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Cross Sections in the Three-Body Problem

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In *Dynamical Systems*, Birkhoff gave a clear formulation of a cross section, suggested a possible generalization to cross sections with boundary, and raised the question of whether or not such cross sections exist in the three-body problem. In this work, we explicitly develop Birkhoff's notion of a generalized cross section, formulate homological necessary conditions for the existence of a cross section or generalized cross section, and show that these conditions are not satisfied in the three-body problem.

KEY WORDS: Birkhoff's problem; cross section; three-body problem.

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1. STATEMENT OF RESULTS

1.1. Introduction

The existence of a global cross section to a flow places global geometric restrictions not only on the flow but on the space that underlies it. The nonexistence of a cross section suggests a certain level of complexity for the flow. Unfortunately, it is not always easy to determine whether or not a given flow admits a cross section, since the existence or nonexistence depends on both the topology of the space and the dynamics of the flow.

Poincaré first realized the importance of cross sections in his studies of the restricted three-body problem [20]. He was able to reduce the problem of the existence of periodic solution of the restricted problem to the problem of finding fixed points of a cross section with boundary. Although Poincaré was able to establish the existence of fixed points in several

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special cases, it was left to Birkhoff [3] to establish the general fixed point theorem. Much of Birkhoff's classic memoir [2] is devoted to questions about cross sections in the three body problem.

Even though it is popular to assume the existence of global cross sections in dynamical systems, Reeb observed that they do not appear in classical mechanics [21]. Here is one consequence of Reeb's results.

Consider a classical Hamiltonian system defined on the cotangent bundle of a manifold Q with Hamiltonian $H=K+V$, where K is kinetic energy (a Riemannian metric) and $V: Q \to \mathbb{R}$ is potential energy. Let M be the level set where $H=h$, then the Hamiltonian flow defined by H on M does not admit a compact global cross section.

The applications we have in mind are to the problems of celestial mechanics and, in particular, to the three-body problem. In these problems the level sets are not compact in general. Also, these problems start as classical systems, but they are no longer classical systems when studied on the reduced space where all the integrals and symmetries have been eliminated. Thus Reeb's result does not apply to our studies.

We develop some necessary conditions for a flow on a manifold to admit a global cross section or a cross section with boundary. Then using our computations of the cohomology of the integral manifolds of the spatial three-body problem by McCord *et al.* [17] and the planar three-body problem in Section 4, we discuss the existence of cross section in both senses. This discussion answers in the negative a question raised by Birkhoff in his classic text on dynamical systems [2].

1.2. Global Cross Sections

Let M be a connected manifold of dimension m without boundary, $\Phi: \mathbb{R} \times M \to M$ a flow, and C a submanifold of M of dimension $m-1$ without boundary. Then C is a *global cross section* if

1. For each point $p \in M$ there is a $t(p) > 0$ such that $\Phi(t(p), p) \in C$.
2. There is a continuous function $\tau: C \to \mathbb{R}$ such that
   (a) $\Phi(t, p) \notin C$ for all $p \in C$ and $0 < t < \tau(p)$.
   (b) $\Phi(\tau(p), p) \in C$ for all $p \in C$.
3. There is an open neighborhood U of $C \times \{0\}$ in $C \times \mathbb{R}$ such that $\Phi|_U$ is a homeomorphism from U to an open neighborhood of C in X.

The function τ is called the *return time*. The function $P: C \to C: p \to \Phi(\tau(p), p)$ is a homeomorphism and is called the *Poincaré map* or *first return map*.

If a flow admits a cross section, it admits infinitely many. For example, if C is a cross section with return time function τ, and ρ: $C \to \mathbb{R}$ is any continuous function, then $C' = \{\Phi(\rho(p)\,\tau(p), p) \mid p \in C\}$ is also a cross section with return time function

$$\tau'(\Phi(\rho(p)\,\tau(p), p)) = \tau(p) + \rho(P(p))\,\tau(P(p)) - \rho(p)\,\tau(p)$$

Clearly, $h(p) = \Phi(\rho(p)\,\tau(p), p)$ defines a homeomorphism h: $C \to C'$ which conjugates P and P'. But M may admit other cross sections (C'', P'') which are not conjugate. For example, on the two-dimensional torus T^2 with flow

$$\dot{\theta}_1 = 1$$
$$\dot{\theta}_2 = 0,$$

each $C_n = \{\theta_2 = n\theta_1\}$, $n \geqslant 1$ is a cross section with return map $P_n(\theta_1) = \theta_1 + 1/n$. These are clearly not conjugate to each other.

The point is, if a flow admits a cross section, then it may generate several inequivalent diffeomorphisms as Poincaré maps. On the other hand, suppose $\tilde{P}$: $\tilde{C} \to \tilde{C}$ is a homeomorphism of a manifold $\tilde{C}$. Let $\tilde{M} = (\mathbb{R} \times \tilde{C})/\sim$ where $\sim$ is the equivalence relation on $\mathbb{R} \times \tilde{C}$ defined by $(t+1, p) \sim (t, \tilde{P}(p))$. Define a flow on $\tilde{M}$ by Φ: $\mathbb{R} \times \tilde{M} \to \tilde{M}$: $(s, [(t, p)]) \to [(t+s, p)]$ where $[\cdot]$ denotes an equivalence class. $\tilde{\Phi}$ is called the *suspension* of $\tilde{P}$. The flow $\tilde{\Phi}$ admits $[0 \times \tilde{C}] \equiv \tilde{C}$ as a global cross section.

Thus, every cross section on a flow produces a diffeomorphism on a manifold of one dimension less, and every diffeomorphism produces a flow with a cross section on a manifold of one dimension higher. If one begins with a cross section, obtains the Poincaré map, and constructs its suspension, the resulting manifold and flow are equivalent to the original. If one begins with a diffeomorphism and constructs its suspension, then any slice $C_t = \{t = t_0\}$ will be a section with return map conjugate to the original diffeomorphism (though other, nonconjugate return maps may also exist).

The first and fundamental question for cross sections is existence: Given a flow Φ on a manifold M, does there exist a cross section? In particular, are there *computable* necessary or sufficient conditions for the existence of a cross section? A related but distinct problem is to classify all conjugacy classes of cross sections and Poincare maps. While this second question has received considerable attention [10, 11, 29], the first has not received as much attention as one might expect. Some simple necessary conditions for the existence of a global cross section are formulated in the following:

Theorem 1.1. *If the flow Φ: $\mathbb{R} \times M \to M$ on the manifold M admit a global cross section C, then*

- *M is a fiber bundle over S^1 with fiber C.*
- *There is a long exact homology sequence*

$$\to H_{k+1}(M) \to H_k(C) \xrightarrow{id - P_*} H_k(C) \to H_k(M) \to$$

- *If C is of finite type, there exists an integer polynomial $Q(t)$ with $0 \leqslant Q(t) \leqslant P_C(t)$ such that $P_M(t) = (1+t)\, Q(t)$.*
- *If C is of finite type, then $\chi(M) = 0$ (the Euler characteristic of M is zero).*
- *If C is of finite type, $H_1(M; \mathbb{Z})$ has a factor $\mathbb{Z}$.*
- *The flow has no equilibrium points.*

Remark. *Of finite type* means that the homology of M is finitely generated. In that setting, $P_M(t)$ is the *Poincaré polynomial* of M: a formal polynomial whose nth coefficient is the nth Betti number of M. The Euler characteristic is the alternating sum of the Betti numbers, and is also the value of the Poincaré polynomial evaluated at $t = -1$. The inequality $0 \leqslant Q(t) \leqslant P_C(t)$ should be interpreted term by term: each coefficient q_n is nonnegative and less than or equal to the nth Betti number of C.

