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Comet like periodic orbits in the N-body problem

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Abstract

Take any nonresonant relative equilibrium solution of the N-body problem. Then there is a periodic solution of the $(N+1)$-body problem where N of the particles remain close to the relative equilibrium solution and the remaining particle is close to a circular orbit of the Kepler problem encircling the center of mass of the N-particle system.

Keywords: Periodic orbit; N-body problem; Relative equilibrium; Hamiltonian

1. Introduction

The main result of this paper is the existence of a family of periodic solutions of the planar $(N+1)$-body problem where one of the particles is at a great distance from the other N particles. This distant particle will be called the *comet*. In this family of periodic solutions the other N particles, called the *primaries*, move approximately on a nonresonant relative equilibrium solution of the N-body problem. The comet moves approximately on a circular orbit of the Kepler problem about the center of mass of the primary system.

The existence of this family of periodic solutions is established by the small parameter method called Poincaré's continuation method. The small parameter used is a scale parameter whose smallness indicates that the distances between the primaries are small relative to the distance to the comet. The scaling is a symplectic transformation with multiplier. None of the masses are assumed to be small.

There is a vast literature on periodic solutions of the N-body problem, so only the literature on comet-like periodic solutions obtained by perturbation methods will be discussed here. For the three-body problem these solutions correspond to Hill-type periodic solutions since in a typical Hill-type solution two particles are close and one is far away. These periodic solutions of

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the three-body problem were established in [3,10,12]. For the four-body problem Crandall [4] established the existence of this family when the relative equilibrium of the primaries was the equilateral triangular solution of Lagrange.

This family was established in [8] for the general $(N+1)$-body problem under the additional assumption that the comet had small mass. Analogs of this family in the restricted $(N+1)$-body were discussed in [7]. The present paper uses essentially the same method as the previous papers. However, this problem has a different degeneracy due to the existence of elliptic periodic orbits near a relative equilibrium. This degeneracy requires some variations in the old arguments, see the discussion in Section 3. See [7,8] for further references to the literature and see [9] for background information where similar notation is used.

2. Equations of motions, symmetries and integrals

Except for an occasional side remark, only the planar N-body problem will be considered here. Most of the work on this problem will be done in rotating coordinates, but some of the discussion in this section and the next require fixed coordinates. Fixed coordinates will be in the bold-face font.

Let $\boldsymbol{q}_1,\dots,\boldsymbol{q}_N \in \mathbb{R}^2$ be the position vectors, $\boldsymbol{p}_1,\dots,\boldsymbol{p}_N \in \mathbb{R}^2$ the momentum vectors of N particles of masses $m_1,\dots,m_N$ in a Newtonian reference frame. Let $\boldsymbol{q}=(\boldsymbol{q}_1,\dots,\boldsymbol{q}_N)$, $\boldsymbol{p}=(\boldsymbol{p}_1,\dots,\boldsymbol{p}_N)$ and $\boldsymbol{Z}=(\boldsymbol{q},\boldsymbol{p})$. Let the distance between the jth and kth particles be denoted by $d_{jk}=\|\boldsymbol{q}_j-\boldsymbol{q}_k\|$. The Hamiltonian for the N-body problem is total energy and in these rectangular coordinates is

$$\boldsymbol{H}_N=\boldsymbol{H}=\sum_{j=1}^{N}\frac{\|\boldsymbol{p}_j\|^2}{2m_j}-\boldsymbol{U}_N(\boldsymbol{q}), \tag{2.1}$$

where

$$\boldsymbol{U}_N=\boldsymbol{U}=\sum_{1\leqslant j<k\leqslant N}\frac{m_jm_k}{d_{jk}} \tag{2.2}$$

is the self-potential. The equations of motion are

$$\dot{\boldsymbol{q}}_j=\frac{\boldsymbol{p}_j}{m_j},\quad \dot{\boldsymbol{p}}_j=\partial_j\boldsymbol{U}_N,\qquad j=1,\dots,N, \tag{2.3}$$

where ∂_j denotes the partial derivative with respect to the jth variable, i.e., $\partial_j=\partial/\partial\boldsymbol{q}_j$. Eq. (2.3) implies the Newtonian equations

$$m_j\ddot{\boldsymbol{q}}_j=\partial_j\boldsymbol{U}_N(\boldsymbol{q}),\quad j=1,\dots,N. \tag{2.4}$$

By introducing the $4N\times 4N$ skew symmetric matrix $J=\left(\begin{smallmatrix}0 & -E\\ E & 0\end{smallmatrix}\right)$ where here E is the identity matrix and 0 is the zero matrix of size $2N\times 2N$, (2.3) can be written in the usual compact form

$$\dot{\boldsymbol{Z}}=J\,\nabla\boldsymbol{H}(\boldsymbol{Z}). \tag{2.5}$$

It will be necessary to discuss the problem in rotating coordinates also. Introduce rotating coordinates $Z = (q, p) = (q_1, \dots, p_n;\ p_1, \dots, p_N)$ by

$$\boldsymbol{q}_i = \exp(-\omega K t) q_i, \qquad \boldsymbol{p}_i = \exp(-\omega K t) q_i, \tag{2.6}$$

where the 2×2 matrix $K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of an infinitesimal rotation and ω is the frequency. Later ω will be set to 1. The Hamiltonian H_N of the N-body problem in rotating coordinates is

$$H_N = H = \sum_{j=1}^{N} \left(\frac{\| p_j \|^2}{2m_j} - \omega q_i^{\mathrm{T}} K p_i \right) - U_N(q), \tag{2.7}$$

where $U_N(q) = \boldsymbol{U}_N(\boldsymbol{q})$. The equations of motion are

$$\dot{q}_j = \frac{p_j}{m_j} - \omega K q_j, \quad \dot{p}_j = \partial_j U_N - \omega K p_j, \qquad j = 1, \dots, N. \tag{2.8}$$

