J Dyn Diff Equat (2010) 22:367–380
DOI 10.1007/s10884-010-9180-3

Bifurcations of Heteroclinic Orbits

Kenneth R. Meyer · Patrick McSwiggen · Xiaojie Hou

Received: 10 February 2010 / Published online: 15 July 2010

Abstract The search for traveling wave solutions of a semilinear diffusion partial differential equation can be reduced to the search for heteroclinic solutions of the ordinary differential equation $\ddot{u} - c\dot{u} + f(u) = 0$, where c is a positive constant and f is a nonlinear function. A heteroclinic orbit is a solution $u(t)$ such that $u(t) \to \gamma_1$ as $t \to -\infty$ and $u(t) \to \gamma_2$ as $t \to \infty$ where γ_1, γ_2 are zeros of f. We study the existence of heteroclinic orbits under various assumptions on the nonlinear function f and their bifurcations as c is varied. Our arguments are geometric in nature and so we make only minimal smoothness assumptions. We only assume that f is continuous and that the equation has a unique solution to the initial value problem. Under these weaker smoothness conditions we reprove the classical result that for large c there is a unique positive heteroclinic orbit from 0 to 1 when $f(0) = f(1) = 0$ and $f(u) > 0$ for $0 < u < 1$. When there are more zeros of f, there is the possibility of bifurcations of the heteroclinic orbit as c varies. We give a detailed analysis of the bifurcation of the heteroclinic orbits when f is zero at the five points $-1 < -\theta < 0 < \theta < 1$ and f is odd. The heteroclinic orbit that tends to 1 as $t \to \infty$ starts at one of the three zeros, $-\theta, 0, \theta$ as $t \to -\infty$. It hops back and forth among these three zeros an infinite number of times in a predictable sequence as c is varied.

Keywords Heteroclinic orbits · Bifurcation · Traveling wave

To Jack, my teacher, my coauthor and my life long friend. KRM.

K. R. Meyer (✉) · P. McSwiggen
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA
e-mail: Ken.Meyer@uc.edu

P. McSwiggen
e-mail: Pat.McSwiggen@uc.edu

X. Hou
Department of Mathematics and Statistics, North Carolina University Wilmington, Wilmington, NC 28401, USA
e-mail: Houx@uncw.edu

Mathematics Subject Classification (2000) 34C23 · 34C37

1 Introduction

We consider the equation

$$\ddot{u} - c\dot{u} + f(u) = 0 \tag{1}$$

where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous and c is a real parameter. The independent variable is t and dot denotes differentiation with respect to t, i.e. $\dot{} = d/dt$. Rewrite Eq. 1 as a system by introducing $v = \dot{u}$, so that

$$\dot{u} = v, \qquad \dot{v} = cv - f(u). \tag{2}$$

The equilibrium points of (2) are of the form $(\gamma, 0)$ where $f(\gamma) = 0$. A *heteroclinic orbit* of (2) is a solution $\phi(t) = (u(t), v(t))$ such that

$$\lim_{t\to-\infty}(u(t), v(t)) = (\gamma_1, 0), \qquad \lim_{t\to+\infty}(u(t), v(t)) = (\gamma_2, 0) \tag{3}$$

where $\gamma_1 \neq \gamma_2$ are zeros of f. We sometimes say the orbit is *heteroclinic from* $(\gamma_1, 0)$ *to* $(\gamma_2, 0)$. A heteroclinic orbit $\phi(t) = (u(t), v(t))$ is *positive* if $u(t) > 0$ for all $-\infty < t < \infty$ and is *positive and negative* if it takes both signs in the same interval. We are interested in the existence of heteroclinic orbits of (2) and their evolution and bifurcations as the parameter c is varied.

Our initial motivation came from the vast literature on traveling wave solutions of semi-linear diffusion equations of the form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + f(U). \tag{4}$$

This is sometimes called a real Ginzburg-Landau equation [3] and is also known as the Allen-Cahn equation in phase field theory [8]. Making the ansatz that U is of the form $U(t, x) = u(x + ct)$ yields Eq. 1 and a heteroclinic orbit of (1) gives a traveling wave solution of (4) (see [1,4,5,7] and the references therein).

In much of this traveling wave literature the function f is defined for $0 \leq u \leq 1$, is positive for $0 < u < 1$ and is zero at 0 and 1. In this case the only heteroclinic orbit would be from the equilibrium point (0, 0) to the equilibrium point (0, 1). In [1], Aronson and Weinberger introduce a genetics model of the form (4) where f has an addition zero between 0 and 1. In such a model there are two types of heteroclinic orbits that can occur and there is the possibility that the heteroclinic orbit pops from one equilibrium point to the other as the parameter c varies. Ultimately, our attention turned to the fascinating complexities of these bifurcations and the extent this could be analyzed using only geometric arguments.

To emphasize the geometric nature of the arguments, we make minimal smoothness assumptions. We only assume that f is continuous and that Eq. 2 have a unique solution to the initial value problem

$$u = u_0, \ v = v_0 \ \text{when} \ t = 0.$$

This insures that the solutions of the initial value problem $u(t, u_0, v_0, c),\ v(t, u_0, v_0, c)$ are continuous in the displayed arguments [6, p. 94]. Since these solutions satisfy (2) the partials $u_t(t, u_0, v_0, c),\ v_t(t, u_0, v_0, c)$ are also continuous. This last remark allows one to use the Implicit Function Theorem to define continuous cross section maps as found in [6,12].

To be specific, we make the following standing assumptions:

(i) $c > 0$
(ii) $f : [-1, 1] \to \mathbb{R}$ is continuous;
(iii) the Eq. 2 has a unique solution to the initial value problem;
(iv) $f(-1) = f(0) = f(1) = 0$;
(v) $f(-u) = -f(u)$;
(vi) $\int_0^1 f(u)du = F_1 > 0$.

This first assumption is for convenience. With f odd, the reversal of time $t \to -t$ together with $u \to -u$ preserves (2), but changes the sign of c, so there is no loss of generality in this assumption. Assuming f is odd allows us to use ideas from dynamical systems. For now we make no assumptions about the sign of f beyond what is implied by assumption (vi).

