phi_1, \phi_2)$ be any solution of the planar Kepler problem, r the length of ϕ, and c its angular momentum, so

$$\ddot{\phi} = -\frac{\phi}{\|\phi\|^3}, \qquad r = \sqrt{\phi_1^2 + \phi_2^2}, \qquad c = \phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1 \tag{1}$$

where the independent variables is t, time, and $\dot{} = d/dt$, $\ddot{} = d^2/dt^2$. Rule out collinear solutions by assuming that $c \neq 0$ and then scale time so that $c = 1$. The units distance and mass have been chosen so that all other constants are one. In polar coordinates (r, θ), the equations become

$$\ddot{r} - r\dot{\theta}^2 = -1/r^2, \qquad d(r^2\dot{\theta})/dt = dc/dt = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{2}$$

Using the fact that $c = 1$ is a constant of motion yields

$$\ddot{r} - 1/r^3 = -1/r^2 \tag{3}$$

Equation (3) is reduced to a harmonic oscillator $u'' + u = 1$ by letting $u = 1/r$ and changing from time t to the true anomaly τ by $dt = r^2\, d\tau$ and $' = d/d\tau$. The general solution is then

$$r = r(\tau) = 1/(1 + \varepsilon\cos(\tau - \omega)) \tag{4}$$

where ε and ω are integration constants, ε is the eccentricity, and ω is the argument of the pericenter. $\varepsilon = 0$ is a circle, $0 < \varepsilon < 1$ is an ellipse, $\varepsilon = 1$ is a parabola, and $\varepsilon > 1$ is a hyperbola. There is no harm in assuming that the argument of the pericenter is zero, so henceforth $\omega = 0$.

Define a matrix A by

$$A=\begin{pmatrix}\phi_1 & -\phi_2\\ \phi_2 & \phi_1\end{pmatrix} \tag{5}$$

so $A^{-1}=(1/r^2)\,A^{\mathrm{T}}$ and $A^{-\mathrm{T}}=(A^{\mathrm{T}})^{-1}=(1/r^2)A$, where A^{T} denotes the transpose of A.

Consider the planar N-body problem given by the Hamiltonian

$$H=H_N=\sum_{i=1}^{N}\frac{\|p_i\|^2}{2m_i}-U(q), \qquad U(q)=\sum_{1\leqslant i<j\leqslant N}\frac{m_i m_j}{\|q_i-q_j\|} \tag{6}$$

The vectors q_i, $p_i\in\mathbb{R}^2$ are the position and momentum of the ith particle with mass $m_i>0$ where $i=1,..., N$. U is the negative of the potential and is called the self potential.

Apollonius coordinates are the symplectic coordinates defined below by two symplectic coordinate changes. First, make the symplectic change of coordinates

$$q_i=AX_i, \qquad p_i=A^{-\mathrm{T}}Y_i=(1/r^2)\,AY_i, \qquad i=1,..., N \tag{7}$$

Recall that if $H(z)$ is a Hamiltonian and $z=T(t)u$ is a linear, symplectic change of coordinates, then the Hamiltonian becomes $H(u)+(1/2)\,u^{\mathrm{T}}W(t)u$, where W is the symmetric matrix, $W=JT^{-1}\dot{T}$. Compute

$$\begin{aligned}W_i&=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}r^{-2}A^{\mathrm{T}} & 0\\ 0 & A^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\dot{A} & 0\\ 0 & (r^{-2}\dot{A}-2r^{-3}\dot{r}A)\end{pmatrix}\\ &=\begin{pmatrix}0 & -r^{-2}(A^{\mathrm{T}}\dot{A})^{\mathrm{T}}\\ -r^{-2}A^{\mathrm{T}}\dot{A} & 0\end{pmatrix}\end{aligned} \tag{8}$$

(recall that W is symmetric or use $A^{\mathrm{T}}A=r^2I$ to get the 1, 2 position). Now

$$-r^{-2}A^{\mathrm{T}}\dot{A}=r^{-2}\begin{pmatrix}-r\dot{r} & 1\\ -1 & -r\dot{r}\end{pmatrix} \tag{9}$$

Note that $\|AX\|=r\,\|X\|$, so the Hamiltonian becomes

$$H=\frac{1}{r^2}\sum_{i=1}^{N}\frac{\|Y_i\|^2}{2m_i}-\frac{1}{r}\,U(X)-\frac{\dot{r}}{r}\sum_{i=1}^{N}X_i^{\mathrm{T}}Y_i-\frac{1}{r^2}\sum_{i=1}^{N}X_i^{\mathrm{T}}JY_i \tag{10}$$