1.3. General Cross Sections

Reeb's theorem indicates that global cross sections are not common. Theorem 1.1 shows that, among other things, no flow with an equilibrium point can admit a global cross section. But there are situations in which a global cross section "almost" exists. As the simplest possible example, consider the flow

$$\dot{\theta} = 1$$

$$\dot{r} = 0$$

on $\mathbb{R}^2$ where (r, θ) are polar coordinates. Let C be a closed ray emanating from the origin. Then $C \backslash \partial C$ is a cross-section for the flow on $\mathbb{R}^2 \backslash \{0\}$.

Now let M and Φ be as above but now C is a submanifold of M of dimension $m-1$ with boundary ∂C of dimension $m-2$. Let int $C = C \backslash \partial C$ be the interior of C. Then C is a *cross section with boundary* if

1. The boundary ∂C of C is invariant under the flow Φ.
2. $C \backslash \partial C$ is a cross section for the flow on $X \backslash \partial C$.
3. The return time and Poincaré map on $C \backslash \partial C$ extend continuously to ∂C.

Reeb's theorem says that global cross sections in an energy surface do not exist for classical dynamical systems but cross sections with boundary often exist. Generically on a compact manifold, the Hamiltonian has a non-degenerate minimum of general elliptic type [15], and for a two-degree of freedom system the flow near such an equilibrium point admits a cross section with boundary on an energy surface. We illustrate this with a simplified example.

Consider a Hamiltonian system on $\mathbb{R}^4$ with Hamiltonian

$$H=\frac{\omega_1}{2}(x_1^2+y_1^2)+\frac{\omega_2}{2}(x_2^2+y_2^2)$$

where $\omega_1, \omega_2>0$ are constants (frequencies). H has a minium at the origin and $H=h>0$ is an ellipsoid homeomorphic to S^3. Change to action-angle coordinates I_1, I_2, ψ_1, ψ_2 by $I_i=\frac{1}{2}(x_i^2+y_i^2)$, $\psi_i=\arctan y_i/x_i$ so that

$$H=\omega_1 I_1+\omega_2 I_2$$

and the equations of motion become

$$\begin{aligned} \dot{I}_1&=0, \qquad & \dot{\psi}_1&=-\omega_1 \\ \dot{I}_2&=0, \qquad & \dot{\psi}_2&=-\omega_2 \end{aligned}$$

A geometric model for S^3 can be obtained from these coordinates. Consider the set where $H=\omega_1>0$, which we call S^3. Since $\omega_1 I_1+\omega_2 I_2=\omega_1$, we can ignore the I_2 coordinate and use I_1, ψ_1, ψ_2 as coordinates on S^3, but remember that $0\leqslant I_1\leqslant 1$ and ψ_1, ψ_2 are angles defined modulo 2π. The closed unit disk is coordinatized by (I_1, ψ_1), $0\leqslant I_1\leqslant 1$ using the usual conventions of action-angle (polar) coordinates. For each point in the open unit disk, there is a circle with coordinate ψ_2 (defined modulo 2π), but when $I_1=1$, $I_2=0$; so the circle collapses to a point over the boundary of the disk. Thus a geometric model for S^3 is two solid cones with points on the boundary identified as show in Fig. 1. There are always two periodic orbits, namely, where $I_1=0$ and $I_1=1$, and these would be called the normal modes by engineers.

The closed disk where $\psi_2=0 \bmod 2\pi$ is a cross section with boundary, since its boundary is the periodic orbit where $I_1=1$ and all other solutions cross the open disk ($\dot{\psi}_2=-\omega_2\neq 0$). The return time is $2\pi/\omega_2$. This cross section is shaded in Fig. 2.

There is another cross section which is an annulus with both the periodic orbits as boundaries. This cross section is defined by $\psi_1+\psi_2=0 \bmod 2\pi$

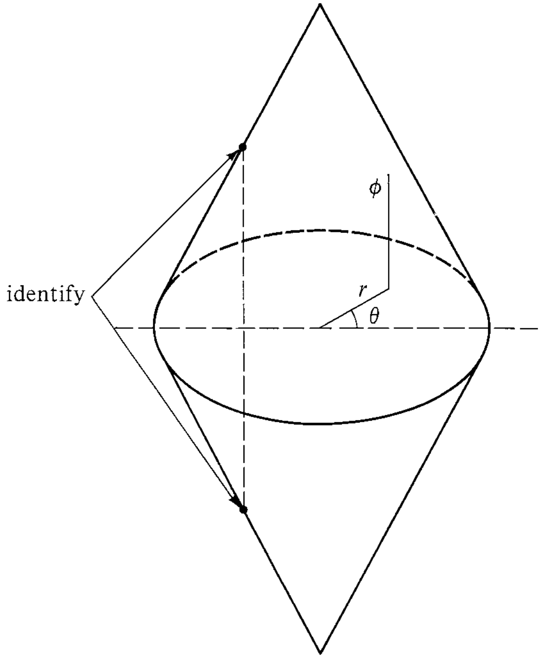

Fig. 1. Model of S^3.

with return time $2\pi/(\omega_1+\omega_2)$. Conley [7] showed that the flow near a primary in the restricted three-body problem after regularization of collisions is but a perturbation of this examples and was able to use this observation to invoke Birkhoff's fixed point theorem to establish the existence of long-period periodic solutions.

The same basic existence and uniqueness questions arise for cross sections with boundary. The natural generalization of Theorem 1.1 gives some verifiable necessary conditions for the existence of a cross section with boundary.

Theorem 1.2. *If the flow Φ: $\mathbb{R}\times M \to M$ on the manifold M admits a cross section with boundary C, then*

- *$M\backslash\partial C$ is a fiber bundle over S^1 with fiber $C\backslash\partial C$.*
- *There is a long exact homology sequence*

$$\to H_{k+1}(M, \partial C) \to H_k(C, \partial C) \xrightarrow{id - P_*} H_k(C, \partial C) \to H_k(M, \partial C) \to$$

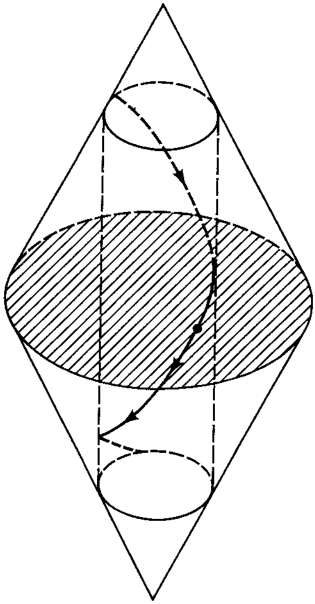

Fig. 2. Orbits on S^3.

- *If C and ∂C are of finite type, then there exists a polynomial $Q(t)$ with*

$$-\min\{P_{\partial C}(t), tP_{(C, \partial C)}(t)\} \leqslant Q(t) \leqslant P_{(C, \partial C)}(t)$$

  such that $P_M(t) - P_{\partial C}(t) = (1+t)\, Q(t)$.
- *If C and ∂C are of finite type, then $\chi(M) = \chi(\partial C)$.*
- *All equilibrium points of the flow must lie in ∂C.*

This theorem is proved in Section 2.

1.4. Applications to the Three-Body Problem

Here we apply the above results on cross sections to the three-body problem. The three-body problem is a system of differential equations describing the motion of three mass points moving in a Newtonian inertial frame under the influence of their mutual gravitational attraction. Let the particles have masses m_1, m_2, m_3, positions u_1, u_2, u_3, and velocities v_1, v_2, v_3, respectively. The masses are positive constants and the positions and velocities are two- or three-dimensional vectors depending on whether we are discussing the planar or the spatial problem.