In Sect. 2 we use a Liapunov function argument to find a region in phase space where all solutions of interest tend to an equilibrium as $t \to -\infty$. In Sect. 3 we give the local phase plane structure near an equilibrium using geometric methods. Section 4 is devoted to establishing the existence of a positive heteroclinic orbit in the classical case when f is positive on $0 < u < 1$. Finally in Sect. 5 we study the intricate bifurcations of heteroclinic orbits as the parameter c is varied when f has an additional zero in $0 < u < 1$.

2 Global Stability

The standard Liapunov function for (2), is

$$V = \frac{1}{2}v^2 + F(u), \qquad F(u) = \int\limits_0^u f(\tau)d\tau, \tag{5}$$

whose derivative along the solutions of (2) is

$$\dot{V} = \frac{\partial V}{\partial v}\dot{v} + \frac{\partial V}{\partial u}\dot{u} = v(cv - f(u)) + f(u)v = cv^2 \geq 0. \tag{6}$$

Lemma 2.1 *Any solution of (2) which is bounded for $t \leq 0$ approaches an equilibrium point as $t \to -\infty$.*

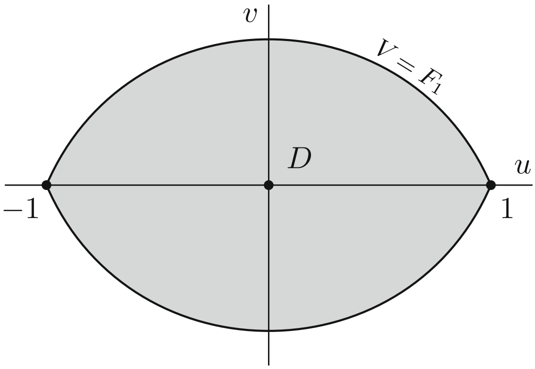
Proof The classical Liapunov type theorems deal with behavior as $t \to +\infty$. Therefore, there will be sign shifts in the following arguments.

Since $\dot{V} \geq 0$, it follows from LaSalle's Theorem (Theorem 2, p. 282 of [9]) that a solution which is bounded for $t \leq 0$ tends to the largest invariant set in $Z = \{(u, v) : \dot{V}(u, v) = 0\}$ as $t \to -\infty$ (see also [10,11]). By (6) the set Z is defined by $v = 0$, since $c > 0$. On Z we have $\dot{v} = -f(u)$, so to remain in Z we must have $f(u) = 0$ and the largest invariant set is the set of equilibria. □

Let $D = \{(u, v) : -1 \leq u \leq 1,\ V \leq F_1\}$. Since $F(\pm 1) = F_1$, the only points of D with $u = \pm 1$ are $(\pm 1, 0)$. Consequently, the boundary of D is contained in the level set $V = F_1$. Figure 1 is a representation of D in the case where F is increasing on $0 \leq u \leq 1$ (i.e., $f(u) \geq 0$). If F_1 is not the maximum value of F on $0 \leq u \leq 1$, then D will have multiple components.

Proposition 2.1 *Any solution of (2) which starts in the region D when $t = 0$ remains in D for all $t \leq 0$ and tends to an equilibrium point as $t \to -\infty$.*

Fig. 1 The region D



Proof Since $\dot{V} \geq 0$, solutions cross level sets of V in the direction of larger values of V as t increases. Consequently, a solution that starts in D remains in D for $t \leq 0$ and, since D is a bounded set, the Proposition follows from Lemma 2.1. □

3 Local Analysis

In this section we look at the local phase portrait of Eq. (2) near an equilibrium point, which we assume has been translated to the origin. Specifically, let $x = u - \gamma$, $y = v$ where $f(\gamma) = 0$ and $g(x) = f(x + \gamma)$ so that Eq. 2 becomes

$$\dot{x} = y, \qquad \dot{y} = cy - g(x). \tag{7}$$

The translate of V (up to a constant) is

$$W = \frac{1}{2}y^2 + G(x), \qquad G(x) = \int_0^x g(\tau)d\tau,$$

so $\dot{W} = cy^2 \geq 0$. For the equilibrium of Eq. (2) at $(1, 0)$, Eq. 7 is defined on only one side of the origin. For simplicity we can make an odd extension of f across $u = 1$, so that g is also odd. However, we will see that we can work equally well with g defined on only one side.

The form of these equations and the sign of g near the origin is enough to determine the general nature of the orbit structure near the equilibrium without assuming that $g'(0)$ is nonzero or that it even exists. If g is odd, there are two main cases.

Case 1 *Saddle point.* We will call the equilibria a *saddle point* if there is an $\eta > 0$ with

$$xg(x) < 0 \qquad \text{for} \qquad 0 < |x| \leq \eta. \tag{8}$$

Thus we are assuming that locally the graph of g is in the second and fourth quadrant which would certainly be true if g had a negative derivative at the origin.

Proposition 3.1 *Under the assumption (8) the local phase portrait near the equilibrium of equation (7) looks like a traditional saddle point. That is to say, there is a neighborhood N of the equilibrium in phase space such that:*

(1) *[stable curves] there is a unique orbit in each of the second and fourth quadrants which stays in N for $t > 0$ and tends to the equilibrium as $t \to +\infty$;*

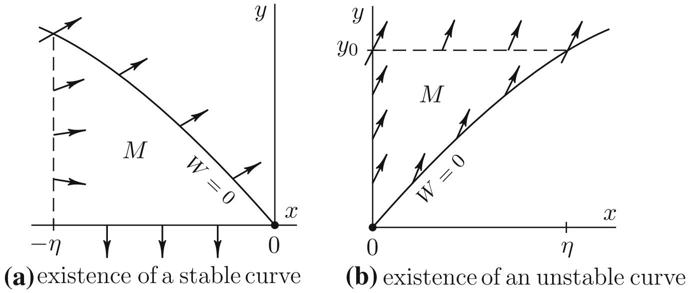


Fig. 2 The Region M

(2) *[unstable curves] there is a unique orbit in each of the first and third quadrants which stays in N for $t < 0$ and tends to the equilibrium as $t \to -\infty$;*
(3) *all other orbits leave N as $t \to \pm\infty$.*

If g is defined on only one side of the origin, then these conclusions still hold in the appropriate half plane.