Change the independent variable from time, t, to the true anomaly of the Kepler problem, τ, by $dt=r^2\,d\tau$, $'=d/d\tau$, $H\to r^2H$, so

$$H=\sum_{i=1}^{N}\frac{\|Y_i\|^2}{2m_i}-rU(X)-\frac{r'}{r}\sum_{i=1}^{N}X_i^{\mathrm{T}}Y_i-\sum_{i=1}^{N}X_i^{\mathrm{T}}JY_i \tag{11}$$

The second symplectic change of variables changes only the momentum, by letting

$$X_i = x_i, \qquad Y_i = y_i + \alpha_i x_i \tag{12}$$

where the $\alpha_i = \alpha_i(\tau)$ are to be determined. This defines the Apollonius coordinates (x_i, y_i), $i = 1, \ldots, N$. To compute the remainder term, consider

$$R_i = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\alpha_i I & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \alpha_i' & 0 \end{pmatrix} = \begin{pmatrix} \alpha_i' & 0 \\ 0 & 0 \end{pmatrix} \tag{13}$$

Thus the remainder term is $(1/2) \sum \alpha_i'(\tau)\, x_i^{\mathrm{T}} x_i$ and the Hamiltonian becomes

$$\begin{aligned} H = \sum_{i=1}^{N} \frac{\|y_i\|^2}{2m_i} - rU(x) + \left(\frac{\alpha_i}{m_i} - \frac{r'}{r}\right) \sum_{i=1}^{N} x_i^{\mathrm{T}} y_i - \sum_{i=1}^{N} x_i^{\mathrm{T}} J y_i \\ + \sum_{i=1}^{N} \left(\frac{1}{2}\alpha_i' + \frac{1}{2}\frac{\alpha_i^2}{m_i} - \frac{r'}{r}\alpha_i\right) x_i^{\mathrm{T}} x_i \end{aligned} \tag{14}$$

Choose α_i so that the third term on the right in (14) vanishes, i.e., take $\alpha_i = m_i r'/r$. To compute the coefficient of $x_i^{\mathrm{T}} x_i$ in the last sum in (14), note that

$$\left(\frac{r'}{r}\right)' - \left(\frac{r'}{r}\right)^2 = \frac{rr'' - 2r'^2}{r^2} = r\frac{d}{d\tau}\left(\frac{r'}{r^2}\right) = r\frac{d\dot{r}}{d\tau} = r^3 \ddot{r} = 1 - r \tag{15}$$

where the last equality comes from formula (3). Thus the Hamiltonian of the N-body problem in Apollonius coordinates is

$$H = \sum_{i=1}^{N} \frac{\|y_i\|^2}{2m_i} - rU(x) - \sum_{i=1}^{N} x_i^{\mathrm{T}} J y_i + \frac{(1-r)}{2} \sum_{i=1}^{N} m_i x_i^{\mathrm{T}} x_i \tag{16}$$

and the equations of motion are

$$\begin{aligned} x_i' &= \frac{y_i}{m_i} - Jx_i \\ y_i' &= r\frac{\partial U}{\partial x_i} - Jy_i - (1-r)\, m_i x_i \end{aligned} \tag{17}$$

These are particularly simple equations considering complexity of the coordinate change.

III. CENTRAL CONFIGURATIONS AND RELATIVE EQUILIBRIUM

A *central configuration of the N-body problem* is a solution $(x_1, \ldots, x_N)$ of the system of nonlinear algebraic equations

$$\frac{\partial U}{\partial x_i} + \lambda m_i x_i = 0, \qquad i = 1, \ldots, N \tag{1}$$

for some scalar λ. By scaling the distance, λ may be taken as 1. Thus a central configuration is a geometric configuration of the N particles so that the force on the ith is proportional to m_i times the position. This is the usual definition of a central configuration. Define *a relative equilibrium* as a critical point of the Hamiltonian of the N-body problem in Apollonius coordinates. This is slightly different from the usual definition of a relative equilibrium.

Proposition 1. *The relative equilibria are central configurations.*

Proof. The critical points of (II.16) satisfy

$$\partial H/\partial x_i = -r\, \partial U/\partial x_i + J y_i + (1-r)\, m_i x_i = 0, \qquad \partial H/\partial y_i = y_i/m_i - J x_i = 0 \tag{2}$$

From the second equation $y_i = m_i J x_i$. Plugging this into the first equation gives

$$-r\, \partial U/\partial x_i - m_i x_i + (1-r)\, m_i x_i = -r\{\partial U/\partial x_i + m_i x_i\} = 0 \tag{3}$$

Since r is positive (2) is satisfied if and only if $\partial U/\partial x_i + m_i x_i = 0$. ∎

This simple fact was observed already by Waldvogel (1973).