Eq. (2.8) gives the Newtonian equations

$$m_j \left(\ddot{q}_j + 2\omega K \dot{q}_j - \omega^2 q_j \right) = \partial_j U_N(q), \quad j = 1, \dots, N. \tag{2.9}$$

These equations can be written in the usual compact form

$$\dot{Z} = J\ \nabla H(Z). \tag{2.10}$$

A *relative equilibrium* is an equilibrium point in these rotating coordinates. That is, a relative equilibrium is a solution of the equations

$$0 = \frac{p_j}{m_j} - \omega K q_j, \quad 0 = \partial_j U_N - \omega K p_j, \qquad j = 1, \dots, N, \tag{2.11}$$

or equivalently,

$$-\omega^2 m_j q_j = \partial_j U_N(q), \quad p_j = -\omega m_j K q_j, \qquad j = 1, \dots, N. \tag{2.12}$$

The eigenvalues of linearization of the equations of motion about a relative equilibrium are called the *exponents of the relative equilibrium* and the characteristic polynomial $p(\lambda)$ is called the *characteristic polynomial of the relative equilibrium*.

The problem discussed in this paper is highly degenerate due to symmetries and other special properties on the N-body problem. In this section, a brief discussion of the classical symmetries and integrals is given, see [8, Section II.B]. The next section gives a new set of coordinates to deal with a different degeneracy which caused some new difficulties in the present problem. The particular degeneracy that causes difficulties in the problem considered in this paper is the fact that the characteristic polynomial of a relative equilibrium $p(\lambda)$ has a factor $\lambda^2(\lambda^2 + 1)^3$, i.e., it has the exponent 0 with multiplicity at least two and the exponents $\pm\mathrm{i}$ with multiplicity at least three. This fact is well known, see [13]. It follows easily from the special coordinates discussed in the next section.

The Hamiltonian H of the N-body problem is invariant under the symplectic extension of the group of Euclidean motions of the plane and this introduces certain degeneracies which must be accounted for in a perturbation analysis. The symplectic extension of the Euclidean motions of the plane is

$$(\boldsymbol{q}_1,\dots,\boldsymbol{q}_N;\ \boldsymbol{p}_1,\dots,\boldsymbol{p}_N) \to (A\boldsymbol{q}_1+\boldsymbol{b},\dots,A\boldsymbol{q}_N+\boldsymbol{b};\ A\boldsymbol{p}_1,\dots,A\boldsymbol{p}_N), \tag{2.13}$$

where $A \in \mathrm{SO}(2, \mathbb{R})$ is a rotation matrix and $\boldsymbol{b} \in \mathbb{R}^2$. This transformation carries periodic orbits into periodic orbits and so periodic orbits of the N-body problem are not isolated even in an energy surface. By a theorem of [6], the algebraic multiplicity of the multiplier $+1$ of a periodic solution must be at least 8 (at least 12 in $\mathbb{R}^3$). In a like manner this symmetry gives rise to the fact that the characteristic polynomial $p(\lambda)$ of a relative equilibrium has $\lambda^2(\lambda^2+1)^2$ as a factor. Unless this degeneracy is taken into account, the standard methods of perturbation theory cannot be applied.

By a classical theorem, the translational symmetry implies that the equations of motion admit total linear momentum

$$\boldsymbol{L} = \boldsymbol{p}_1 + \cdots + \boldsymbol{p}_N \tag{2.14}$$

as an integral, and the rotational symmetry implies that the equations of motion admit total angular momentum

$$\boldsymbol{O} = \boldsymbol{q}_1^{\mathrm{T}} K \boldsymbol{p}_1 + \cdots + \boldsymbol{q}_N^{\mathrm{T}} K \boldsymbol{p}_N \tag{2.15}$$

as an integral. Thus part, but not all, of these degeneracies can be eliminated by holding these integrals fixed; that is, consider the equations on the invariant set $\mathscr{B}' \subset \mathbb{R}^{4N}$, where $\boldsymbol{L}$ and $\boldsymbol{O}$ are fixed. $\boldsymbol{L}$ fixed, say at zero, defines a linear subspace and since $\nabla \boldsymbol{O} \neq 0$ when $\boldsymbol{Z} = (\boldsymbol{q}, \boldsymbol{p}) \neq 0$, $\mathscr{B}$ is an invariant submanifold when we take $\boldsymbol{L} = 0$ and $\boldsymbol{O} = 1$. Since $\mathscr{B}$ is odd-dimensional, it cannot be a symplectic manifold. A periodic orbit restricted to $\mathscr{B}'$ would have the multiplier $+1$ of algebraic multiplicity at least 5.

Again $\mathscr{B}'$ is invariant under the symplectic extension Euclidean motions given in (2.13), and so periodic orbits are not isolated in $\mathscr{B}'$, even in an energy surface. However, if one identifies points in $\mathscr{B}$ which are carried into one another by (2.13), one obtains the quotient space $\mathscr{D}' = \mathscr{B}'/\sim$, which is of dimension $4N-6$. The action is free and proper, so by the reduction theorem of [6], the quotient space $\mathscr{D}'$ is a symplectic manifold. Indeed, $\mathscr{D}'$ inherits its symplectic structure in a natural way from $\mathbb{R}^{4N}$, and the Hamiltonian H with its flow naturally drop down to $\mathscr{D}'$. This reduction drops the minimum multiplicity of the multiplier $+1$ to 2. This reduction in the setting of the N-body problem is of course classical but is a good example of the general theorem found in [6].