Written as a system of first-order equations, the equations of motion are

$$\begin{aligned} \dot{u}_i &= v_i \\ m_i \dot{v}_i &= \frac{\partial U}{\partial u_i}, \qquad i = 1, 2, 3 \end{aligned} \tag{1}$$

where the dot represents the derivative with respect to time, $\cdot = d/dt$, U is the self-potential

$$U = \sum_{1 \leqslant i < j \leqslant 3} \frac{G m_i m_j}{\|u_i - u_j\|} \tag{2}$$

and G is the universal gravitational constant. The self-potential is the negative of the potential energy. Since there are six vectors in (1), the phase space of the spatial problem is $\mathbb{R}^{18} \backslash \Delta$, and for the planar problem the phase space is $\mathbb{R}^{12} \backslash \Delta$, where Δ is the collision set $\{(u_1, u_2, u_3, v_1, v_2, v_3) : u_i = u_j, i \neq j\}$. Here and below Δ is the generic symbol for the collision set.

The equations of motion for the spatial problem admit the 10 known integrals. There are six integrals of linear momentum expressing the fact that the center of mass of the system moves with uniform velocity in a straight line. We assume that our original Newtonian reference frame has the center of mass fixed at the origin, so that in this frame the linear momentum integrals are

$$m_1 u_1 + m_2 u_2 + m_3 u_3 = 0 \tag{3}$$

$$m_1 v_1 + m_2 v_2 + m_3 v_3 = 0 \tag{4}$$

Three more integrals are the three components of the angular momentum integral

$$m_1 u_1 \times v_1 + m_2 u_2 \times v_2 + m_3 u_3 \times v_3 = \mathbf{c} \tag{5}$$

Lastly, there is the energy integral,

$$\sum_{i=1}^{3} \tfrac{1}{2} m_i \|v_i\|^2 - U = h \tag{6}$$

Here $\mathbf{c}$ is a constant vector which we assume to be nonzero; see Cabral [5] for a detailed discussion of the case when $\mathbf{c}$ is zero. The reference frame will be taken so that $\mathbf{c} = ck$, where k is the unit vector $(0, 0, 1)^T$ and c is a scalar. Thus the *invariant plane*, the plane perpendicular to the angular momentum vector, is the x–y plane. In the planar case (3) and (4) give two integrals each, and (5) and (6) give one integral each, for a total of six integrals.

The first algebraic sets we consider are

$$\mathfrak{M}(c, h) = \{(u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^{18} \backslash \Delta : (3), (4), (5), (6)\} \tag{7}$$

and its planar analogue

$$\mathfrak{m}(c, h) = \{(u_1, u_2, u_3, v_1, v_2, v_3) \in \mathbb{R}^{12} \backslash \Delta : (3), (4), (5), (6)\} \tag{8}$$

These are called the *integral manifolds*. The integral manifolds admit the SO_2 symmetry of rotation about the z-axis since this action leaves the integrals fixed. Thus we also study the sets

$$\mathfrak{M}_R(c, h) = \mathfrak{M}(c, h)/SO_2$$

$$\mathfrak{m}_R(c, h) = \mathfrak{m}(c, h)/SO_2$$

We call these sets *reduced integral manifolds*. We note that $\mathfrak{M}(c, h)$ is 8-dimensional, $\mathfrak{M}_R(c, h)$ is 7-dimensional, $\mathfrak{m}(c, h)$ is 6-dimensional, and $\mathfrak{m}_R(c, h)$ is 5-dimensional.

In a natural way the three-body problem defines a flow on these reduced manifolds where all the integrals and symmetries have been eliminated. It is on these manifolds that the three-body problem should be studied. Indeed, Birkhoff [2, p. 287] said, "The manifold M_7 [$\mathfrak{M}_R(c, h)$] has fundamental importance for the problem of three bodies...."

There has been considerable research into the nature of these manifolds and the corresponding Hill's regions; see the works by Albouy [1], Cabral [5], Chen [6], Easton [8, 9], Iacob [14], McCord *et al.* [17], Saari [22–24], Simo [25], and Smale [26, 27]. A detailed discussion of the history can be found by McCord *et al.* [17], with further references.

For fixed masses, the quantity $v = -c^2 h$ is the sole bifurcation parameter and there are nine special values of this parameter, $0 = v_1 < v_2 \leqslant v_3 \leqslant v_4 < v_5 < v_6 < v_7 \leqslant v_8 \leqslant v_9$. For the spatial problem eight of these values give rise to bifurcation of the topological type of these manifolds. (One, v_5, is an artifact of the method of analysis but does not give rise to a new topological bifurcation. We keep this numeration to be consistent with our previous work.) For the planar problem five of these values give rise to bifurcations of these manifolds, namely, v_1, v_6, v_7, v_8, v_9. Thus for the

Table I. Cohomology Groups for $\mathfrak{M}_R(c, h)$

$H^p(\mathfrak{M}_R)$	0	1	2	3	4	5	6	7	$\chi(\mathfrak{M}_R)$
I	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	0	0	0	0	5
II	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}^5$	0	$\mathbb{Z}^2$	0	12
III	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}^3$	0	$\mathbb{Z}^2$	0	10
IV	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	8
V	$\mathbb{Z}$	0	$\mathbb{Z}^4$	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	6
VII	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	0	0	0	$\mathbb{Z}^3$	0	6
VIII	$\mathbb{Z}$	0	$\mathbb{Z}^3$	0	0	0	$\mathbb{Z}^2$	0	6
IX	$\mathbb{Z}^2$	0	$\mathbb{Z}^3$	0	0	0	$\mathbb{Z}$	0	6
X	$\mathbb{Z}^3$	0	$\mathbb{Z}^3$	0	0	0	0	0	6

spatial problem there are nine parameter ranges, which we denote by Roman numerals. Specifically,

$$\mathrm{I} = (-\infty, \nu_1), \qquad \mathrm{II} = (\nu_1, \nu_2), \qquad \mathrm{III} = (\nu_2, \nu_3)$$
$$\mathrm{IV} = (\nu_3, \nu_4), \qquad \mathrm{V} = (\nu_4, \nu_6), \qquad \mathrm{VII} = (\nu_6, \nu_7)$$
$$\mathrm{VIII} = (\nu_7, \nu_8), \qquad \mathrm{IX} = (\nu_8, \nu_9), \qquad \mathrm{X} = (\nu_9, \infty)$$

Note that by McCord *et al.* [17] the value ν_5 divided the region V into two regions, V and VI. For the planar problem there are six parameter ranges:

$$\mathrm{i} = (-\infty, \nu_1), \qquad \mathrm{ii} = (\nu_1, \nu_6), \qquad \mathrm{vii} = (\nu_6, \nu_7)$$
$$\mathrm{viii} = (\nu_7, \nu_8), \qquad \mathrm{ix} = (\nu_8, \nu_9), \qquad \mathrm{x} = (\nu_9, \infty)$$

We compute the cohomology of these manifolds by McCord *et al.* [17] for the spatial problem and in Section 4 for the planar problem. For the discussion of the cross sections, the essential information is given in Tables I and II. Note that the homology for Case I is computed incorrectly by

Table II. Cohomology Groups for $\mathfrak{m}_R(c, h)$

$H^p(\mathfrak{m}_R)$	0	1	2	3	4	5	$\chi(\mathfrak{m}_R)$
i	$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	0
ii	$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	0
vii	$\mathbb{Z}$	$\mathbb{Z}^4$	0	0	$\mathbb{Z}^3$	0	0
viii	$\mathbb{Z}$	$\mathbb{Z}^3$	0	0	$\mathbb{Z}^2$	0	0
ix	$\mathbb{Z}^2$	$\mathbb{Z}^3$	0	0	$\mathbb{Z}$	0	0
x	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	0	0	0

McCord *et al.* [17]. The correct values are given in Table I, with the computations given in the Appendix.