IV. THE RESTRICTED PROBLEM

Consider the $(N+1)$-body problem with particles indexed from 0 to N. Let H_{N+1} and U_{N+1} be the Hamiltonian and potential of the $(N+1)$-body problem written in Apollonius coordinates. Also consider the N-body problem with particles indexed from 1 to N. Let H_N and U_N be the Hamiltonian and potential of the N-body problem written in Apollonius coordinates.

$$\begin{aligned} H_{N+1} &= \sum_{i=0}^{N} \frac{\|y_i\|^2}{2m_i} - rU_N(x) - \sum_{i=0}^{N} x_i^{\mathrm{T}} J y_i + \frac{(1-r)}{2} \sum_{i=0}^{N} m_i x_i^{\mathrm{T}} x_i \\ &= \frac{\|y_0\|^2}{2m_0} - r \sum_{j=1}^{N} \frac{m_0 m_j}{\|x_0 - x_j\|} - x_0^{\mathrm{T}} J y_0 + \frac{(1-r)}{2} m_0 x_0^{\mathrm{T}} x_0 + H_N \end{aligned} \tag{1}$$

Assume that one mass is small by setting $m_0 = v^2$. It is known as the *infinitesimal* and the other N bodies are known as the *primaries*. Let Z be the $4N$ coordinate vector for the N-body problem, so $Z = (x_1,..., x_N, y_1,..., y_N)$, and let $Z^* = (a_1,..., a_N, b_1,..., b_N)$ be any central configuration for the N-body problem. By Proposition 1, grad $H_N(Z^*) = 0$. The Taylor expansion for H_N is

$$H_N(Z) = H_N(Z^*) + \tfrac{1}{2}(Z = Z^*)^{\mathrm{T}} S(\tau)(Z - Z^*) + \cdots \tag{2}$$

where $S(\tau)$ is the Hessian of H_N at Z^*. Forget the constant term $H(Z^*)$.

Change coordinate by

$$x_0 = \xi, \qquad y_0 = v^2\eta, \qquad Z - Z^* = vV \tag{3}$$

This is a symplectic transformation with multiplier v^{-2}. Making this change of coordinates in (1) yields

$$H_{N+1} = R + \tfrac{1}{2}V^{\mathrm{T}}S(\tau)V + O(v) \tag{4}$$

where R is the Hamiltonian of the *conic* (*elliptic, parabolic, etc.*) *restricted* $(N+1)$*-body problem* given by

$$R = \frac{1}{2}\|\eta\|^2 - r\sum_{i=1}^{N}\frac{m_i}{\|\xi - a_i\|} - \xi^{\mathrm{T}}J\eta + \frac{(1-r)}{2}\xi^{\mathrm{T}}\xi \tag{5}$$

To zeroth order the equations of motion are

$$\xi' = \eta + J\xi, \qquad \eta' = -r\sum_{i=1}^{N}\frac{m_i(\xi - a_i)}{\|\xi - a_i\|^3} + J\eta - (1-r)\xi \tag{6}$$

$$V' = D(\tau)V, \qquad D(\tau) = JS(\tau) \tag{7}$$

The equations in (6) are the equations of the restricted problem and those in (7) are the linearized equations of motion about the relative equilibrium.

When $\varepsilon = 0$ both Eq. (6) and Eq. (7) are time independent and (5) is the Hamiltonian of the (circular) restricted N-body problem. In this case a periodic solution of (6) is called *nondegenerate* if exactly two of its multipliers are $+1$. When $0 < \varepsilon < 1$ both Eq. (6) and Eq. (7) are 2π-periodic in τ and (5) is the Hamiltonian of the elliptic, restricted N-body problem. In this case a $2k\pi$-periodic solution of (6) is called *nondegenerate* if all four of its multipliers are different from $+1$.

In the classical, elliptic restricted three-body problem the masses of the primaries are $m_1 = 1 - \mu$, $m_2 = \mu$, and they are located at $a_1 = (-\mu, 0)$,

$a_2 = (1 - \mu, 0)$. The parameter μ is called the *mass ratio parameter*. Thus the Hamiltonian of the classical, elliptic three-body problem is

$$R = \frac{1}{2}\eta^2 - r\left(\frac{1-\mu}{d_1} + \frac{\mu}{d_2}\right) - \xi^T J\eta + \frac{(1-r)}{2}\xi^T\xi \tag{8}$$

where

$$\begin{gathered} d_1 = \{(\xi_1 + \mu)^2 + \xi_2^2\}^{1/2}, \qquad d_2 = \{(\xi_1 - 1 + \mu)^2 + \xi_2^2\}^{1/2} \\ r = r(\tau) = 1/(1 + \varepsilon \cos \tau), \qquad 0 < \varepsilon < 1 \end{gathered} \tag{9}$$