Instead of making these reductions in the Newtonian frame, we shall make the analogous reductions in the rotating frame. Define the center of mass C, total mass M, total linear momentum and angular momentum to be

$$\begin{aligned} C &= \frac{m_1 q_1 + \cdots + m_N q_N}{M}, & M &= m_1 + \cdots + m_N, \\ L &= p_1 + \cdots + p_N, & O &= q_i^{\mathrm{T}} K p_1 + \cdots + q_N^{\mathrm{T}} K p_N. \end{aligned} \tag{2.16}$$

It is easy to see that

$$\dot{C} = KC + \frac{L}{M}, \qquad \dot{L} = KL, \qquad \dot{O} = 0. \tag{2.17}$$

Thus $C = L = 0$ defines an invariant, linear space $\mathscr{B}_1 \subseteq \mathbb{R}^{4N}$ of dimension $4N-4$. The system of equations for C and L in (2.17) is linear and the characteristic polynomial is $(\lambda^2+1)^2$. Thus the characteristic polynomial $p(\lambda)$ of a relative equilibrium restricted to $\mathscr{B}_1$ has the factor $(\lambda^2+1)^2$ removed. $\mathscr{B}_1$ is a symplectic linear space, see [8, p.10].

The system on $\mathscr{B}_1$ is still invariant under rotations and so still admits angular momentum O as an integral. Consider the equivalence relation

$$(q_1,\dots,q_N;\ p_1,\dots,p_N) \sim\sim (Aq_1,\dots,Aq_N;\ Ap_1,\dots,Ap_N), \tag{2.18}$$

where $A \in \mathrm{SO}(2, \mathbb{R})$ is a rotation matrix. Let $c \neq 0$, $\mathscr{B}_2 = O^{-1}(c) \subseteq \mathscr{B}_1$, and $\sim$ be the restriction of $\sim\sim$ to $\mathscr{B}_2$ and $\mathscr{D} = \mathscr{B}_2/\sim$. $\mathscr{D}$ is a symplectic manifold of dimension $4N-6$. Using the classical method of reducing equations using an integral, we can show that there are nice, local, symplectic coordinates at a relative equilibrium, see [15]. In these coordinates it is clear that the characteristic polynomial of a relative equilibrium on $\mathscr{D}$ has the factor λ^2 removed, see [8]. There still remains a factor of λ^2+1 which needs to be taken into account. It turns out that it is better to postpone the reduction due to the rotational symmetry until after the new reduction discussed in the next section.

3. Elliptic solutions and new symplectic coordinates

A *central configuration* is a solution $\boldsymbol{q}_1 = \boldsymbol{a}_1, \dots, \boldsymbol{q}_N = \boldsymbol{a}_N$ of the algebraic equations

$$-\lambda m_j \boldsymbol{q}_j = \partial_j U(\boldsymbol{q}_1,\dots,\boldsymbol{q}_N), \tag{3.1}$$

for some constant λ. The usual interpretation of a central configuration is a critical point of the potential restricted to a level set of the moment of inertia, $\boldsymbol{I} = \frac{1}{2}\Sigma m_j \|\boldsymbol{q}_j\|^2$. Here λ is the Lagrange multiplier. By letting $\lambda = \omega^2$, one sees from (2.12) that a central configuration gives rise to a relative equilibrium. This is true in the plane only since there are central configurations in $\mathbb{R}^3$ that are not relative equilibria. If $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_N)$ is a central configuration, then so is $(\alpha \boldsymbol{A}\boldsymbol{a}_1,\dots,\alpha \boldsymbol{A}\boldsymbol{a}_N)$ where α is a nonzero scalar and $\boldsymbol{A} \in \mathrm{SO}(2, \mathbb{R})$ is any 2×2 rotation matrix. Since the origin in $\mathbb{R}^{2N}$ is a limit of central configurations, we shall consider it a central configuration also. Thus a central configuration begets the set of central configurations $C_a = \{(\alpha \boldsymbol{A}\boldsymbol{a}_1,\dots,\alpha \boldsymbol{A}\boldsymbol{a}_N)\colon \alpha \in \mathbb{R},\ \boldsymbol{A} \in \mathrm{SO}(2, \mathbb{R})\}$. Not all the $\boldsymbol{a}_j$'s are zero, so assume that $\boldsymbol{a}_1$ is nonzero. Then $\{\alpha \boldsymbol{A}\boldsymbol{a}_1\colon \alpha \in \mathbb{R},\ \boldsymbol{A} \in \mathrm{SO}(2, \mathbb{R})\}$ is clearly a plane and this set is isomorphic to C_a. Thus C_a is a two-dimensional linear subspace of $\mathbb{R}^{2N}$.

The equations of motion admit angular momentum

$$\boldsymbol{O} = \sum_{j=1}^{N} \boldsymbol{q}_j^{\mathrm{T}} K \boldsymbol{p}_j \tag{3.2}$$

as an integral. Define the critical set $\mathscr{K} \subseteq \mathbb{R}^{4N}$ as the set where $\nabla \boldsymbol{H}$ and $\nabla \boldsymbol{O}$ are dependent, i.e.,

$$\mathscr{K} = \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{4N}\colon \alpha\ \nabla \boldsymbol{H}(\boldsymbol{q}, \boldsymbol{p}) + \beta\ \nabla \boldsymbol{O}(\boldsymbol{q}, \boldsymbol{p}) = 0,\ \alpha, \beta \in \mathbb{R},\ \alpha^2+\beta^2 = 1\}. \tag{3.3}$$

Since $\nabla \boldsymbol{H}$ is never zero and $\nabla \boldsymbol{O}$ is zero only at the origin, it is enough to look for solutions where both α and β are nonzero. The point $(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{a}, \boldsymbol{b})$ is in the critical set $\mathscr{K}$ if and only if

$$\alpha \partial_j U + \beta K \boldsymbol{p}_j = 0, \qquad \frac{\alpha \boldsymbol{p}_j}{m_j} - \beta K \boldsymbol{q}_j = 0. \tag{3.4}$$

If $(\boldsymbol{a}, \boldsymbol{b}) \in \mathscr{K}$, then $\boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ is a central configuration and $(\boldsymbol{a}, \boldsymbol{b})$ is a relative equilibrium.