These curves are traditionally called *separatrices* [6,12] because they often form the boundary between domains of attraction for sinks and domains of repulsion for sources, but we shall call them *stable* and *unstable curves* or the *asymptotic curves* in general.

Proof Consider first the second quadrant. Let M be the region defined by

$$M = \{(x, y) : -\eta \le x \le 0,\ y \ge 0,\ W(x, y) \le 0\}.$$

See Fig. 2a. The vectors defined by Eq. 7 point into M on the left boundary and point out of M on the top and bottom boundaries. This region does not quite satisfy the hypothesis of Ważewski's theorem [6,14] since there are both ingress and egress points on the boundary. This region is not quite an isolating block in the sense of Conley and Easton [2], since there is an equilibrium point on the boundary. However, the ideas of these classic works readily apply. The set of ingress points are the points on the boundary of M where the trajectories enter M, i.e. the ingress points are $I = \{(-\eta, y) : y > 0, W(-\eta, y) < 0\}$. The set of ingress points, I, is homeomorphic to an open interval.

The set of egress points are the points on the boundary of M where the trajectories exit M, i.e. $E = \{(x, 0) : -\eta < x < 0\} \cup \{(x, y) : -\eta < x < 0, y > 0, W(x, y) = 0\}$. The set of egress points E is homeomorphic to the union of two open intervals. (The intervals are separated by the equilibrium point $(0, 0)$.)

Suppose that no trajectory starting on the ingress set I approaches $(0, 0)$. Then all trajectories enter at the ingress set and exit at the egress set. The ingress and egress sets are cross sections to the flow and cross section maps define a homeomorphism. However, this is a contradiction since the ingress set is connected and the egress set is disconnected. Therefore, there is at least one trajectory in the second quadrant that approaches the origin.

Since $\dot{x} = y > 0$ in the interior of M, a solution which tends to the origin defines a graph of a continuous function of x which is C^1 on $[-\eta, 0)$. Suppose there were two different solutions tending to the origin given by the graphs $y = \psi_1(x)$ and $y = \psi_2(x)$ with $\psi_1(-\eta) > \psi_2(-\eta)$. Since $(\psi_1 - \psi_2)(-\eta) > 0$ and $(\psi_1 - \psi_2)(0) = 0$, by the mean value theorem there must be a $\xi \in (-\eta, 0)$ such that $(\psi_1 - \psi_2)'(\xi) < 0$. On the other hand, these are solution curves which cannot cross, so $\psi_1(x) > \psi_2(x)$ for $-\eta \le x < 0$. But $dy/dx = c - g(x)/y$ by (7), so for fixed $x = \xi$ the slope of solution curves increase with y, which implies $\psi_1'(\xi) > \psi_2'(\xi)$. This is a contradiction, so the solution is unique. Note that the argument was restricted to the second quadrant, so the conclusion still holds if g is defined on only the left half-plane.

For the fourth quadrant we use $M = \{(x, y) : 0 \leq x \leq \eta,\ y \leq 0,\ W(x, y) \leq 0\}$, but the argument is the same. For the first and third quadrants the arguments are also similar, but with time reversed. See Fig. 2b for the choice of M in the first quadrant. Here $M = \{(x, y) : x \geq 0,\ 0 \leq y \leq y_0,\ W(x, y) \geq 0\}$, where $y_0 > 0$ is chosen sufficiently small that $x \leq \eta$ on M. In this case the ingress set is homeomorphic to the union of two disjoint open intervals and the egress set is connected. Following the flow backwards in time we again get a homeomorphism from a connected set onto a disconnected set in the absence of a solution curve emanating from the origin, which sets up the contradiction. □

Proposition 3.2 *Moreover, the local asymptotic curves are continuous in the parameter c with those in the first and fourth quadrant strictly increasing in c and those in the second and third quadrant strictly decreasing in c.*

Pictorially, this proposition says that the positions of the asymptotic curves rotate strictly counterclockwise as c increases while staying in their respective quadrants.

Proof Let $0 < d < c$ be two parameter values and the graphs $y = \psi_c(x)$ and $y = \psi_d(x)$ be the corresponding asymptotic curves in the second quadrant. We now mimic the existence proof. Let $M' = \{(x, y) : -\eta \leq x \leq 0,\ y \geq 0,\ y \leq \psi_d(x)\}$. The vector field defined by (7) with parameter value c points into M' on the left boundary and points out of M' on the bottom boundary. Since $c > d$, the vector field also points out of M' on the top boundary. By the existence proof argument, the asymptotic curve $y = \psi_c(x)$ lies in M' and so $\psi_c(x) < \psi_d(x)$ for $-\eta \leq x < 0$. The other quadrants are analogous.

To show continuity from below, let $P = (-\eta, \psi_c(-\eta))$ be the point where the stable curve for parameter c crosses the line $x = -\eta$. Choose a point Q on $x = -\eta$ above P which is sufficiently close to P that the solution through Q for parameter c stays as close as we wish to the stable curve before exiting the second quadrant along the positive y-axis. Continuity of the solutions in the parameter guarantees that we can choose $\epsilon > 0$ such that for any $d < c$ with $c - d < \epsilon$, the d-solution through Q still exits the second quadrant along the positive y-axis. We now make the following observations. Since $d < c$, the stable curve for d is strictly above the stable curve for c. Since the d-solution through Q is above the origin at $x = 0$, the stable curve for d must be strictly below the d-solution through Q. Moreover, as the parameter decreases the tangent vectors rotate clockwise, which implies the d-solution through Q sits below the c-solution through Q. Consequently, the stable curve for parameter d sits strictly between the stable curve for c and the c-solution through Q, which were chosen to be as close as we wished. There is a mirror image argument for continuity from above and analogous arguments for the other asymptotic curves. □

For our applications $(1, 0)$ is a saddle, in which case we have the following.