From Theorem 1.1 and Tables I and II, we have:

Theorem 1.3. *The flow of the spatial three-body problem on $\mathfrak{M}_R(c, h)$ does not admit a global cross section of finite type. In cases i–ix, the flow of the planar three-body problem on $\mathfrak{m}_R(c, h)$ does not admit a global cross section of finite type.*

Proof. For the spatial problem we see in Table I that, in all parameter ranges, $\chi(\mathfrak{M}_R(c, h)) \neq 0$. Thus Theorem 1.1 shows that there does not exist a global cross section of finite type.

For the planar problem, we require a more subtle argument, since in all parameter ranges $H_1(\mathfrak{m}_R(c, h))$ has a factor of $\mathbb{Z}$ and $\chi(\mathfrak{m}_R(c, h)) = 0$. Instead, we make use of Morse inequalities. From Table II, the Poincaré polynomial for $\mathfrak{m}_R(c, h)$ is

Case	$P_{\mathfrak{m}}(t)$	$(1+t)\,Q(t)$
i	$1+2t+t^3+2t^4$	$(1+t)(1+t-t^2+2t^3)$
ii	$1+2t+t^3+2t^4$	$(1+t)(1+t-t^2+2t^3)$
vii	$1+4t+3t^4$	$(1+t)(1+3t-3t^2+3t^3)$
viii	$1+3t+2t^4$	$(1+t)(1+2t-2t^2+2t^3)$
ix	$2+3t+t^4$	$(1+t)(2+t-t^2+t^3)$
x	$3+3t$	$(1+t)\,3$

In all cases but case x, the polynomial $Q(t)$ has a negative coefficient, in contradiction to Theorem 1.1. □

The possibility remains that a cross section of finite type might exist in the planar manifold in case x. We will see in Proposition 4.2 that, for v in region x, each component of $\mathfrak{m}_R(c, h)$ is homeomorphic to the product of a 4-disk and S^1. Certainly, there are no topological obstacles to the existence of a cross section. But does the flow on this five-dimensional cylinder admit a cross section? At the moment, this is an open question.

At the end of Birkhoff's [2, p. 288] discussion of these integral manifolds, he said

> In conclusion it may be observed that the states of motion in which the three bodies move constantly in a plane through the center of gravity perpendicular to the angular momentum vector, corresponds to an invariant sub-manifold M_5 [$\mathfrak{m}_R(c, h)$] within M_7 [$\mathfrak{M}_R(c, h)$],.... So far as dimensionality is concerned, this manifold M_5 would be suited to form the complete boundary of a surface of section (Chap. 5) of the properly extended type.

In other words, Birkhoff asks if $\mathfrak{m}_R(c, h)$ is the boundary of a cross section of $\mathfrak{M}_R(c, h)$ since it is an invariant submanifold of codimension 2. We answer Birkhoff in the negative.

Theorem 1.4. *The invariant manifold $\mathfrak{m}_R(c,h)$ is not the boundary of a cross section of finite type of the flow of the three-body problem on $\mathfrak{M}_R(c,h)$.*

Proof. Refer to Tables I and II and note that, in all parameter ranges, $\chi(\mathfrak{M}_R(c,h)) \neq \chi(\mathfrak{m}_R(c,h))$. Thus Theorem 1.2 implies that $\mathfrak{m}_R(c,h)$ is not the boundary of a cross section of finite type. $\square$

2. THE TOPOLOGY OF FLOWS WITH CROSS SECTIONS

This section is devoted to the proof of Theorems 1.1 and 1.2. Theorem 1.1 is simply the special case $\partial C = \emptyset$, so it suffices to prove Theorem 1.2. Further, the bundle structure on $M \backslash \partial C$ and the corresponding absence of equilibria are plain, so we need only to verify the homological statements. The proof is essentially the same as that by McCord *et al.* [16].

Both homological statements are derived from a Mayer–Vietoris decomposition of the space. Let

$$U_0 = \left\{ x \cdot t \in M \;\middle|\; x \in C, 0 \leqslant t \leqslant \frac{\tau(x)}{2} \right\}$$

$$U_1 = \left\{ x \cdot t \in M \;\middle|\; x \in C, -\frac{\tau(x)}{2} \leqslant t \leqslant 0 \right\}$$

Let $C' = \{ \Phi(\tau(x)/2, x) \mid x \in C \}$. Then $U_0 \cup U_1 = M$ and $U_0 \cap U_1 = C \cup_{\partial C} C'$. Clearly, flowing forward by $\tau/2$ defines homeomorphisms $C' \cong C$ and $U_0 \cong U_1$. Moreover, using the flow lines, we have strong deformation retractions from both U_0 and U_1 onto C, and onto C'. That is, we can flow everything in U_0 forward to C' and backward to C; we can flow everything in U_1 backward to C' and forward to C.

We then have a Mayer–Vietoris sequence for the pair $(M, \partial C)$

$$\to H_k(C \cup_{\partial C} C', \partial C) \xrightarrow{\left[\begin{smallmatrix} \iota_{0*} \\ \iota_{1*} \end{smallmatrix}\right]} H_k(U_0, \partial C) \oplus H_k(U_1, \partial C) \xrightarrow{[j_{0*} \;\; -j_{1*}]} H_k(M, \partial C) \to$$

Further, if we take the obvious Mayer–Vietoris decomposition of $(C \cup_{\partial C} C', \partial C)$, we see that there is an isomorphism,

$$H_k(C, \partial C) \oplus H_k(C', \partial C) \to H_k(C \cup_{\partial C} C', \partial C)$$

If we let $\iota_i \colon C \to U_i$ and $\iota_i' \colon C' \to U_i$ denote the inclusions, then the sequence for $(M, \partial C)$ can be rewritten

$$\xrightarrow{\left[\begin{smallmatrix} \partial \\ \partial' \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C', \partial C) \xrightarrow{\left[\begin{smallmatrix} \iota_{0*} & \iota_{0*}' \\ \iota_{1*} & \iota_{1*}' \end{smallmatrix}\right]} H_k(U_0, \partial C) \oplus H_k(U_1, \partial C)$$

$$\xrightarrow{[j_{0*} \;\; -j_{1*}]} H_k(M, \partial C) \to$$

The maps ι_{i*} and ι'_{i*} are all isomorphisms. We can use ι_{i*} to identify $H_k(U_i, \partial C)$ with $H_k(C, \partial C)$, and $\iota'^{-1}_{0*}\iota_{0*}$ to identify $H_k(C', \partial C)$ with $H_k(C, \partial C)$. With these identifications, the homomorphism

$$H_k(C, \partial C) \oplus H_k(C', \partial C) \xrightarrow{\left[\begin{smallmatrix} \iota_{0*} & \iota'_{0*} \\ \iota_{1*} & \iota'_{1*} \end{smallmatrix}\right]} H_k(U_0, \partial C) \oplus H_k(U_1, \partial C)$$

becomes

$$H_k(C, \partial C) \oplus H_k(C, \partial C) \xrightarrow{\left[\begin{smallmatrix} id & id \\ id & \iota_{1*}^{-1}\iota'_{1*}\iota'^{-1}_{0*}\iota_{0*} \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C)$$