Let $\boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N)$ be a specific central configuration scaled so that $\Sigma m_j \| \boldsymbol{a}_j \| = 1$. Let $C' = \{(\alpha A \boldsymbol{a}_1, \ldots, \alpha A \boldsymbol{a}_N)\colon \alpha \in \mathbb{R},\ A \in \mathrm{SO}(2, \mathbb{R})\}$ as above. Define $\mathscr{C}'$ as the subset of $\mathbb{R}^{4N}$ defined by

$$\mathscr{C}' = \{(\alpha A \boldsymbol{a}_1, \ldots, \alpha A \boldsymbol{a}_N;\ \beta B m_1 \boldsymbol{a}_1, \ldots, \beta B m_N \boldsymbol{a}_N)\colon \alpha, \beta \in \mathbb{R},\ A, B \in \mathrm{SO}(2, \mathbb{R})\}. \tag{3.5}$$

Lemma 1. $\mathscr{C}'$ is a *four-dimensional, invariant, linear, symplectic subspace of* $\mathbb{R}^{4N}$.

Proof. A symplectic basis for $\mathscr{C}'$ is

$$\begin{aligned} u_1 &= (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N; 0, \ldots, 0), & u_2 &= (K\boldsymbol{a}_1, \ldots, K\boldsymbol{a}_N; 0, \ldots, 0), \\ v_1 &= (0, \ldots, 0; m_1 \boldsymbol{a}_1, \ldots, m_N \boldsymbol{a}_N), & v_2 &= (0, \ldots, 0; K m_1 \boldsymbol{a}_1, \ldots, K m_N \boldsymbol{a}_N). \end{aligned} \tag{3.6}$$

So $\mathscr{C}'$ is a four-dimensional, linear symplectic subspace of $\mathbb{R}^{4N}$.

For the moment think of the vectors $\boldsymbol{q}_j$, $\boldsymbol{p}_j$ etc. as complex numbers. Then the set $\mathscr{C}'$ is defined by

$$\mathscr{C}' = \{(z\boldsymbol{a}_1, \ldots, z\boldsymbol{a}_N;\ w m_1 \boldsymbol{a}_1, \ldots, w m_N \boldsymbol{a}_N)\colon z, w \in \mathbb{C}\}. \tag{3.7}$$

Let $(z_0 \boldsymbol{a}_1, \ldots, z_0 \boldsymbol{a}_N;\ w_0 m_1 \boldsymbol{a}_1, \ldots, w_0 m_N \boldsymbol{a}_N)$, z_0, $w_0 \in \mathbb{C}$, $z_0 \neq 0$, be any point in $\mathscr{C}'$ and $z(t)$, $w(t)$ be the solutions of the Kepler problem

$$\dot{z} = w, \qquad \dot{w} = -\frac{z}{|z|^3}, \tag{3.8}$$

and starting at z_0, w_0 when $t = 0$. Then it is easy to verify that

$$V(t) = (\boldsymbol{q}(t), \boldsymbol{p}(t)) = (z(t)\boldsymbol{a}_1, \ldots, z(t)\boldsymbol{a}_N;\ w(t) m_1 \boldsymbol{a}_1, \ldots, w(t) m_N \boldsymbol{a}_N) \tag{3.9}$$

is a solution of the equations of motion of the N-body problem in fixed coordinates (2.3), and clearly $V(t) \in \mathscr{C}'$ for all t. This proves that $\mathscr{C}'$ is invariant. □

Lemma 2. *There exist symplectic coordinates* (z, w) *for* $\mathscr{C}'$ *and symplectic coordinates* (Z, W) *for* $\mathscr{E}' = \{x \in \mathbb{R}^{4N}\colon \{x, \mathscr{C}'\} = 0\}$ *so that* $(z, Z; w, W)$ *are symplectic coordinates for* $\mathbb{R}^{4N}$ *and the Hamiltonian of the N-body problem* $H(z, Z, w, W)$ *has the properties*

$$\begin{gathered} \frac{\partial H(z, 0, w, 0)}{\partial Z} = 0, \qquad \frac{\partial H(z, 0, w, 0)}{\partial W} = 0, \\ H(z, 0, w, 0) = H_K(z, w) = \tfrac{1}{2}\|w\|^2 - \frac{1}{\|z\|}. \end{gathered} \tag{3.10}$$

Proof. Since $\mathscr{C}'$ is symplectic, $\mathscr{E}'$ is also and $\mathbb{R}^{4N} = \mathscr{C}' \oplus \mathscr{E}'$, see [9, p.43]. Thus the vectors in

(3.6) can be extended to a symplectic basis $u_1, \dots, u_{2N}$; $v_1, \dots, v_{2N}$ with $u_3, \dots, u_{2N}$; $v_3, \dots, v_{2N}$ a symplectic basis for $\mathscr{E}'$. Given this basis, let $\boldsymbol{z}$, $\boldsymbol{w}$ be symplectic coordinates for $\mathscr{E}'$ as in the proof of Lemma 1 and Z, W be symplectic coordinates for $\mathscr{E}'$. The first two equations in (3.10) simply say that $\mathscr{E}'$ is invariant and the second says that in the $(\boldsymbol{z}, \boldsymbol{w})$ coordinates in $\mathscr{E}'$ the motion is that of the Kepler problem. $\boldsymbol{H}_K$ is just the Hamiltonian of the Kepler problem. □

Remark. The above discussion heavily used complex multiplication and thus is valid for the planar problem only. One can use real numbers for the general case of $\mathbb{R}^n$. In this case the corresponding $\mathscr{E}'$ would be a two-dimensional, invariant, linear, symplectic subspace of $\mathbb{R}^{2nN}$. The coordinates $\boldsymbol{z}$ and $\boldsymbol{w}$ in (3.10) would be one-dimensional.

For $n = 4$ (respectively $n = 8$) one can use Hamilton's quaterians (respectively Cayley numbers). In these cases the corresponding $\mathscr{E}'$ would be an eight- (respectively a sixteen-) dimensional, invariant, linear, symplectic subspace of $\mathbb{R}^{8N}$ (respectively $\mathbb{R}^{16N}$). The coordinates $\boldsymbol{z}$ and $\boldsymbol{w}$ in (3.10) would be four- (respectively eight-) dimensional.