Theorem 3.1 *If $f(u) > 0$ on some interval $1 - \epsilon < u < 1$, then there is a unique heteroclinic orbit $\phi(t)$ tending to $(1, 0)$ as $t \to +\infty$ and it tends to one of the equilibria in the interior of D as $t \to -\infty$.*

Proof From the proof of Proposition 3.1 the unique orbit that approaches $(1, 0)$ from the left lies in D and so it must approach one of the equilibria in D as $t \to -\infty$ by Proposition 2.1. □

Case 2 *Source points*. Now assume that there is an $\eta > 0$ such that

$$xg(x) > 0 \quad \text{for} \quad 0 < |x| \leq \eta. \tag{9}$$

Thus we are assuming that locally the graph of g is in the first and third quadrant, which would certainly be true if g had a positive derivative at the origin.

Proposition 3.3 *Under the assumption (9) the equilibrium of (7) is asymptotically unstable, i.e. it is a source.*

In particular, it is either like a node, with orbits emanating from the origin only through one of the two sectors making up $\mathcal{S} = \{(x, y) : 0 < y/x \leq c\}$, *or a spiral, with orbits winding around the origin infinity often in a counterclockwise direction as* $t \to -\infty$.

Proof The assumption (9) implies that W has a local minimum at the origin. As before $\dot{W} \geq cy^2$. These facts and the same reasoning as given in Proposition 2.1 yield the first assertion.

For this proof 'enters', 'leaves' and 'approaches' refers to backward time. The line $y = cx$ divides the interior of the first quadrant into two sectors with the lower sector constituting one half of the set $\mathcal{S}$. If an orbit meets this line it must enter the upper sector and remain there before exiting through the positive y axis since the slope of the orbit, $dy/dx = c - g(x)/y$, is less than the slope, c, of the line. Thus the orbit must enter the second quadrant.

Continue to trace the orbit backwards. Since g is odd, $g(x) < 0$ in the second quadrant. Therefore, both $\dot{y}$ and $\dot{x}$ are positive and orbits have positive slope at all points in the second quadrant. This implies the orbit cannot converge to $(0, 0)$ through the second quadrant, since convergence would require the orbit to have points with negative slope. Consequently, the orbit must leave the second quadrant through the negative x-axis. The arguments for the third and fourth quadrants repeat those of the first and second, respectively. From this we can conclude that an orbit ultimately approaches the origin through one of the two sectors making up $\mathcal{S}$ or spirals around the origin infinitely often. Since orbits cannot cross, all orbits approaching the origin must behave in the same way. □

4 Classic Case

In this section we will assume that

$$f(u) > 0 \quad \text{for} \quad 0 < u < 1. \tag{10}$$

Since f is odd we have $f(u) < 0$ for $-1 < u < 0$ and f is zero only at $-1, 0, 1$. The equilibria at $(\pm 1, 0)$ are saddle points in the sense of Proposition 3.1 and the equilibrium at $(0, 0)$ is a source in the sense of Proposition 3.3.

In this case the origin is the only equilibrium in D, so by Theorem 3.1 there is a unique heteroclinic orbit from $(0, 0)$ to $(1, 0)$. Let $\phi_c(t) = (u(t), v(t))$ be this heteroclinic orbit.

Theorem 4.1 *If (10) holds and there exists a constant* $a > 0$ *such that*

$$f(u) \leq au \quad \textit{for} \quad 0 \leq u \leq 1, \tag{11}$$

then there exists a real number $c^* > 0$ *such the heteroclinic orbit* $\phi_c(t)$ *is positive for all* $c \geq c^*$ *and it is positive and negative for all* $0 < c < c^*$.

The bound (11) would certainly hold if f had a derivative at the origin. This theorem appears in [1,7] under the additional assumption that f is C^1.

Proof When $c = 0$, the equilibrium point at the origin is a center, i.e. there is a neighborhood of the origin filled with periodic solutions. Therefore, when c is small and positive all small solutions spiral out from the origin and the heteroclinic orbit is both positive and negative.

Let $\phi(t)$ be a heteroclinic orbit that is both positive and negative. It must cross from the second quadrant to the first quadrant by crossing the positive v-axis. The crossing must be transversal since at the crossing $\dot{u} = v > 0$. Since a transversal intersection is an open condition, the set $\{c : \phi_c \text{ is positive and negative}\}$ is open in $\mathbb{R}$.

Choose c with $0 < 4a \leq c^2$. We compare Eq. 2 and the linear equation

$$\dot{u} = v, \qquad \dot{v} = cv - au \tag{12}$$

Equation 12 is a linear unstable node with particular solutions along the eigenvectors of the coefficient matrix, that is along the vectors $(c \pm \sqrt{c^2 - 4a}, 2a)$. Let $\psi(t) = (\mathbf{u}(t), \mathbf{v}(t))$ be one such solution of (12), so $\psi(t)$ remains in the first quadrant for all t, tends to the origin as $t \to -\infty$, and tends to ∞ as $t \to +\infty$.

If the heteroclinic orbit $\phi_c(t) = (u(t), v(t))$ were not positive it would leave the first quadrant (in negative time) by crossing the positive v axis. It would have to cross the curve defined by $\psi(t)$. But at any point in the first quadrant

$$\frac{dv}{du} = c - \frac{f(u)}{v} \geq c - \frac{au}{v} = \frac{d\mathbf{v}}{d\mathbf{u}}.$$

This is impossible since the inequality would have to be in the opposite sense at a crossing.

If $\phi_c(t)$ is positive for $c = \tilde{c}$ then it is positive for all $c \geq \tilde{c}$, since the asymptotic curves are strictly decreasing with c by Proposition 3.2.

Let $c^* = \sup\{\gamma : \phi_c \text{ is positive and negative for all } 0 < c < \gamma\}$. Now $c^* < \infty$ since for large c the heteroclinic orbits is positive. Since the set of c giving rise to positive and negative heteroclinic orbits is open, ϕ_{c^*} must be positive. Thus, for all $c \geq c^*$ we have that ϕ_c is positive. □

We observe that in general the bound (11) is necessary.