V. SYMMETRIES AND REDUCTION

Henceforth, consider the elliptic case only. For the moment consider the N-body problem in the original rectilinear coordinates (q, p) in (II.6). This Hamiltonian is invariant under the symplectic extension of the group of Euclid motions of the plane, i.e., it is invariant under the action $q_i \to Eq_i + b$, $p_i \to Ep_i$ where E is a rotation matrix and b is a vector. This motion carries a periodic solution to a periodic solution and so periodic solutions are not isolated even in an energy level $H = \text{constant}$. A theorem of Meyer (1973) states that due to this symmetry the algebraic multiplicity of the characteristic multiplier $+1$ of a periodic solution of the N-body problem must be at least 8. Unless these degeneracies are eliminated the standard methods of perturbation analysis will fail.

By a classical theorem of Noether the symmetry implies that the equations of motion admits linear and angular momenta as integrals, a total of three integrals. Thus part, but not all, of the degeneracies can be eliminated by holding these integrals fixed. Holding these three integrals fixed reduces the dimension by three, but the total degeneracy can be eliminated by going to the reduced space, which reduces the dimension by 6. This reduction is accomplished in several steps. First fix the center of mass at the origin and hold linear momentum to zero. These are four linear constraints and so defines a linear, invariant subspace $B_1 \subset \mathbb{R}^{4N}$ of dimension $4N-4$. Next hold angular momentum equal to a fixed nonzero number. This defines an invariant submanifold $B_2 \subset B_1$ of dimension $4n-5$. Finally, let B be the quotient space $B = B_2/_\sim$ where $\sim$ is the equivalence relation $(q, p) \sim (q^\dagger, p^\dagger)$ where $q_i = Eq_i^\dagger$, $p_i = Ep_i^\dagger$, E a rotation matrix. B is called the *reduced space* for the N-body problem.

By a theorem of Meyer (1973) the reduced space, B, is a symplectic manifold of dimension $4N-6$ and the Hamiltonian H and the flow defined by this Hamiltonian naturally drop down to this quotient space. This is the natural place to study the N-body problem because all the degeneracies due to the symmetries of the problem have been eliminated. In general a periodic solution would have the multiplier $+1$ with multiplicity 2 on the reduced space which is the generic number; such a periodic solution is called *a nondegenerate periodic solution.* Now turn to the Hamiltonian of the N-body problem in Apollonius coordinates.

Let C be the center of mass, L the total linear momentum, and F the total angular momentum in Apollonius coordinates, i.e.,

$$C = \sum_{l}^{N} m_i x_i, \qquad L = \sum_{l}^{N} y_i, \qquad F = \sum_{l}^{N} x_i^{\mathrm{T}} J y_i \tag{1}$$

From Eqs. (II.17) it follows that

$$\begin{aligned} C' &= -JC + L \\ L' &= -(1-r)C - JL \\ F' &= 0 \end{aligned} \tag{2}$$

From these equations we see that C and L satisfy a time-varying, linear, homogeneous, Hamiltonian system of equations so the set $C = L = 0$ is invariant. From the last equation angular momentum, F, is an integral. The Hamiltonian of the N-body problem in Apollonius coordinates (II.16) is still invariant under rotations and so the reduction can be carried out in these coordinates also. That is, the reduction can be accomplished by setting $C = L = 0$, $F = \text{constant} \neq 0$, and identifying points by $(x, y)_{\sim} (x^{\dagger}, y^{\dagger})$ where $x_i = E x_i^{\dagger} y_i = E y_i^{\dagger}$, E a rotation matrix.

Let (Q, P) be rectangular coordinates in $\mathbb{R}^2 \times \mathbb{R}^2$. If the Hamiltonian $K = (1/2) P^{\mathrm{T}} P$ is written in Apollonius coordinate (C, L), then K becomes $K(C, L) = (1/2) L^{\mathrm{T}} L - C^{\mathrm{T}} JL + ((1-r)/2) C^{\mathrm{T}} C$, which is the Hamiltonian for the first two equations in (2). Thus the first two equations in (2) are just the equations $\dot{Q} = P$, $\dot{P} = 0$, written in Apollonius coordinates and so the characteristic multipliers of this system are all $+1$. Thus fixing the $C = L = 0$ decreases the multiplicity of the multiplier $+1$ by 4. Holding F fixed and going to the quotient space decreases the multiplicity of the multiplier $+1$ by another 2 by the same argument as given by Meyer (1981). Thus going to the reduced space decreases the multiplicity of $+1$ by 6.