The argument given above only uses the homogeneity of the force field and so works for inverse power laws in general.

Now consider the problem in rotating coordinates. Let $\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_N)$ be a central configuration and $C = \{(\alpha A a_1, \dots, \alpha A a_N)\colon \alpha \in \mathbb{R},\ A \in \mathrm{SO}(2, \mathbb{R})\}$ as above. Define $\mathscr{E}$ as the subset of $\mathbb{R}^N$ defined by

$$\mathscr{E} = \{(\alpha A a_1, \dots, \alpha A a_N;\ \beta B m_1 a_1, \dots, \beta B m_N a_N)\colon \alpha, \beta \in \mathbb{R},\ A, B \in \mathrm{SO}(2, \mathbb{R})\}. \tag{3.11}$$

The same reasoning yields the following lemma.

Lemma 3. *$\mathscr{E}$ is a four-dimensional, invariant, linear, symplectic subspace of $\mathbb{R}^{4N}$. Moreover, there exist symplectic coordinates (z, w) for $\mathscr{E}$ and symplectic coordinates (Z, W) for $\mathscr{E} = \{x \in \mathbb{R}^{4N}\colon \{x, \mathscr{E}\} = 0\}$ so that $(z, Z;\ w, W)$ are symplectic coordinates for $\mathbb{R}^{4N}$ and the Hamiltonian of the N-body problem $H(z, Z, w, W)$ has the properties*

$$\begin{gathered}\frac{\partial H(z, 0, w, 0)}{\partial Z} = 0, \qquad \frac{\partial H(z, 0, w, 0)}{\partial W} = 0, \\ H(z, 0, w, 0) = H_K(z, w) = \tfrac{1}{2}\| w \|^2 - z^{\mathrm{T}} K w - \frac{1}{\| z \|}.\end{gathered} \tag{3.12}$$

Now consider angular momentum first in fixed coordinates (3.2) and in rotating coordinate (2.16).

Lemma 4. *In the symplectic coordinates $(z, \boldsymbol{Z}, \boldsymbol{w}, \boldsymbol{W})$ of Lemma 2, angular momentum has the form*

$$\boldsymbol{O} = z^{\mathrm{T}} K w + \boldsymbol{Z}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{W}, \tag{3.13}$$

where $\boldsymbol{B}$ *is a skew symmetric matrix of dimension* $(4N-4)\times(4N-4)$. *In the symplectic coordinates* (z, Z, w, W) *of Lemma* 3 *angular momentum has the form*

$$O = z^{\mathrm{T}}Kw + Z^{\mathrm{T}}BW, \tag{3.14}$$

where B is a skew symmetric matrix of dimension $(4N-4)\times(4N-4)$.

Proof. Use complex notation again and consider the case of rotating coordinates, since both cases are essentially the same. $O = \Sigma q_j^{\mathrm{T}}Kp_j = \Sigma \mathscr{I} \bar{q}_j p_j$. When $Z = W = 0$, we have $q_j = za_j$ and $p_j = m_j w a_j$, and so $O = \mathscr{I}\bar{z}\dot{w}\Sigma m_j |a_j|^2 = \mathscr{I}\bar{z}w$. Here $\mathscr{I}$ stands for the imaginary part. □

Since the Hamiltonian on the invariant subspaces $\mathscr{C}'$ (respectively $\mathscr{C}$) is just the Hamiltonian of the Kepler problem in fixed coordinates (respectively rotating coordinates), there are lots of special coordinates which simplify the equations of motion. One useful special coordinate system is the Poincaré elements, see [9,14]. These coordinates are valid in a neighborhood of the circular orbits. In a nonrotating frame, the Hamiltonian of the Kepler problem in Poincaré elements is

$$\boldsymbol{H}_K(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{P}_1, \boldsymbol{P}_2) = -\tfrac{1}{2}\boldsymbol{P}_1^2, \tag{3.15}$$

and in the rotating frame it is

$$H_K(Q_1, Q_2, P_1, P_2) = -\tfrac{1}{2}P_1^2 - P_1 + (Q_2^2 + P_2^2). \tag{3.16}$$

In the above, $\boldsymbol{Q}_1$ and Q_1 are angular variables defined modulo 2π and the other variables are rectangular variables. Angular momentum in these variables is

$$\boldsymbol{O}_K = \boldsymbol{P}_1 - (\boldsymbol{Q}_2^2 + \boldsymbol{P}_2^2), \qquad O_K + P_1 - (Q_2^2 + P_2^2). \tag{3.17}$$

$\boldsymbol{Q}_2 = \boldsymbol{P}_2 = 0$ and $Q_2 = P_2 = 0$ corresponds to the circular orbits of the Kepler problem.

Lemma 5. *Consider a fixed central configuration a of the N-body problem in nonrotating coordinates. This central configuration gives rise to a periodic solution* $\boldsymbol{Z}_a(t)$ *where the bodies uniformly rotate about the center of mass on circular orbits. Let the period of* $\boldsymbol{Z}_a(t)$ *be* $\boldsymbol{T}_a$ *and the energy of* $\boldsymbol{Z}$ *be* $\boldsymbol{H}_a$. *In the fixed energy surface* $\boldsymbol{H} = \boldsymbol{H}_a$, *there is an invariant, three-dimensional manifold containing* $\boldsymbol{Z}_a$, *filled with periodic solutions all of period* $\boldsymbol{T}_a$. *In fact, this set is the subset of* $\mathscr{C}'$ *where* $\boldsymbol{H} = \boldsymbol{H}_K$ *is fixed at* $\boldsymbol{H}_a$.

Proof. It is well known that the period of the elliptic solutions depends only on the value of the energy. Or integrate the problem in Poincaré elements. □

Polar coordinates are convenient also. The Hamiltonian of the Kepler problem in a rotating frame in polar coordinates is

$$H_K = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{1}{r}. \tag{3.18}$$

The circular orbit when angular momentum Θ is $+1$ is given by θ arbitrary, $r=1$, $\Theta=1$, $R=0$. The linearized equations about this periodic solution are

$$\dot{\theta}=\Theta, \qquad \dot{\Theta}=0, \qquad \dot{r}=R, \qquad \dot{R}=-r. \tag{3.19}$$

The characteristic equation of this system is $\lambda^2(\lambda^2+1)$. Thus, we have proved the following lemma.