But the compositions $\iota_0'^{-1}\iota_0$ and $\iota_1^{-1}\iota_1'$ are both simply flowing forward by time $\tau/2$, so the composition $\iota_1^{-1}\iota_1'\iota_0'^{-1}\iota_0$ is the Poincaré map $P\colon C \to C$. That is, the matrix is

$$\begin{bmatrix} id & id \\ id & P_* \end{bmatrix}$$

and the Mayer–Vietoris sequence is

$$\xrightarrow{\left[\begin{smallmatrix} \partial \\ \partial \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C) \xrightarrow{\left[\begin{smallmatrix} id & id \\ id & P_* \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C)$$

$$\xrightarrow{[i_* \quad -i_*]} H_k(M, \partial C) \to$$

The maps $\left[\begin{smallmatrix} \partial \\ \partial \end{smallmatrix}\right]$ and $[i_* \quad -i_*]$ can be factored as

$$H_{k+1}(M, \partial C) \xrightarrow{\partial} H_k(C, \partial C) \xrightarrow{\left[\begin{smallmatrix} id \\ id \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C)$$

and

$$H_k(C, \partial C) \oplus H_k(C, \partial C) \xrightarrow{[id \quad -id]} H_k(C, \partial C) \xrightarrow{i_*} H_k(M, \partial C)$$

Now, the composition

$$H_k(C, \partial C) \xrightarrow{\left[\begin{smallmatrix} id \\ id \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C) \xrightarrow{\left[\begin{smallmatrix} id & id \\ id & P_* \end{smallmatrix}\right]} H_k(C, \partial C) \oplus H_k(C, \partial C)$$

$$\xrightarrow{[id \quad -id]} H_k(C, \partial C) \to$$

reduces to $H_k(C, \partial C) \xrightarrow{id - P_*} H_k(C, \partial C)$, so the Mayer–Vietoris sequence can be rewritten as

$$\xrightarrow{\partial} H_k(C, \partial C) \xrightarrow{id - P_*} H_k(C, \partial C) \xrightarrow{i_*} H_k(M, \partial C) \xrightarrow{\partial}$$

It is a routine algebraic exercise to verify that this sequence is exact.

Given any long exact sequence

$$\xrightarrow{\Delta_n} A_n \to B_n \to C_n \xrightarrow{\Delta_{n-1}}$$

with the groups $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ finitely generated, the Poincaré polynomials satisfy the relation

$$P_A(t) + P_C(t) = P_B(t) + (1+t)\, R(t)$$

where the nth coefficient of $R(t)$ is the rank of Δ_n. In particular, $0 \leqslant R(t) \leqslant \min\{tP_A(t), P_C(t)\}$. Applying this first to the exact sequence

$$H_k(C, \partial C) \xrightarrow{id - P_*} H_k(C, \partial C) \to H_k(M, \partial C) \to H_{k-1}(C, \partial C)$$
$$\xrightarrow{id - P_*} H_{k-1}(C, \partial C)$$

we have

$$(1+t)\, P_{(C, \partial C)}(t) = P_{(C, \partial C)}(t) + (1+t)\, R(t)$$

where $R_2(t)$ is the polynomial whose nth coefficient is the rank of $id - P_{n*}$. This reduces to

$$P_{(M, \partial C)}(t) = (1+t)\, K(t)$$

where $R_2(t)$ is the polynomial whose nth coefficient is the nullity of $id - P_{n*}$. In particular, $0 \leqslant K(t) \leqslant P_{(C, \partial C)}(t)$.

Applying the polynomial formula next to the exact sequence of the pair $(M, \partial C)$, we have

$$P_{(M, \partial C)}(t) + P_{\partial C}(t) = P_M(t) + (1+t)\, R'(t)$$

with $0 \leqslant R'(t) \leqslant \min\{tP_{\partial C}(t), P_{(M, \partial C)}(t)\}$. Combining these, we have

$$P_M(t) = P_{\partial C}(t) + (1+t)(K(t) - R'(t))$$

If $Q(t) = K(t) - R'(t)$, then

$$K(t) - \min\{tP_{\partial C}(t), (1+t)\, K(t)\} \leqslant Q(t) \leqslant K(t)$$

which can be rewritten

$$-\min\{tP_{\partial C}(t), tP_{(C,\partial C)}(t)\} \leqslant Q(t) \leqslant P_{(C,\partial C)}(t)$$

The Euler characteristic formula follows at once from the Poincaré polynomial formula.

3. DECOMPOSING THE INTEGRAL MANIFOLDS

For completeness we describe the integral manifolds, the Hill's regions, and the configuration space for the planar problem just as we did for the spacial problem. Our development follows the lines suggested by Easton and is included because it represents a simplification of some of the computations by McCord *et al.* [17]. Most of what is done in this section can be gleaned from the works by Chen [6], Easton [8, 9], Iacob [14], Saari [22–24], and Smale [26, 27]. This description is used in Section 4 to compute the tables of cohomology groups.

In this section the vectors u_i, v_i, etc., are in $\mathbb{R}^2$. Write the angular momentum integral

$$m_1 u_1^T J v_1 + m_2 u_2^T J v_2 + m_3 u_3^T J v_3 = c, \qquad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

Also, define the moment of inertia as

$$I = \tfrac{1}{2}(m_1 \|u_1\|^2 + m_2 \|u_2\|^2 + m_3 \|u_3\|^2) \tag{10}$$

The Hill's region is the projection of the integral manifold onto configuration space, e.g.,

$$\begin{aligned} \mathfrak{h}(c, h) &= \{(u_1, u_2, u_3) : \exists v_1, v_2, v_3 \text{ s.t. } (u_1, u_2, u_3, v_1, v_2, v_3) \in \mathfrak{m}(c, h)\} \\ \mathfrak{h}_R(c, h) &= \mathfrak{h}(c, h)/SO_2 \end{aligned} \tag{11}$$

This characterization of the Hill's region as a projection of the integral manifold is useful for relating the two spaces but not for understanding the structure of the Hill's region itself. For that, we make use of a direct formulation of the Hill's region.

Proposition 3.1. *The Hill's region, $\mathfrak{h}(c, h)$, is the set of $u=(u_1, u_2, u_3)$ such that*

$$U(u)+h \geqslant \frac{c^2}{4I(u)} \tag{12}$$

and $\partial\mathfrak{h}(c, h)$, the boundary of $\mathfrak{h}(c, h)$, is the set on which equality holds in (12).

Also, $\mathfrak{m}(c, h)$ is a singular fiber bundle over $\mathfrak{h}(c, h)$ where the fiber is a 2-sphere over int $\mathfrak{h}(c, h)$ and a point over $\partial\mathfrak{h}(c, h)$.

Proof. The energy relation is $K=U+h$, where $K=\frac{1}{2}\sum m_i \|v_i\|^2$ is the kinetic energy. A level sets of K is a 5-dimensional ellipsoid. Since the linear momentum constraint (4) defines two planes through the origin in v-space, the intersection of such a level set and (4) is a 3-dimensional ellipsoid.

Fix u and use Lagrange multipliers to find the minimum of K on the constraint (9) to be $c^2/4I$. Thus the 3-ellipsoid and the plane (9) intersect if (12) hold. The intersection is a point if equality holds and otherwise it is a 2-dimensional ellipsoid (so topologically a 2-sphere). □

The next step in the decomposition of the manifolds is to scale the distances by projecting onto the ellipsoid with $I=1$. Let

$$\begin{aligned} \mathfrak{k}(c, h) &= \{u \in \mathfrak{h}(c, h)\} : I(u)=1\} \\ \mathfrak{k}_R(c, h) &= \mathfrak{k}(c, h)/SO_2 \end{aligned} \tag{13}$$

Define a projection

$$\Omega : \mathfrak{h}(c, h) \to \mathfrak{k}(c, h) : (u_1, u_2, u_3) \to \left(\frac{u_1}{\rho}, \frac{u_2}{\rho}, \frac{u_3}{\rho}\right), \qquad \text{where} \quad \rho=\sqrt{I(u)} \tag{14}$$

Proposition 3.2. *The set $\mathfrak{k}(c, h)$ is the set of u with $I(u)=1$ and such that*

$$U(u) \geqslant v \tag{15}$$

and the boundary of $\mathfrak{k}(c, h)$ is where equality holds in (15).