A relative equilibrium becomes an equilibrium for the Hamiltonian on the reduced space. The nontrivial multipliers of the relative equilibrium are defined in the following way. First, consider the linear variational equation about the relative equilibrium on the reduced spaces—this is a linear, 2π-periodic system of dimension $4N-6$. In general, the multiplier $+1$ will have multiplicity 2. The remaining $4N-8$ multipliers are called the *nontrivial multipliers of the relative equilibrium.*

A solution of the $(N+1)$-body problem is called *reduced periodic with period T* if its projection on the reduced space is periodic of period T. A reduced periodic solution of the $(N+1)$-body problem is called *nondegenerate* if its projection onto the reduced space is a periodic solution with multiplier $+1$ of multiplicity 2.

VI. THE CONTINUATION THEOREM

There are many theoretic and numeric investigations of periodic solutions in the elliptic three-body problem. See Brouke (1969, 1971), Moulton (1920), Schubart (1956a, 1966), Sergysels-Lamy (1975), Shelus (1972), Szebehely and Giacaglia (1964), and their references. Consider a system of 2π-periodic equations $\xi' = f(\tau, \xi, v)$ depending on a parameter v and let $\chi(\tau)$ be a $2k\pi$-periodic solution when $v=0$. The solution $\chi(\tau)$ *can be continued* if there is a smooth one-parameter family of $2k\pi$-periodic solutions $\chi^{\dagger}(\tau, v)$ defined for v small such that $\chi^{\dagger}(\tau, 0) = \chi(\tau)$.

Theorem 2. *Let $(\phi(\tau), \psi(\tau))$ be a nondegenerate $2k\pi$-periodic solution of the elliptic, restricted $(N+1)$-body problem in (IV.6) with Hamiltonian (IV.5). Let the nontrivial multipliers of the relative equilibrium not be kth roots of unity. Then the $2k\pi$-periodic solution $\xi=\phi(\tau)$, $\eta=\psi(\tau)$, $V=0$ of (IV.6), (IV.7) can be continued into the full $(N+1)$-body problem as a nondegenerate reduced periodic solution for small values of $m_0 = v^2$.*

Proof. Consider the $(N+1)$-body problem using the notation in Section IV. Let $V=(u_1,\dots,u_N, v_1,\dots,v_N)$ so $x_i = a_i - vu_i$, $y_i = b_i - vv_i = -m_iJa_i - vv_i$. Since the center of mass of the relative equilibrium is fixed at the origin $\sum_i^N m_ia_i = 0$ and

$$
\begin{aligned}
C &= v^2\xi + v\{m_1u_1 + \cdots + m_Nu_N\} \\
L &= v^2\eta + v\{v_1 + \cdots + u_N\} \qquad (1) \\
A &= v^2\xi^{\mathrm{T}}J\eta + \sum_i^N (a_i - vu_i)^{\mathrm{T}} J(b_i - vv_i)
\end{aligned}
$$

From these formulas it follows that the reduced space depends smoothly on the parameter v and the Hamiltonian on the reduced space also is smooth in v.

Remember that the $(N+1)$-body problem is time independent and a periodic solution can be continued if the eigenvalue $+1$ has multiplicity 2. [This is a simple consequence of the implicit function theorem applied to the Poincaré map in an energy level; see Abraham and Marsden (1967).] By the assumptions above the $2k\pi$-periodic solution $\xi = \phi(\tau)$, $\eta = \psi(\tau)$, $V = 0$ when $\nu = 0$ has the multiplier $+1$ with multiplicity 2 on the reduced space. ▮

Corollary 3. *Let $(\phi(\tau), \psi(\tau))$ be a nondegenerate $2k\pi$-periodic solution of the classical, elliptic, restricted three-body problem with Hamiltonian (IV.8). Then the $2k\pi$-periodic solution $\xi = \phi(\tau)$, $\eta = \psi(\tau)$, $V = 0$ of (IV.6), (IV.7) can be continued into the full three-body problem as a nondegenerate reduced periodic solution for small values of $m_0 = \nu^2$.*

Proof. The two-body problem is eight dimensional and its reduced space is two dimensional. Therefore, there are no nontrivial multipliers of the relative equilibrium and so no restriction on them. ▮

VII. JACOBI COORDINATES

This presentation of Jacobi coordinates is taken from Meyer (1981); see that paper for more details. Consider the $(N+1)$-body problem with the index running from 0 to N. Define the Jacobi coordinates by a sequence of symplectic coordinate changes starting with $g_0 = x_0$, $G_0 = y_0$, $\mu_0 = m_0$, and henceforth

$$\begin{aligned} u_k &= x_k - g_{k-1}, & v_k &= (m_{k-1}/\mu_k)\, y_k - (m_k/\mu_k)\, G_{k-1} \\ g_k &= (1/\mu_k)(m_k x_k + \mu_{k-1} g_{k-1}), & G_k &= y_k + G_{k-1} \\ \mu_k &= m_k + \mu_{k-1} \end{aligned} \tag{1}$$