Lemma 6. *The characteristic polynomial $p(\lambda)$ of a relative equilibrium has the factor $\lambda^2(\lambda^2+1)^3$.*

Let $p(\lambda)=\lambda^2(\lambda^2+1)^3 r(\lambda)$. If $r(\lambda)$ does not have zero as a root, then the relative equilibrium will be called *nondegenerate* and if $r(\lambda)$ does not have a zero of the form $n\mathrm{i}$ where n is an integer, then the relative equilibrium is called *nonresonant*. For the two-body problem, $p(\lambda)=\lambda^2(\lambda^2+1)^3$ and so the relative equilibrium is nonresonant. Moulton's collinear relative equilibrium is nondegenerate for all N, see [11]. The Lagrange equilateral triangle relative equilibria and the Euler collinear relative equilibria for the three-body problem are nonresonant by the analysis in [13, Section 18].

Henceforth, assume that $(\alpha_1,\dots,\alpha_{N-1}, \beta_1,\dots,\beta_{N-1})$ is a nonresonant relative equilibrium with exponents 0, 0, $+\mathrm{i}$, $-\mathrm{i}$, $\lambda_5,\dots,\lambda_{4N-4}$ where λ_j is not an integer multiple of i.

4. Jacobi coordinates and scaling

With these preliminaries established, it is time to consider the main problem. It is necessary to consider both the N-body and the $(N+1)$-body problems together. For the $(N+1)$-body problem, use the notation of the previous section, except start the index from zero, so the masses are m_0, $m_1,\dots,m_N$, etc.

Since the main assumption to be made in this paper is that the distance of one of the particles, say the $(N+1)$st, to the center of mass of the other N particles is large; it is convenient to use Jacobi coordinates because one of the Jacobi coordinates, u_N, is precisely the vector from the center of mass of N of the particles to the $(N+1)$st particle, see Fig. 1. Also one of the Jacobi coordinates is the center of mass $g=m_0q_0+m_1q_1+\cdots+m_Nq_N$ of the whole system and its conjugate momentum is the total linear momentum of the system $G=p_0+p_1+\cdots+p_N$. The center of mass will be fixed at the origin and total linear momentum will be set to zero by putting $g=G=0$. Having so fixed the center of mass and the linear momentum, the Hamiltonian of the $(N+1)$-body problem in rotating Jacobi coordinates is

$$H_{N+1}=\sum_{j=1}^{N}\left(\frac{\| v_j\|^2}{2M_j}-u_j^{\mathrm{T}}Kv_j\right)-U_{N+1}, \tag{4.1}$$

where the M_i are constants depending only on the masses ($M_k=m_k\mu_{k-1}/\mu_k$, $\mu_k=m_0+m_1+\cdots+m_k$). See [7,9]. Write this Hamiltonian as

$$H_{N+1}=H_2+H_N+H_{\mathrm{E}}, \tag{4.2}$$

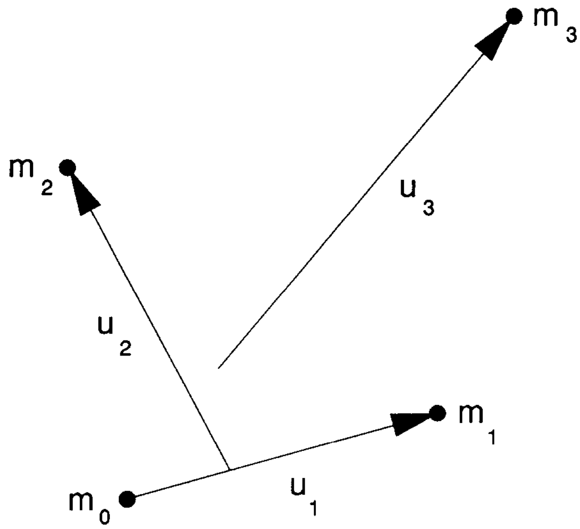

Fig. 1. Jacobi coordinates.

where

$$H_2 = \frac{\| v_N \|^2}{2M_N} - u_N^{\mathrm{T}} K v_N - \frac{\mu_{N-1} m_N}{\| u_N \|}, \tag{4.3}$$

$$H_N = \sum_{j=1}^{N-1} \left(\frac{\| v_j \|^2}{2M_j} - u_j^{\mathrm{T}} K v_j \right) - U_N, \tag{4.4}$$

$$H_{\mathrm{E}} = m_N \sum_{j=1}^{N} m_j \left\{ \frac{1}{d_{jN}} - \frac{1}{\| u_N \|} \right\}. \tag{4.5}$$

In the above, H_2 is the Hamiltonian of the Kepler problem (central force problem) in rotating coordinates. Recall that u_N is the position vector of the Nth particle relative to the center of mass of the first N particles. H_N is the Hamiltonian of the N-body problem (the first N particles) in rotating Jacobi coordinates. Lastly, H_{E} is an error term which is small if the distance between the first N particles is small.