Proposition 4.1 *If (10) holds and f has a vertical tangent at the origin, then for all $c > 0$ the heteroclinic orbit $\phi_c(t)$ is both positive and negative. In fact, the heteroclinic orbit spirals to the origin as $t \to -\infty$.*

Proof Assume the heteroclinic orbit is positive, so by Proposition 3.3 it must lie completely in the set $\mathcal{S}$ with $u > 0$. Consider the curve $\mathcal{C} : v = c^{-1} f(u)$, where $dv/du = 0$. The slope of any orbit at points below this curve is negative. Since f has a vertical tangent at the origin, $f(u) > c^2 u$ for all sufficiently small u, which implies that for small u the curve $\mathcal{C}$ lies above the line $v = cu$ by $c^{-1} f(u) > cu$. Therefore, the heteroclinic orbit must ultimately lie below $\mathcal{C}$ as it approaches the origin and so all points on the orbit near the origin must have negative slope. On the other hand, since it is converging to the origin through the first quadrant, it must have points with positive slope close to the origin. This is a contradiction and so the heteroclinic orbit cannot be positive. Repeat as in Proposition 3.3 to see that the heteroclinic orbit must be a spiral. □

5 Heteroclinic Bifurcations

In this section we assume that there is a real number θ, $0 < \theta < 1$, such that $f(u) < 0$ for $0 < u < \theta$, $f(\theta) = 0$, and $f(u) > 0$ for $\theta < u < 1$. Also assume

$$F_1 = \int_0^1 f(\tau)d\tau > 0.$$

We continue to assume $c > 0$, that $f(u)$ is odd, and $f(\pm 1) = f(\pm\theta) = f(0) = 0$. Each of these zeros of f give rise to an equilibrium point of Eq. 2. By Theorem 3.1 there is a unique heteroclinic orbit $\phi(t)$ tending to $(1, 0)$ from the left as $t \to +\infty$ and it tends to one of the equilibria $(\theta, 0)$, $(0, 0)$, or $(-\theta, 0)$ as $t \to -\infty$.

Let

$$R_n = \{c \in \mathbb{R}_+ : \phi(t) \to (\theta, 0) \text{ as } t \to -\infty \text{ and crosses the negative } v\text{-axis } n \text{ times}\}$$

$$L_n = \{c \in \mathbb{R}_+ : \phi(t) \to (-\theta, 0) \text{ as } t \to -\infty \text{ and crosses the negative } v\text{-axis } n \text{ times}\}$$

$$A_n = \{c \in \mathbb{R}_+ : \phi(t) \to (0, 0) \text{ in the first quadrant as } t \to -\infty \\ \text{and crosses the negative } v\text{-axis } n \text{ times}\}$$

$$B_n = \{c \in \mathbb{R}_+ : \phi(t) \to (0, 0) \text{ in the third quadrant as} t \to -\infty \\ \text{and crosses the negative } v\text{-axis } n \text{ times}\}$$

(R, L, A, B stand for right, left, above, and below).

Theorem 5.1 *The sets A_n, B_n are singletons and the sets R_n, L_n are open intervals. Let $A_n = \{a_n\}$ and $B_n = \{b_n\}$, then*

$$\infty = b_{-1} > a_0 > b_0 > a_1 > b_1 > a_2 > b_2 > a_3 > b_3 \cdots \to 0$$

and

$$R_n = (a_n, b_{n-1}), \qquad L_n = (b_n, a_n).$$

Theorem 5.1 is illustrated in Fig. 3, where only the stable curve of $(1, 0)$ is shown for the parameter c in these various ranges. Note that $(\pm\theta, 0)$ are likely locally to be spirals, so for c in R_n or L_n, the left end of the curve may not go directly to the equilibrium as show in the figure.

Our method of proof is reminiscent of Sir William Matthew Flinders Petrie's method used to find the chronological order of the pyramids of Giza [13]. Long before carbon dating could give good estimates on the exact dates, Petrie in the late 1800s found the order by examining potter shards. Potter styles evolve over time so pyramids with similar shards will be of about the same date. Using this idea he gleaned information about the nearest neighbors. Also he assumed that the simplest shards were the oldest and the most stylized were the youngest. With these three simple principles he was able to order the pyramids.

With Petrie, we do not give the exact values of the a's and b's, but only their order. We also use three principles or lemmas. The first, Lemma 5.1, gives nearest neighbor information, the second, Lemma 5.2, tells what happens for large c, and the last, Lemma 5.3, tells what happens for small c. With these three lemmas we found the order.

Before stating the lemmas we make some general remarks. Our standing assumption is $c > 0$, but if we take $c = 0$ we have $\dot{V} = 0$, so the function V is an integral for Equation