By Meyer (1981) it is shown that

$$\begin{aligned} \sum_{i=0}^{N} \frac{\|y_i\|^2}{2m_i} &= \frac{\|G_N\|^2}{2\mu_N} + \sum_{i=1}^{N} \frac{\|v_i\|^2}{2m_i} \\ \sum_{i=0}^{N} x_i^{\mathrm{T}} J y_i &= g_N^{\mathrm{T}} J G_N + \sum_{i=1}^{N} u_i^{\mathrm{T}} J v_i \\ M_k &= m_k \mu_{k-1}/\mu_k \end{aligned} \tag{2}$$

In a like manner we can show that

$$\mu_{k-1}\, \|g_{k-1}\|^2 + m_k\, \|x_k\|^2 = \mu_k\, \|g_k\|^2 + M_k\, \|u_k\|^2 \tag{3}$$

and so by induction

$$\sum_{i=0}^{N} m_i\, \|x_i\|^2 = \mu_N\, \|g_N\|^2 + \sum_{i=1}^{N} M_i\, \|u_i\|^2 \tag{4}$$

Thus the Hamiltonian of the $(N+1)$-body problem in Jacobi-Apollonius coordinates is

$$H_{N+1} = \frac{\|G_N\|^2}{2\mu_N} + \sum_{i=1}^{N} \frac{\|v_i\|^2}{2M_i} - rU(u) + \left\{ g_N^{\mathrm{T}} J G_N + \sum_{i=1}^{N} u_i^{\mathrm{T}} J v_i \right\}$$
$$+ \frac{(1-r)}{2} \left\{ \mu_N \|g_N\|^2 + \sum_{i=1}^{N} M_i \|u_i\|^2 \right\} \tag{5}$$

The equations for g_N and G_N are

$$g_N' = G_N/\mu_N - Jg_N, \qquad G_N' = -(1+r)\,\mu_N g_N - JG_N \tag{6}$$

These linear equations are the same as the equations for C and L in (V.2), in particular, the set $g_N = G_N = 0$ is invariant. In fact, g_N and G_N are just scalar multiplies of C and L. We can set $g_N = G_N = 0$ in (6) to get a simpler system.

VIII. THE TWO-BODY PROBLEM

Consider the two-body problem in Apollonius coordinates, i.e.,

$$T = H_2 = \sum_{i=1}^{2} \frac{\|y_i\|^2}{2m_i} - rU(x) - \sum_{i=1}^{2} x_i^{\mathrm{T}} J y_i + \frac{(1-r)}{2} \sum_{i=1}^{2} m_i x_i^{\mathrm{T}} x_i \tag{1}$$

For this section let $m_1 + m_2 = 1$, $M = m_1 m_2$, and introduce Jacobi coordinates by

$$\begin{aligned} u_1 &= x_2 - x_1, & x_1 &= u_2 - m_2 u_1 \\ u_2 &= m_2 x_2 + m_1 x_1, & x_2 &= u_2 + m_1 u_1 \\ v_1 &= m_1 y_2 - m_2 y_1, & y_1 &= m_1 v_2 - v_1 \\ v_2 &= y_2 + y_1, & y_2 &= m_2 v_2 + v_1 \end{aligned} \tag{2}$$

The Hamiltonian becomes

$$T = \frac{\|v_1\|^2}{2M} + \frac{\|v_2\|^2}{2} - r\frac{M}{\|u_1\|} + \{u_1^{\mathrm{T}} J v_1 + u_2^{\mathrm{T}} J v_2\}$$
$$+ \frac{(1-r)}{2} \{M \|u_1^2\| + \|u_2^2\|\} \tag{3}$$

The equations for u_2, v_2 are

$$u_2' = v_2 - Ju_2, \qquad v_2' = -Jv_2 - (1-r)\,u_2 \tag{4}$$

and so $u_2 = v_2 = 0$ is an invariant set. Henceforth, assume that $u_2 = v_2 = 0$, and let $u = u_1$, $v = v_1$. Then the Hamiltonian becomes

$$T = \frac{\|v\|^2}{2M} - r\frac{M}{\|u\|} + u^{\mathrm{T}} J v + \frac{(1-r)}{2} M \|u\|^2 \tag{5}$$