The space $\mathfrak{h}(c, h)$ a singular line bundle over $\mathfrak{k}(c, h)$ with the line collapsing to a point over $\partial\mathfrak{k}(c, h)$. Over int $\mathfrak{k}(c, h)$ the fiber is a closed interval when $v>0$ $(h<0)$ and a half-closed interval when $v<0$ $(h>0)$.

Remark. In comparing (15) with the similar inequality in [8, 17], a factor of 2 seems to be missing, but it can be found in (10).

Proof. Let $u=(u_1, u_2, u_3)\in \mathfrak{h}(c, h)$, $\rho=\sqrt{I(u)}$, and $q=(q_1, q_2, q_3)=\rho^{-1}(u_1, u_2, u_3)$. By Proposition 3.1,

$$U(u)+h\geqslant\frac{c^2}{4I(u)}$$

$$\frac{U(q)}{\rho}+h\geqslant\frac{c^2}{4\rho^2}$$

$$h\rho^2+U(q)\,\rho-\frac{1}{4}c^2\geqslant 0$$

The last inequality is a quadratic in ρ which always has a solution of the form $[a, +\infty)$ for $h>0$. For $h<0$ this last inequality has a solution if the discriminant is positive, i.e.,

$$U(q)^2+hc^2\geqslant 0 \qquad \text{or} \qquad U(q)^2\geqslant v$$

In this case, the solution is a closed interval if the strict inequality holds and a single point if equality holds. □

An element of $\mathfrak{k}_R(c, h)$ is a scaled triangle. It is routine to verify that, if a scaled triangle $p\in\mathfrak{k}_R(c, h)$, then so is its mirror image. It is convenient to identify these triangles which differ from one another by a reflection. Let $\mathfrak{c}(c, h)$ denote the resulting quotient space, and $\psi\colon \mathfrak{k}_R(c, h)\to\mathfrak{c}(c, h)$ be the quotient map. The quotient map ψ is one-to-one on the collinear configurations and two-to-one on the rest of $\mathfrak{k}_R(c, h)$. Clearly, $\partial\mathfrak{k}_R(c, h)=\psi^{-1}(\partial\mathfrak{c}(c, h))$.

To describe $\mathfrak{m}_R(c, h)$ and $\mathfrak{h}_R(c, h)$, the only step remaining is to determine the structure of $\mathfrak{c}(c, h)$ as a function of v. Thus we must study the function U on the set $I=1$. Following Lagrange and Easton, we use the mutual distances as coordinates. Let

$$s_1=\|u_2-u_3\|, \qquad s_2=\|u_3-u_1\|, \qquad s_3=\|u_1-u_2\|$$

so that

$$\begin{aligned} U&=G\left(\frac{m_1m_2}{s_3}+\frac{m_2m_3}{s_1}+\frac{m_3m_1}{s_2}\right)\\ I&=\frac{1}{2M}(m_1m_2s_3^2+m_2m_3s_1^2+m_3m_1s_2^2)\end{aligned} \tag{16}$$

where $M = m_1 + m_2 + m_3$. Since s_1, s_2, s_3 are the sides of a triangle or a collinear configuration, they must satisfy

$$s_1 > 0, \quad s_2 > 0, \quad s_3 > 0, \quad s_1 + s_2 \geqslant s_3, \quad s_2 + s_3 \geqslant s_1, \quad s_3 + s_1 \geqslant s_2 \tag{17}$$

Elements of $\mathfrak{c}(c, h)$ are triangles with unit moment of inertia with the two orientations identified, thus can be coordinatized by points in

$$D = \{(s_1, s_2, s_3) : I(s) = 1 \text{ and } (17)\}$$

Thus the domain D of $U(s)$ is a "spherical" triangle on the ellipsoid $I = 1$ in the first octant as illustrated in Fig. 3. Since these coordinates do not betray the orientation of the triangle, they can be considered as coordinates for $\mathfrak{c}(c, h)$. That is $\mathfrak{c}(c, h) \subset D$.

Both I and U are positive convex functions—their Hesians are positive definite. Using Lagrange multipliers we find that the unique minimum of U^2 is

$$v_6 = \frac{G^2}{2} \frac{(m_1 m_2 + m_2 m_3 + m_3 m_1)^3}{(m_1 + m_2 + m_3)} \tag{18}$$

which occurs at the equilateral triangle configuration where

$$s_1^2 = s_2^2 = s_3^2 = 2(m_1 + m_2 + m_3)/(m_1 m_2 + m_2 m_3 + m_3 m_1)$$

Thus, $\mathfrak{c}(c, h) \equiv D$ in the range $v < v_6$. As v increases beyond v_6, the set of points not satisfying (15) is a monotonically increasing domain in D

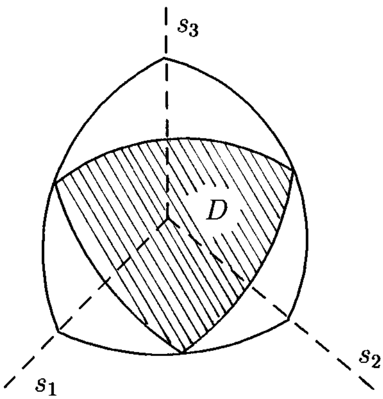

Fig. 3 The domain of $U(s)$.

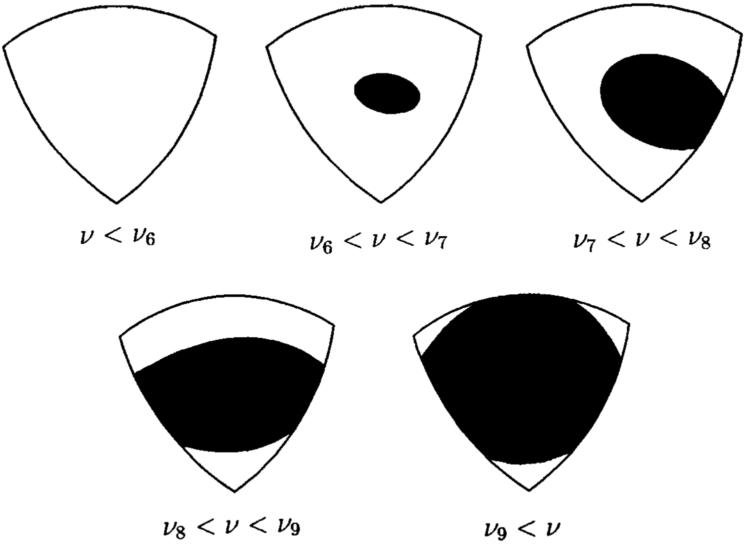

Fig. 4. The Admissible region.

growing out from the equilateral configuration. See Fig. 4, where the set of nonadmissible configurations is shown as the black region. The non-admissible region grows as v increases until it hits one boundary of D (a collinear configuration) when $v = v_7$, then the next boundary at $v = v_8$, and then the last boundary at $v = v_9$.