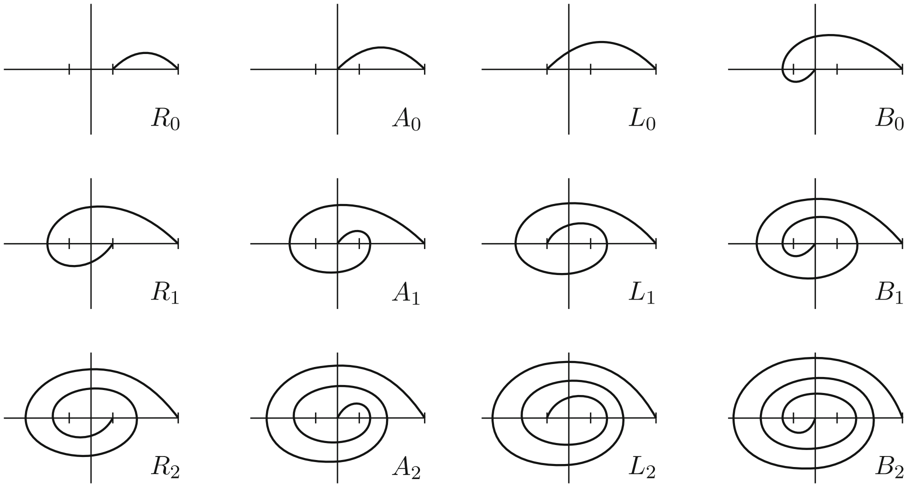


Fig. 3 The unfolding sequence as c decreases

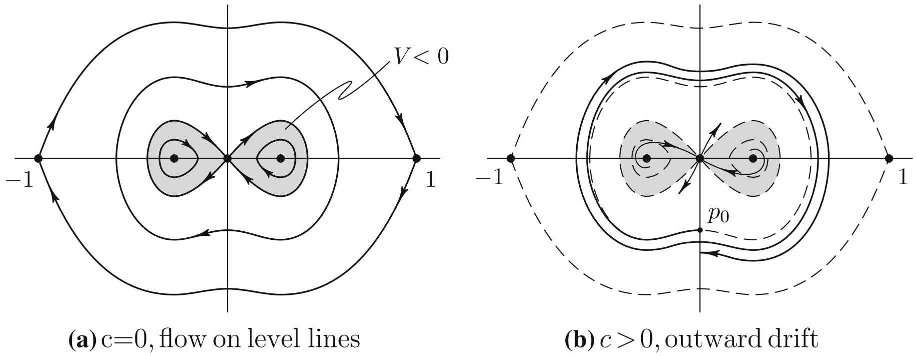
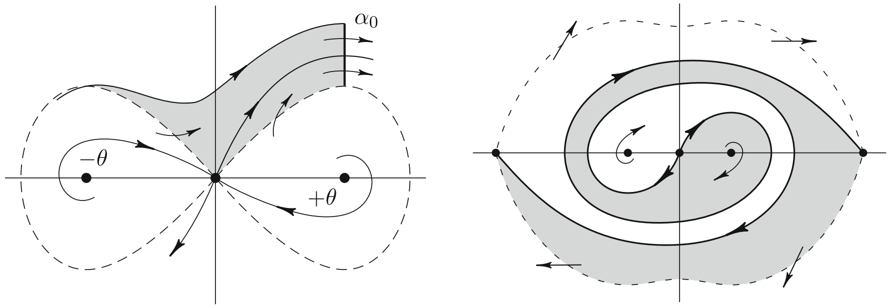


Fig. 4 Flow lines compared to the level lines of V

(2) and the flow is along the level lines of V—see Fig. 4a. By (5) the level lines of V are qualitatively accurate as shown. In particular, the region where $V < 0$, marked in gray, consists of two disjoint neighborhoods of the equilibria at $(\pm\theta, 0)$ with the maximum value of v on its boundary occurring at $u = \pm\theta$, where f changes sign. The equilibria at $(\pm\theta, 0)$ are centers and the outer region of D consists of periodic solutions which limit on the two orbits asymptotic to the equilibria at $(\pm 1, 0)$. As soon as $c > 0$, V becomes a Liapunov function and the flow drifts across the level lines in the outward direction making the equilibria at $(\pm\theta, 0)$ sources. Note that since the function V does not depend on c, the region where $V < 0$ is contained in the domains of repulsion for the equilibria $(\pm\theta, 0)$ for all $c > 0$. We will call the set where $V < 0$ and $u > 0$ (respectively $u < 0$) the *core domain of repulsion of* $(+\theta, 0)$ (respectively $(-\theta, 0)$). Because the equilibria at $(\pm\theta, 0)$ are sources for $c > 0$, it follows that the sets R_n and L_n are open.

Since V increases along solutions and $V = 0$ at the origin, the stable curves of $(0, 0)$ are contained in the core domains of repulsion of $(\pm\theta, 0)$, while the unstable curves are contained in the region where $V > 0$ (see Fig. 4b). This can also be seen directly from the choice of M in Fig. 2. In our discussion we shall refer to the two unstable curves of $(0, 0)$ as



(a) The region $N0$. **(b)** The regions of repulsion when $\tilde{c} \in A_1$.

Fig. 5 **a** The region N_0. **b** The regions of repulsion when $\tilde{c} \in A_1$

the A-curve and the B-curve. The A-curve is the unstable curve of $(0, 0)$ starting in the first quadrant; the B-curve is the unstable curve of $(0, 0)$ starting in the third quadrant.

Let $\bar{v}$ be the maximum value of v on the set where $V = 0$ (the boundary of the core domains) which, as observed above, occurs at $u = \pm\theta$. For any $c > 0$ consider the solution through the point $(-\theta, \bar{v})$. Since $\dot{V} > 0$, the solution is bounded away from the u-axis on the strip $-\theta \leq u \leq +\theta$. Consequently, $\dot{u}$ is bounded away from 0 and the solution extends across the strip and intersects the line $u = \theta$. Let $v = \phi(u)$, $-\theta \leq u \leq +\theta$, be the graph of this solution through $(-\theta, \bar{v})$. The A-curve must also cross $u = \theta$ and at a value intermediate between $\bar{v}$ and $\phi(\theta)$. Define a neighborhood N_0 by

$$N_0 = \{(u, v) : |u| < \theta, 0 \leq v < \phi(u), V(u, v) > 0\}$$

and let α_0 be the subsegment of $u = \theta$ satisfying $\bar{v} < v < \phi(\theta)$ (see Fig. 5a). N_0 forms a funnel. The only ingress sets for N_0 are from the core domains of repulsion of $(\pm\theta, 0)$, so every solution crossing α_0 which is not the A-curve came directly from the core domain of either $(-\theta, 0)$, if it crossed above the A-curve, or $(+\theta, 0)$, if it crossed below the A-curve. We can construct a similar picture for the B-curve.

For $c > 0$ the A- and B-curves must leave D as t increases or tend to $(\pm 1, 0)$ as $t \to +\infty$ for if not they would be bounded and tend to an invariant set in D. The only invariant sets in D are $(0, 0)$, $(\pm\theta, 0)$. The A- and B-curves cannot tend to $(\pm\theta, 0)$, which are sources, and cannot return to $(0, 0)$, since V is increasing. Thus the union of the A- and B-curve separate D into two regions which are the domains of repulsion of $(\pm\theta, 0)$ in D. (They are the separatrices.)

We will call the stable curve of $(1, 0)$ starting in $\{u < 1, v > 0\}$ the S-curve. Figure 5b illustrates the heteroclinic orbit in the region D when $\tilde{c} \in A_1$. In this case the A-curve and the S-curve coincide. Also shown is the symmetric heteroclinic orbit from $(0, 0)$ to $(-1, 0)$—the B-curve. As stated above, the A- and B-curves separate D into the domains of repulsion of the sources at $(\pm\theta, 0)$ for all $c > 0$. In Fig. 5b the domain of repulsion of $(+\theta, 0)$ is shaded, whereas the domain for $(-\theta, 0)$ is not.