Now change to polar coordinates by

$$\begin{aligned} u_1 &= \rho \cos\theta, \qquad & v_1 &= R\cos\theta - (\Theta/\rho)\sin\theta \\ u_2 &= \rho\sin\theta, \qquad & v_2 &= R\sin\theta - (\Theta/\rho)\cos\theta \end{aligned} \tag{6}$$

so in these coordinates the Hamiltonian becomes

$$T = \frac{1}{2M}\left\{R^2 + \frac{\Theta^2}{\rho^2}\right\} - r\frac{M}{\rho} + \Theta + \frac{(1-r)}{2} M\rho^2 \tag{7}$$

and the equations of motion are

$$\begin{aligned} \theta' &= \frac{\Theta}{M\rho^2} + 1, \qquad & \Theta' &= 0 \\ \rho' &= \frac{R}{M}, \qquad & R' &= \frac{\Theta^2}{M\rho^3} - \frac{rM}{\rho^2} - (1-r)M\rho \end{aligned} \tag{8}$$

The critical points are at

$$\theta = \text{anything}, \qquad \Theta = -M, \qquad \rho = 1, \qquad R = 0 \tag{9}$$

and the linearized equations about this critical point are

$$\begin{aligned} \theta' &= 2\rho + \frac{\Theta}{M}, \qquad & \Theta' &= 0 \\ \rho' &= \frac{R}{M}, \qquad & R' &= -2\Theta + \{3r - 4\} M\rho \end{aligned} \tag{10}$$

IX. HILL'S LUNAR EQUATIONS

Consider the three-body problem in Jacobi-Apollonius coordinates with the center of mass at the origin and linear momentum set to zero. Think of m_0, m_1, and m_2 as the mass of the earth, moon, and sun, respectively. The Hamiltonian is

$$\begin{aligned} H = \sum_{i=1}^{2} \left\{\frac{\|v_i\|^2}{2M_i} - u_i^{\mathrm{T}} J v_i\right\} - r\left\{\frac{m_0 m_1}{\|u_1\|} + \frac{m_1 m_2}{\|u_2 - \alpha_0 u_1\|} + \frac{m_0 m_2}{\|u_2 - \alpha_1 u_1\|}\right\} \\ + \frac{(1-r)}{2} \sum_{i=1}^{2} v_i \|u_i\|^2 \end{aligned} \tag{1}$$

where

$$M_1 = \frac{m_0 m_1}{(m_0 + m_1)}, \qquad M_2 = \frac{m_2(m_0 + m_1)}{(m_0 + m_1 + m_2)}$$

$$\alpha_0 = \frac{m_0}{(m_0 + m_1)}, \qquad \alpha_1 = \frac{m_1}{(m_0 + m_1)} \tag{2}$$

$$v_1 = m_1 m_0, \qquad v_2 = m_2(m_0 + m_1)$$

Assume that the masses of the earth and moon are small compared to the mass of the sun, but are of the same order of magnitude, by setting

$$m_0 = \nu^6 \delta_0, \qquad m_1 = \nu^6 \delta_1, \qquad m_2 = 1 \tag{3}$$

and scale by $v_i \to \nu^6 v_i$ (symplectic with multiplier ν^6) to get

$$\begin{aligned} H &= H_1 + H_2 + O(\nu^6) \\ H_1 &= \frac{\|v_1\|^2}{2\kappa_1} - u_1^{\mathrm{T}} J v_1 - \nu^6 r \frac{\delta_0 \delta_1}{\|u_1\|} + \nu^6 \frac{(1-r)}{2} \delta_0 \delta_1 \|u_1\|^2 \\ H_2 &= \frac{\|v_2\|^2}{2\kappa_2} - u_2^{\mathrm{T}} J v_2 - \frac{r\delta_1}{\|u_2 - \beta_0 u_1\|} - \frac{r\delta_0}{\|u_2 - \beta_1 u_1\|} \\ &\quad + \frac{(1-r)}{2} (\delta_0 + \delta_1) \|u_2\|^2 \end{aligned} \tag{4}$$

$$\beta_0 = \frac{\delta_0}{\delta_0 + \delta_1}, \qquad \beta_1 = \frac{\delta_1}{\delta_0 + \delta_1}, \qquad \kappa_1 = \delta_0 \delta_1, \qquad \kappa_2 = \frac{\delta_0 \delta_1}{\delta_0 + \delta_1}$$