The values v_7, v_8, v_9 are the minima of $U(s)$ on the three boundary curves of D. Using Lagrange multipliers, we find

$$v_7 = \frac{G^2}{4} \frac{(m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2)}{(m_1 + m_2 + m_3)} \left\{ \frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} \right\}^2$$

where r_{ij} is the distance between the ith and the jth particles in the Euler collinear central configuration. v_8 and v_9 are defined by the same formula as v_7 with a permutation of the masses and the corresponding r_{ij}.

Proposition 3.3. *The admissible regions,* $\mathfrak{c}(c, h)$, *are show in white in Fig. 4 for the various ranges of* v.

Assembling all of this, we have the complete diagram of spaces and maps.

$$
\begin{tikzcd}
\mathfrak{m} \arrow[rr, "\Phi_m"] \arrow[d, "\Pi"'] & & \mathfrak{m}_R \arrow[d, "\pi"] \\
\mathfrak{h} \arrow[rr, "\Phi_h"] \arrow[d, "\Omega"'] & & \mathfrak{h}_R \arrow[d, "\omega"] \\
\mathfrak{k} \arrow[rr, "\Phi_k"] \arrow[dr, "\Psi"'] & & \mathfrak{k}_R \arrow[dl, "\psi"] \\
& \mathfrak{c} &
\end{tikzcd}
$$

The dimensions of the various spaces are $\dim \mathfrak{m}(c, h) = 6$, $\dim \mathfrak{m}_R(c, h) = 5$, $\dim \mathfrak{h}(c, h) = 4$, $\dim \mathfrak{h}_R(c, h) = 3$, $\dim \mathfrak{k}(c, h) = 3$, and $\dim \mathfrak{k}_R(c, h) = \dim \mathfrak{c}(c, h) = 2$. The information in Propositions 3.1 and 3.2 can be collated to give the following description of $\mathfrak{m}_R(c, h) \xrightarrow{\omega\pi} \mathfrak{k}_R(c, h)$.

Corollary 3.1. *The fiber $\pi^{-1}\omega^{-1}(k)$ in $\mathfrak{m}_R(c, h)$ over $k \in \mathfrak{k}_R(c, h)$ is*

$$
\pi^{-1}\omega^{-1}(k) \cong \begin{cases} S^3, & v > 0, \quad k \in \operatorname{int} \mathfrak{k}_R(c, h) \\ *, & v > 0, \quad k \in \partial \mathfrak{k}_R(c, h) \\ B^3, & v < 0, \quad k \in \operatorname{int} \mathfrak{k}_R(c, h) \end{cases}
$$

This, combined with the information about $\mathfrak{k}_R(c, h)$ and $\mathfrak{c}(c, h)$ encoded in Proposition 3.3 and Fig. 4, will enable us to establish the homotopy types of $\mathfrak{m}(c, h)$ and $\mathfrak{m}_R(c, h)$, and so determine their homology.

4. THE HOMOLOGY OF THE INTEGRAL MANIFOLDS

With this analysis in hand, it is now a relatively simple matter to determine the homotopy types of the Hill's regions and integral manifolds. The first step is to observe that each of these open manifolds has a strong deformation retraction onto a lower-dimensional compact subcomplex.

Proposition 4.1. *For each range of v, there is a 1-complex $\mathfrak{l}(c, h) \subset \mathfrak{k}_R(c, h)$ such that $\mathfrak{k}_R(c, h)$ has a strong deformation retraction onto $\mathfrak{l}(c, h)$ and $\mathfrak{m}_R(c, h)$ has a strong deformation retraction onto $\pi^{-1}\omega^{-1}(\mathfrak{l}(c, h))$. The sets $\mathfrak{l}(c, h)$ are shown in Fig. 5.*

Proof. First, it is clear that each $\mathfrak{k}_R(c, h)$ has a strong deformation retraction $\rho\colon \mathfrak{k}_R(c, h) \times [0, 1] \to \mathfrak{k}_R(c, h)$ onto $\mathfrak{l}_R(c, h)$. The only restriction to be observed in constructing ρ is to require

$$
\partial \mathfrak{k}_R(c, h) \times [0, 1] \subset \rho^{-1}(\partial \mathfrak{k}_R(c, h)) \subset (\partial \mathfrak{k}_R(c, h) \times [0, 1])) \cup (\mathfrak{k}_R(c, h) \times \{1\})
$$

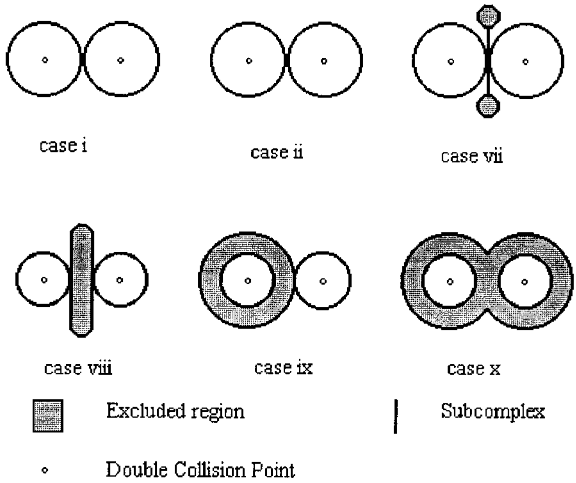

Fig. 5. The 1-complex $\mathfrak{l}(c, h)$.

This lifts to a strong deformation retraction $\tilde{\rho}$: $\mathfrak{m}_R(c, h) \times [0, 1] \to \mathfrak{m}_R(c, h)$ with $\tilde{\rho}_1(\mathfrak{m}_R(c, h)) \subset \pi^{-1}\omega^{-1}(\mathfrak{l}(c, h))$. To see that such a lift exists, let $A = \rho^{-1}(\operatorname{int} \mathfrak{k}_R(c, h))$. By our choice of ρ, A is dense in $\mathfrak{k}_R(c, h) \times [0, 1]$, and over both B and $\rho(A)$, the projection $\omega \circ \pi$ is a fibration. The homotopy lifting condition guarantees the existence of a lift $\tilde{\rho}$ over A. On the complement of A, ρ maps into $\partial \mathfrak{k}_R(c, h)$, and $\omega \circ \pi$ is one-to-one there. Thus, the unique continuous extension to all of $\mathfrak{k}_R(c, h) \times [0, 1]$ is defined by setting

$$\tilde{\rho} = (\omega \circ \pi)^{-1} \circ \rho \circ \omega \circ \pi$$

on the complement of A. □

With this, we can now determine the homotopy types of the integral manifolds, Hill's regions and reduced spaces. The homological values of Table II follow immediately.

Theorem 4.1. *The homtopy types of* $\mathfrak{m}(c, h)$, $\mathfrak{m}_R(c, h)$, $\mathfrak{h}(c, h)$ *and* $\mathfrak{h}_R(c, h)$ *are given in Table III.*

Proof. Combining Propositions 3.2 and retraction, it is clear that $\mathfrak{h}_R(c, h)$ is homotopic to the wedge of circles $\mathfrak{l}(c, h)$. Similarly, the homotopy type of $\mathfrak{m}_R(c, h)$ is obtained by combining the results of Corollary 3.1 and Proposition 4.1. In all cases, $\mathfrak{l}_R(c, h)$ consists of arcs that either lie entirely in $\partial \mathfrak{k}_R(c, h)$, entirely in $\operatorname{int} \mathfrak{k}_R(c, h)$, or lie in $\operatorname{int} \mathfrak{k}_R(c, h)$ with end points in

Table III. Homtopy Types of $\mathfrak{m}(c,h)$, $\mathfrak{m}_R(c,h)$, $\mathfrak{h}(c,h)$, and $\mathfrak{h}_R(c,h)$