The domains of repulsion can be quite complicated especially when c is small. However, if we use the direction of increasing t along a solution to provide an orientation, then the domain of repulsion of $(+\theta, 0)$ is always on the right of the A-curve as t increases and the domain of repulsion of $(-\theta, 0)$ is on the left. On the other side, the domain of repulsion of $(+\theta, 0)$ is on the left of the B-curve as t increases and domain of repulsion of $(-\theta, 0)$ is on

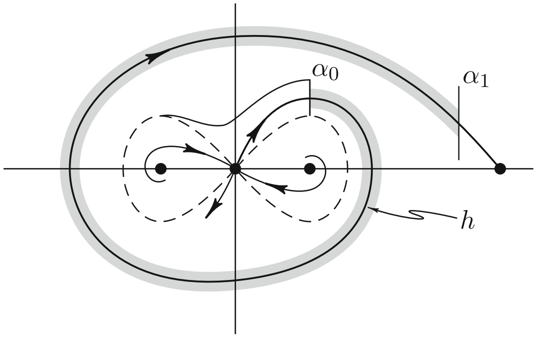


Fig. 6 The tubular neighborhood of h

the right. As c increases the vector field defined by (2) rotates counterclockwise or veers to the left relative to an orbit for a smaller value of c (for increasing t).

Lemma 5.1 *If $\tilde{c} \in A_n$, then $\tilde{c}$ is the right endpoint of an interval in L_n and a left endpoint of an interval in R_n.*

If $\tilde{c} \in B_n$, then $\tilde{c}$ is the right endpoint of an interval in R_{n+1} and a left endpoint of an interval in L_n.

In particular, the set $(\cup_n A_n) \cup (\cup_n B_n)$ contains only isolated points.

Proof Let $\tilde{c} \in A_n$. For any $c > 0$ we can construct a region N_0 as in Fig. 5a where only the top boundary changes with c. Choose $c_0 < \tilde{c}$ sufficiently close to $\tilde{c}$ that the solution through $(-\theta, \bar{v})$ for $c = c_0$ still crosses $u = \theta$ above the A-curve for $c = \tilde{c}$. Fix N_0 at $c = c_0$ and consider solutions for $c > c_0$ relative to this fixed N_0. The top boundary is no longer impervious to the flow. Rather for $c > c_0$ the vector field points out across the top boundary so the top boundary is an egress set. Consequently, the only ingress sets are still the two arcs where $V = 0$ and we can draw the same conclusion as before: any solution which intersects the cross section α_0 is either the A-curve for that parameter value or came directly from the core basins of repulsion of $(\pm\theta, 0)$ without crossing the negative v-axis. This holds even if c is so large that the A-curve crosses above α_0, although in that case all solutions crossing α_0 would be coming exclusively from the core basin of repulsion of $(+\theta, 0)$.

The intersection of the A-curve with α_0 is continuous and strictly increasing as a function of c by Proposition 3.2. Let N_1 be a neighborhood of $(1, 0)$ given by Propositions 3.1 and 3.2 such that the S-curve is strictly decreasing in N_1 as c increases. Let $\alpha_1 \subset N_1$ be a small open arc transverse to the heteroclinic orbit when $c = \tilde{c}$ and let h be the segment of this heteroclinic orbit between α_0 and α_1. Since $\tilde{c} \in A_n$, h crosses the negative v-axis exactly n times. Let Q be a small tubular neighborhood of h which is sufficiently small that any solution which stays in Q also crosses the negative v-axis exactly n times (see Fig. 6). Since solutions are continuous in all of the variable, including the parameter, we can choose an open subarc $\alpha_1' \subset \alpha_1$ and an $\epsilon > 0$ with $\epsilon < \tilde{c} - c_0$, such that for all c with $|c - \tilde{c}| < \epsilon$, any solution for parameter c which intersects α_1' stays in Q in backwards time until it intersects α_0. By possibly choosing ϵ smaller, we can also assume that the S-curve for parameter c intersects α_1' for all c with $|c - \tilde{c}| < \epsilon$.

As c increases from $\tilde{c}$ the vector field points to the left across h as one traverses the orbit with increasing t. Thus solutions for parameter c which stay in Q can only cross h from right to left with increasing t, or from left to right in backwards time. Since the S-curve is strictly decreasing as c increases, for $\tilde{c} < c < \tilde{c} + \epsilon$ the S-curve crosses α_1' strictly below (to the right) of h. Consequently, the S-curve stays in Q and is trapped on the right side of

h, crossing α_0 to the right of h. On the other hand, the A-curve is strictly increasing in c, so the A-curve for parameter c crosses α_0 strictly to the left of h. Since the S-curve is a solution for parameter c crossing α_0 below the A-curve for the same parameter, the S-curve came directly from the core basin of repulsion of $(+\theta, 0)$. As a result, the S-curve does not cross the negative v-axis before hitting α_0, crosses exactly n times while in Q, and then does not cross again after α_1'. Thus, $(\tilde{c}, \tilde{c} + \epsilon) \subset R_n$ and $\tilde{c}$ is the left endpoint of an interval in R_n.

As c decreases from $\tilde{c}$ the vector field points to the right across h as one traverses the orbit with increasing t, the A-curve will intersect α_0 to the right of h, and the S-curve will hit α_1' to the left of h, be trapped to the left of h in Q, and ultimately land in the core basin of repulsion of $(-\theta, 0)$. Thus, $(\tilde{c} - \epsilon, \tilde{c}) \subset L_n$ and $\tilde{c}$ is the right endpoint of an interval in L_n.

When $\tilde{c} \in B_n$ the argument is just about the same. Again as c increases from $\tilde{c}$ the S-curve jumps to the left source and as c decreases it jumps to the right source, but jumping to the right source causes it to cross the negative v-axis one more time, so $\tilde{c}$ is the right endpoint of L_n, but the left endpoint of R_{n+1}. □

Lemma 5.2 *There exist a $c^* > 0$ such that $c \in R_0$ for all $c > c^*$.*

Proof Choose $\bar{v} > 0$ sufficiently small that the segment $I = \{(\theta, v) : 0 < v < \bar{v}\}$ is in the core domain of repulsion of $(\theta, 0)$. This does not depend on c. For each $c > 0$, the S-curve is the graph of a function for $\theta \leq u \leq 0$. Let this function be $v_c(u)$. $v_c(u) \to 0$ as $u \to 1^-$. If for some $c > 0$, $v_c(\theta) \leq \bar{v}$, then the S-curve intersects the domain of repulsion of $(\theta, 0)$ and $c \in R_0$.