5. The Kepler problem

Change to polar coordinates (r, θ, R, Θ) in (4.3) by

$$u_N = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}, \qquad v_N = \begin{pmatrix} R \cos \theta - (\Theta/r) \sin \theta \\ R \sin \theta - (\Theta/r) \cos \theta \end{pmatrix}, \tag{5.1}$$

so that H_2 becomes

$$H_2 = \tfrac{1}{2}M_N\left\{R^2 + \frac{\Theta^2}{r^2}\right\} - \Theta - \frac{\mu_{N-1}m_N}{r}, \tag{5.2}$$

and the equations of motion become

$$\dot{\theta} = \frac{\Theta}{M_N r^2} - 1, \qquad \dot{\Theta} = 0, \qquad \dot{r} = \frac{R}{M_N}, \qquad \dot{R} = \frac{\Theta^2}{M_N r^3} - \frac{\mu_{N-1}m_N}{r^2}. \tag{5.3}$$

These equations have an equilibrium point at $R = 0$, $\theta = \theta_0$ where θ_0 is arbitrary, $r_0 = (\mu_{N-1}m_N/M_N)^{1/3}$, $\Theta_0 = M_N^{1/3}(\mu_{N-1}m_N)^{2/3}$. The Hamiltonian of the linear variational equations about this equilibrium point is

$$Q = \frac{1}{2}\left\{\frac{1}{M_n}P^2 + M_N\rho^2 + \frac{\Phi^2}{\Theta_0} - 2\alpha\rho\Phi\right\}, \tag{5.4}$$

and the linear variational equations are

$$\dot{\phi} = \frac{\Phi}{\Theta_0} - \alpha\rho, \qquad \dot{\Phi} = 0, \qquad \dot{\rho} = \frac{P}{M_N}, \qquad \dot{P} = -M_N P + \alpha\Phi, \tag{5.5}$$

where $\phi = \delta\theta$, $\Phi = \delta\Phi$, $\rho = \delta r$, $P = \delta P$ and $\alpha = 2\Theta_0/(\mu_{N-1}m_N)$. The characteristic equation of the linearized equations is $\lambda^2(\lambda^2 + 1)$ and the exponents are 0, 0, $+\mathrm{i}$, -1.

6. The scaled equations and the solutions of the first approximation

Consider the full Hamiltonian H_{N+1} in (4.1) where H_2 is (5.2), H_N is (4.4) and H_{E} is (4.5). Scale the variables, time and the Hamiltonian as follows:

$$\begin{aligned}
&\theta = \theta_0 + \epsilon\phi, \qquad \Theta = \Theta_0 + \epsilon\Phi, \qquad r = r_0 + \epsilon\rho, \qquad R = \epsilon P,\\
&u_j = \epsilon^4\xi_j, \quad v_j = \epsilon^{-2}\eta_j, \qquad \text{for } j = 1, \ldots, N-1,\\
&H = \epsilon^6 H_{N+1}, \qquad t' = \epsilon^{-6}t.
\end{aligned} \tag{6.1}$$

This change of variables is symplectic with multiplier ϵ^{-2}. We shall drop the prime on t in the future. Now ϵ small means that primaries are close together and the comet is near the circular orbit of the Kepler problem. The Hamiltonian becomes

$$\begin{aligned}
H = {}& \sum_{j=1}^{N-1}\left\{\frac{\|\eta_j\|^2}{2M_j} - \epsilon^6\xi_j^{\mathrm{T}}K\eta_j\right\}\\
&- \sum_{0\leqslant j<k\leqslant N-1}\frac{m_j m_k}{d_{jk}} + \tfrac{1}{2}\epsilon^6\left\{\frac{1}{M_n}P^2 + M_N\rho^2 + \frac{\Phi^2}{\Theta_0} - 2\alpha\rho\Phi\right\} + \mathrm{O}(\epsilon^7).
\end{aligned} \tag{6.2}$$

In the above ignore the terms of order ϵ^7 for the present. To that order the Hamiltonian decouples into the sum of two terms, the first of which is the N-body problem in a slowly rotating coordinate system and the second is the Hamiltonian (5.4) of the linear variational equations (5.5). In this case the truncated equations of motion are

$$\begin{aligned} &\dot{\xi}_j = \frac{\eta_j}{M_j} - \epsilon^6 K \xi_j, \quad \dot{\eta}_j = -\partial_j U(\xi) - \epsilon^6 K \eta_j, \qquad j = 1,\dots,N-1, \\ &\dot{\phi} = \epsilon^6 \left\{ \frac{\Phi}{\Theta_0} - \alpha\rho \right\}, \qquad \dot{\Phi} = 0, \qquad \dot{\rho} = \frac{\epsilon^6 P}{M_N}, \qquad \dot{P} = \epsilon^6 \{ -M_N P + \alpha \Phi \}. \end{aligned} \tag{6.3}$$

Let $\alpha = (\alpha_1, \dots, \alpha_{N-1})$ be the nonresonant central configuration for the N-body problem that was discussed in Section 3, i.e., assume that

$$-M_j \alpha_j = \partial_j U(\alpha), \quad \text{for } j = 1,\dots,N-1. \tag{6.4}$$

Define $\beta = (\beta_i, \dots, \beta_{N-1})$ by $\beta_j = M_j K \alpha_j$. Now a periodic solution of these truncated equations (6.3) is

$$\begin{aligned} &\xi_j(t) = e^{\omega K t}\alpha_j, \quad \eta_j(t) = e^{\omega K t}\beta_j, \qquad \text{for } j = 1,\dots,N-1, \\ &\phi = \Phi = \rho = P = 0, \end{aligned} \tag{6.5}$$

where $\omega = 1 - \epsilon^6$ and the period is $2\pi/\omega = 2\pi(1 + \epsilon^6 + \cdots)$.

In order to calculate the multipliers of this periodic solution of (6.3), make the period change of variables

$$\xi_j(t) = e^{\omega K t} w_j, \quad \eta_j(t) = e^{\omega K t} z_j, \qquad \text{for } j = 1,\dots,N-1. \tag{6.6}$$

The first two equations in (6.3) become

$$\dot{w}_j = \frac{z_j}{M_j} - K w_j, \quad \dot{z}_j = -\partial_j U(w) - K z_j, \qquad j = 1,\dots,N-1. \tag{6.7}$$