Case	$\mathfrak{m}(c,h)$	$\mathfrak{m}_R(c,h)$	$\mathfrak{h}(c,h)$	$\mathfrak{h}_R(c,h)$
i	$S^1 \times S^3 \times (S^1 \vee S^1)$	$S^3 \times (S^1 \vee S^1)$	$S^1 \times (S^1 \vee S^1)$	$S^1 \vee S^1$
ii	$S^1 \times S^3 \times (S^1 \vee S^1)$	$S^3 \times (S^1 \vee S^1)$	$S^1 \times (S^1 \vee S^1)$	$S^1 \vee S^1$
vii	$S^1 \times \left(\left(\bigvee_3 S^4\right) \vee \left(\bigvee_4 S^1\right)\right)$	$\left(\bigvee_3 S^4\right) \vee \left(\bigvee_4 S^1\right)$	$S^1 \times \left(\bigvee_4 S^1\right)$	$\bigvee_4 S^1$
viii	$S^1 \times \left(\left(\bigvee_2 S^4\right) \vee \left(\bigvee_3 S^1\right)\right)$	$\left(\bigvee_2 S^4\right) \vee \left(\bigvee_3 S^1\right)$	$S^1 \times \left(\bigvee_3 S^1\right)$	$\bigvee_3 S^1$
ix	$S^1 \times ((S^4 \vee S^1) \sqcup S^1)$	$(S^4 \vee S^1) \sqcup S^1$	$S^1 \times ((S^1 \vee S^1) \sqcup S^1)$	$(S^1 \vee S^1) \sqcup S^1$
x	$S^1 \times (S^1 \sqcup S^1 \sqcup S^1)$	$S^1 \sqcup S^1 \sqcup S^1$	$S^1 \times (S^1 \sqcup S^1 \sqcup S^1)$	$S^1 \sqcup S^1 \sqcup S^1$

$\partial \mathfrak{k}_R(c, h)$. In the first case, the fiber in $\mathfrak{m}_R(c, h)$ is itself an arc; in the second case, it is $S^3 \times I$; and in the third case, it is S^4 (i.e., $S^3 \times I$, with the ends collapsed to points).

In cases i and ii, the 1-complex $\mathfrak{l}_R(c, h)$ lies entirely in int $\mathfrak{k}_R(c, h)$, so its preimage in $\mathfrak{m}_R(c, h)$ is simply $S^3 \times \mathfrak{l}_R(c, h)$. In case x, $\mathfrak{l}_R(c, h)$ lies entirely in $\partial \mathfrak{k}_R(c, h)$, and so is homeomorphic to its preimage. In the remaining cases, the arcs in $\mathfrak{l}_R(c, h)$ with end points in $\partial \mathfrak{k}_R(c, h)$ effectively attach a 4-sphere at two points to $\partial \mathfrak{k}_R(c, h)$. This is homotopic to attaching the wedge product of an arc and a 4-sphere.

The homotopy types of $\mathfrak{h}(c, h)$ and $\mathfrak{m}(c, h)$ are simply the products of S^1 with $\mathfrak{h}(c, h)$ and $\mathfrak{m}(c, h)$, respectively. A homological calculation suffices to justify this. Namely, since $H^2(\mathfrak{m}_R(c, h)) = H^2(\mathfrak{h}_R(c, h)) = 0$ for all v, the Thom classes of all of the S^1-bundles

$$S^1 \to \mathfrak{m}(c, h) \to \mathfrak{m}_R(c, h)$$

$$S^1 \to \mathfrak{h}(c, h) \to \mathfrak{h}_R(c, h)$$

must be trivial. Since the Thom class determines the homotopy type of the total space, $\mathfrak{m}(c, h)$ and $\mathfrak{h}(c, h)$ must have the homotopy types of the trivial (i.e., product) bundles. $\square$

As a final note, we look more carefully at the reduced integral manifold in case x. In this case, $\mathfrak{k}_R(c, h)$ consists of three half-open annuli. For each half-open interval, the end point lies in $\partial \mathfrak{k}_R(c, h)$ (and hence has a single point as its preimage in $\mathfrak{m}_R(c, h)$), while all other points have S^3 as their preimage. The preimage of the entire interval is the cone on S^3, or D^4. Thus we have:

Proposition 4.2. *For $v > v_9$, $\mathfrak{m}_R(c, h)$ is homeomorphic to three disjoint copies of $D^4 \times S^1$.*

APPENDIX. THE SPATIAL THREE-BODY PROBLEM FOR POSITIVE ENERGY

In this appendix, we correct the erroneous computation of the homology of $H_*(\mathfrak{M}_R(c, h))$ for $v < 0$, that was presented by McCord *et al.* [17]. We use the notation and approach of McCord *et al.* [17].

There, it was shown that the integral manifold $\mathfrak{M}_R(c, h)$ could be understood through a series of projections

$$\mathfrak{M}_R(c, h) \xrightarrow{\pi} \mathfrak{H}_R(c, h) \xrightarrow{\omega} \mathfrak{K}_R(c, h) \xrightarrow{\psi} \mathfrak{C}(c, h)$$

For positive energy, the set $\mathfrak{C}(c, h)$ is a triangle with the three vertices deleted, while $\psi^{-1}(c)$ is a 2-sphere for c in the interior of the triangle and a point for c on one of the three boundary lines. Over each point $k \in \mathfrak{K}$, the fiber $\pi^{-1} \circ \omega^{-1}(k)$ is contractible.

The integral manifold $\mathfrak{M}_R(c, h)$ is thus homotopic to $\mathfrak{K}_R(c, h)$. This is in turn homotopic to a singular fiber bundle over a Y (i.e., a 1-complex with three edges all meeting at a single common vertex), with the fiber over each of the three end points a point and the fiber over all other points a 2-sphere. The problem is to identify correctly the limiting behavior of the S^2 fibers as points in the interior of the Y approach the boundary.

It was this step that was incorrectly described by McCord *et al.* [17]. A point in $\mathfrak{C}$ records the shape of the triangle formed by the three masses, while its preimage in $\mathfrak{K}_R$ describes the orientation of the triangle relative to the angular momentum vector. The points in $\partial\mathfrak{C}$ are the collinear configurations. In the presence of nonzero angular momentum, collinear configurations must lie in the invariant plane orthogonal to the angular momentum vector. Thus the non-collinear configurations that limit onto collinear must also lie in or asymptotically approach the invariant plane. That is, as points $c \in \operatorname{int}(\mathfrak{C})$ approach $c_0 \in \partial\mathfrak{C}$, it is not the entire 2-sphere $\psi^{-1}(c)$ that limits onto the single point $\psi^{-1}(c_0)$, but just the equator.

Let $A = \psi^{-1}(\operatorname{int}(\mathfrak{C}))$ and B be a small neighborhood of $\psi^{-1}(\partial\mathfrak{C})$. Then $A \simeq S^2$, each of the three components of B is contractible, and each of the three components of $A \cap B$ is homotopic to a circle. The Mayer–Vietoris sequence is then

$$0 \to \mathbb{Z} \to H_2(\mathfrak{M}_R) \to \mathbb{Z}^3 \to 0 \to H_1(\mathfrak{M}) \to \mathbb{Z}^3 \to \mathbb{Z}^4 \to H_0(\mathfrak{M}_R)$$

from which the values for Case I in Table I follow. In fact, for $v < 0$, it is not hard to identify the homotopy type of $\mathfrak{M}_R(c, h) \simeq \mathfrak{K}_R(c, h)$. The fiber in $\mathfrak{K}_R(c, h)$ over each of the arms of the Y is homotopic to a 2-disk, so the entire space has the homotopy type of a 2-sphere with three disks sewn onto the equator, which is homotopic to a wedge of four 2-spheres.

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