Let k be the maximum of $f(u)$ on $\theta \leq u \leq 1$ and set $c^* = k/\bar{v}$. For $c > c^*$, the slope of the solution curve through any point (u, v) in the half strip $H = \{(u, v) : \theta \leq u \leq 1, v \geq \bar{v}\}$ is

$$\frac{dv}{du} = c - \frac{f(u)}{v} \geq c - \frac{k}{\bar{v}} > 0.$$

Therefore, $dv_c/du > 0$ at any point on the graph of $v_c(u)$ which is inside H. Consequently, if the graph were to start in H, $v_c(\theta) \geq \bar{v}$, then v_c would be increasing on $\theta \leq u \leq 1$. However, this would contradict that $v_c(u) \to 0$ as $u \to 1^-$. Consequently, $v_c(\theta) < \bar{v}$ and $c \in R_0$ for all $c > c^*$. □

Lemma 5.3 *For any large n there is a small c such that the heteroclinic orbit crosses the negative v axis at least n times.*

Proof When $c = 0$ the flow is along the level lines of V—see Fig. 4a. Exterior to the stable and unstable curves of the origin and within D all solutions are periodic and cross the negative v-axis. Each of these periodic orbits are Jordan curves that separates the critical points $(0, 0)$, $(\pm\theta, 0)$ from the critical points $(\pm 1, 0)$.

Let p_0 be a point on the negative v axis within D, so the solution $\psi_0(t)$ of Eq. 2 is periodic with period T. Let $c > 0$ be so small that the solution $\psi_c(t)$ of (2) through p_0 with this c meets the negative v axis at least $n + 1$ times. The solution $\psi_c(t)$ meets the negative v axis in successive points $p_0 = (0, y_0)$, $p_1 = (0, y_1), \cdots, p_n = (0, y_n)$. Since $\dot{V} \geq 0$ it follows that $0 > y_0 > y_1 \cdots > y_n$. This is illustrated in Fig. 4b for $n = 2$. Let $I_k = \{(y, 0) : y_{k-1} \geq y > y_k\}$.

The segment of the solution $p_c(t)$ from p_0 to p_1 union with the interval I_1 is a Jordan curve which separates the critical points $(0, 0)$, $(\pm\theta, 0)$ from the critical points $(\pm 1, 0)$. The heteroclinic orbit must intersect I_1 and then successively $I_2, I_3, \ldots, I_n$ and so it must cross the negative v axis at least n times. □

Proof of Theorem 5.1 By Propositions 3.1 and 3.3 and Theorem 3.1, the sets R_n, L_n, A_n, and B_n partition $(0, +\infty)$. By Lemma 5.2 R_0 is nonempty and contains an interval $(c^*, +\infty)$. Let $a_0 = \inf\{a : (a, +\infty) \subset R_0\}$. R_0 is open, so $(a_0, +\infty) \subset R_0$. Moreover, by Lemma 5.3, R_0 is not everything, so $a_0 > 0$. Since the sets R_n and L_n are open, a_0 must be in some A_n or B_n. By Lemma 5.1, the only possibility is for a_0 to be an isolated point of A_0. We also know from Lemma 5.1 that there exists a nonempty interval $(b, a_0) \subset L_0$. Define $b_0 = \inf\{b : (b, a_0) \subset L_0\}$. By the same arguments we conclude that $b_0 > 0$, $(b_0, a_0) \subset L_0$, and b_0 is an isolated point of B_0. We continue in this manner to generate a sequence

$$\infty = b_{-1} > a_0 > b_0 > a_1 > b_1 > a_2 > b_2 > a_3 > b_3 \ldots$$

with the properties that for each n, $a_n \in A_n$, $b_n \in B_n$, (a_n, b_{n-1}) is a maximal interval in R_n and (b_n, a_n) is a maximal interval in L_n. If this sequence converges to 0 we are done, for then each of the A_n and B_n contain exactly one point and each of the R_n and L_n consist of exactly one interval. Suppose the sequence converges to some $d \geq 0$. d cannot be in R_n or L_n, since these are open, and d cannot be in A_n or B_n, because these sets contain only isolated points. Therefore, $d = 0$. □

References

1. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusion arising in population genetics. Adv. Math. **30**, 33–76 (1978)
2. Conley, C., Easton, R.: Isolated invariant sets and isolating blocks. Trans. Am. Math. Soc. **158**, 137–143 (1971)
3. Doelman, A., Hek, G., Valkhoff, N.: Stabilization by slow diffusion in a real Ginzburg-Landau system. J. Nonlinear Sci. **14**(3), 237–278 (2004)
4. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to traveling front solutions. Arch. Rational Mech. Anal. **65**, 335–361 (1977)
5. Gallay, Th., Raugel, G.: Stability of travelling waves for a damped hyperbolic equation. Z. Angew. Math. Phys. **48**(3), 451–479 (1997)
6. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
7. Hou, X., Li, Y., Meyer, K.R.: Traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities. Discrete Contin. Dyn. Syst. **26**(1), 265–290 (2010)
8. Ilmanen, T.: Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. J. Differ. Geom. **38**(2), 417–461 (1993)
9. LaSalle, J.P.: An invariance principle in the theory of stability. In: Hale, J.K., LaSalle, J.P. (eds.) Differential Equations and Dynamical Systems, pp. 277–286. Academic Press, New York (1967)
10. LaSalle, J.P.: Stability theory for ordinary differential equations. J. Differ. Equ. **4**, 57–65 (1968)
11. LaSalle, J.P., Lefschetz, S.: Stability by Liapunov's Direct Method with Applications. Academic Press, New York (1961)
12. Lefschetz, S.: Differential Equations: Geometric Theory. Interscience, New York (1957)
13. Petrie, W.M.F.: The Pyramids and Temples of Gizeh. Field & Tuer, London (1883)
14. Ważewski, T.: Sur l'unicité et la limitation des intégrales des équations aux dérivées partielles du premier ordre. Atti R. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **18**(6), 372–376 (1933)