Note that the $O(\nu^6)$ depends contains terms in the momenta only. Now assume that the distance between the earth and the moon, $\|u_1\| = \|x_1 - x_0\|$, is small by comparison to the distance from the earth–moon system to the sun by scaling $u_1 \to \nu^2 u_1$. This scaling is not symplectic but it is augmented later so that the total scaling is symplectic. Before this scaling is done the potential terms must be investigated. Following Hill we expand the two potential terms in H_2 in a Legendre series as follows:

$$\frac{\delta_1}{\|u_2 - \beta_0 u_1\|} + \frac{\delta_0}{\|u_2 - \beta_1 u_1\|} = \frac{(\delta_0 + \delta_1)}{\|u_2\|} + \frac{1}{\|u_2\|} \sum_{k=2}^{\infty} b_k \rho^k P_k(\cos\theta) \tag{5}$$

where $\rho = \|u_1\|/\|u_2\|$, $b_k = \delta_1 \beta_0^k + \delta_0(-\beta_1)^k$, θ is the angle between u_1 and u_2, and P_k is the kth Legendre polynomial. Thus (4) becomes

$$H = H_1 + H_3 + \frac{r}{\|u_2\|} \sum_{k=2}^{\infty} b_k \rho^k P_k(\cos\theta) + O(\nu^6) \tag{6}$$

where

$$H_3 = \frac{\|v_2\|^2}{2\kappa_2} - u_2^T J v_2 - \frac{r\kappa_2}{\|u_2\|} + \frac{(1-r)}{2}\kappa_2 \|u_2\|^2 \tag{7}$$

H_3 is just the Hamiltonian of the Kepler problem in Apollonius coordinates where the fixed body of mass 1 is located at the origin and the moving body is of mass $\kappa_2 = \delta_0 + \delta_1$. One can think of the fixed body as the sun and the moving body as the earth–moon system. The latter assumption is that the earth–moon system moves approximately on a solution of the Kepler problem about the sun. As seen above, the Kepler problem in Apollonius coordinates has a critical point at a central configuration. Specifically, H_3 has a critical point at $u_2 = a$, $v_2 = b$, where a is any constant vector satisfying $\|a\|^3 = 1$ and $b = \kappa_2 J a$. Let

$$Z = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \qquad Z^* = \begin{pmatrix} a \\ b \end{pmatrix} \tag{8}$$

so H_3 is a function of Z, grad $H_3(Z^*) = 0$, and H_3 has a Taylor expansion of the form

$$H_3(Z, \tau) = H_3(Z^*) + \tfrac{1}{2}(Z - Z^*)^T S(\tau)(Z - Z^*) + \cdots \tag{9}$$

where $S(\tau)$ is the Hessian of H_3 evaluated at Z^*. Henceforth, drop the constant term in (9) since the equations of motion are independent of the constants in the Hamiltonian.

Now complete the previous scaling, $u_1 \to v^2 u_1$, by $Z - Z^* \to v^2 W$, $v_1 \to v^2 v_1$, which is symplectic with multiplier v^4. The Hamiltonian becomes

$$\begin{aligned} H &= H_4 + \tfrac{1}{2} W^T S(\tau) W + O(v) \\ H_4 &= \frac{\|v_1\|^2}{2\kappa_1} - u_1^T J v_1 - r\frac{\delta_0\delta_1}{\|u_1\|} - r b_2 \|u_1\|^2 P_2(\cos\theta) \end{aligned} \tag{10}$$

In order to reduce the number of parameters in the problem we make one further scaling. Recall that $P_2(x) = (1 - 3x^2)/2$ and let $a = (1, 0)$ so that the abscissa points at the sun and make the symplectic change of coordinates

$$u_1 = (\delta_0 + \delta_1)^{1/3}\xi, \qquad v_1 = (\delta_0 + \delta_1)^{1/3}\kappa_1 \eta, \; W = (\delta_0 + \delta_1)^{1/3}\kappa_1^{1/2} V \tag{11}$$

so that the Hamiltonian becomes

$$H = L + \tfrac{1}{2} V^T S(\tau) V + O(v^2) \tag{12}$$

where

$$L = \frac{\|\eta\|^2}{2} - \xi^T J\eta - \frac{r}{\|\xi\|} + r(3\xi_1^2 - \|\xi\|^2) \tag{13}$$

L is the Hamiltonian of Hill's lunar problem with the earth–moon system moving on a Kepler orbit about the sun. It is called the *conic Hill's lunar problem.* In the classical Hill's lunar problem $c = r = 1$. If we take an elliptic solution of the Kepler problem with $c = 1$, then L is 2π-periodic and we call it the *elliptic Hill's lunar problem.* This problem was used via a singular-perturbation problem by Spirig and Waldvogel (1985).

As before we have the following.

Theorem 4. *Let $(\phi(\tau), \psi(\tau))$ be a nondegenerate $2k\pi$-periodic solution of the elliptic Hill's lunar problem. Then the $2k\pi$-periodic solution $\xi = \phi(\tau)$, $\eta = \psi(\tau)$, $V = 0$ of the equations defined by (12) and (13) when $v = 0$ can be continued into the full three-body problem as a nondegenerate reduced periodic solution for small values of v.*

Proof. The proof is a slight modification of the proof of Theorem 2 or of the similar theorem of Meyer and Schmidt (1982). ▌

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