The periodic solution (6.5) becomes $w_j = \alpha_j$, $z_j = \beta_j$, $\phi = \Phi = \rho = P = 0$. Eqs. (6.7) are the equations of the N-body problem in rotating coordinates and so the variational equations about $w_j = \alpha_j$, $z_j = \beta_j$ give rise to the exponents $0, 0, +\mathrm{i}, -\mathrm{i}, \lambda_5, \dots, \lambda_{4N-4}$. Thus the characteristic multipliers are

$$\begin{aligned} &+1,\ +1,\ \exp\left(\frac{2\pi \mathrm{i}}{\omega}\right),\ \exp\left(-\frac{2\pi \mathrm{i}}{\omega}\right),\ \exp\left(\frac{2\pi\lambda_5}{\omega}\right), \dots, \exp\left(\frac{2\pi\lambda_{4N-4}}{\omega}\right), \\ &+1,\ +1,\ \exp\left(\frac{+2\pi \mathrm{i}\epsilon^6}{\omega}\right),\ \exp\left(-\frac{2\pi \mathrm{i}\epsilon^6}{\omega}\right). \end{aligned} \tag{6.8}$$

The eigenvalues $\lambda_5, \dots, \lambda_{4N-4}$ are assumed not to be an integer multiple of i, so $\exp(\lambda_j 2\pi/\omega) \neq 1$ for small ϵ for $j = 5, \dots, 4N-4$. Since $2\pi/\omega = 2\pi(1 + \epsilon^6 + \cdots)$, it follows that $\exp(\pm 2\pi \mathrm{i}/\omega) = 1 \pm \epsilon^6 2\pi \mathrm{i} + O(\epsilon^{12})$ and $\exp(\pm\epsilon^6 2\pi \mathrm{i}/\omega) = 1 \pm \epsilon^6 2\pi \mathrm{i} + O(\epsilon^{12})$. Thus the multipliers in (6.8) fall into three groups. First there are four $+1$, then there are four of the form $1 \pm 2\pi \mathrm{i}\epsilon^6 + \cdots$, and $4N-8$ of the form $\delta_j + O(\epsilon^6)$, $\delta_j = \exp(2\pi\lambda_j) \neq +1$.

Basically the argument from here on is straightforward application of classical ideas with one variation. First of all the problem still admits rotational symmetry and hence angular momentum is an integral. Passing to the reduced space (see the next section) eliminates two of the multipliers $+1$, leaving two. By considering the cross section map in an energy surface, eliminate the remaining two and so the implicit function theorem can be applied to find a periodic solution of the $(N+1)$-body problem on the reduced space close to the solution (6.5). The tedium comes from the fact that the multipliers differ from $+1$ at different orders. The remaining discussion treats these difficulties.

7. The reduced space

Hamiltonian (4.2) with H_2 as in (5.2) is invariant under the symplectic symmetry of rotation by τ, i.e.,

$$\begin{aligned}&(u_1, v_1, \dots, u_{N-1}, v_{N-1}, \tau, \theta, R, \Theta) \to \\ &\quad \left(e^{Kt}u_1, e^{K\tau}v_1, \dots, e^{K\tau}u_{N-1}, e^{K\tau}v_{N-1}, \tau, \theta+\tau, R, \Theta\right),\end{aligned} \tag{7.1}$$

and so admits total angular momentum

$$O = \sum_{j=1}^{N-1} u_j^{\mathrm{T}} K v_j + \Theta \tag{7.2}$$

as an integral. In the new scaled variables angular momentum becomes

$$O = \epsilon^2 \sum_{j=1}^{N-1} \xi_j^{\mathrm{T}} K \eta_j + \Theta_0 + \epsilon\Phi. \tag{7.3}$$

By fixing angular momentum to be Θ_0, one can solve for Φ to find that

$$\Phi = -\epsilon \sum_{j=1}^{N-1} \xi_j^{\mathrm{T}} K \eta_j. \tag{7.4}$$

By holding O fixed and ignoring the conjugate angle ϕ, one drops to the reduced space. Symplectic coordinates on the reduced space are $\xi_1, \eta_1, \dots, \xi_N, \eta_N, \rho, P$ and the Hamiltonian on the reduced space is

$$R = \sum_{j=1}^{N-1} \left\{ \frac{\| \eta_j \|^2}{2M_j} - \epsilon^6 \xi_j^{\mathrm{T}} K \eta_j \right\} - \sum_{0 \leqslant j < k \leqslant N-1} \frac{m_j m_k}{d_{jk}} + \tfrac{1}{2}\epsilon^2 \left\{ \frac{1}{M_M} P^2 + M_N \rho^2 \right\} + \mathrm{O}(\epsilon^7). \tag{7.5}$$

This is essentially the Hamiltonian (7.2) without the terms in ϕ and Φ. To order ϵ^6 there is a periodic solution

$$\begin{aligned}&\xi_j(t) = e^{\omega K t}\alpha_j, \quad \eta(t) = e^{\omega K t}\beta_j, \qquad \text{for } i = 1, \dots, N-1, \\ &\rho = P = 0.\end{aligned} \tag{7.6}$$

As above, one computes the multipliers to be

$$+1,\ +1,\ \exp\left(\frac{2\pi \mathrm{i}}{\omega}\right),\ \exp\left(-\frac{2\pi \mathrm{i}}{\omega}\right),\ \exp\left(\frac{\lambda_5 2\pi}{\omega}\right),\ldots,\exp\left(\frac{\lambda_{4N-4}2\pi}{\omega}\right),$$
$$\exp\left(+\frac{2\pi \mathrm{i}\epsilon^6}{\omega}\right),\ \exp\left(-\frac{2\pi \mathrm{i}\epsilon^6}{\omega}\right). \tag{7.7}$$

Now the multipliers in (7.7) fall into three groups. First there are two $+1$, then there are four of the form $1 \pm 2\pi \mathrm{i}\epsilon^6 + \cdots$, and $4N-8$ of the form $\delta_j + \mathrm{O}(\epsilon^6)$, $\delta_j = \exp(2\pi\lambda_j) \neq +1$.

8. The perturbation argument

Consider the scaled Hamiltonian R in (7.5) on the reduced space. Up to order ϵ^6 the solutions (7.6) are $(2\pi/\omega)$-periodic with multipliers as in (7.7). Consider the Poincaré map Σ in an energy surface $R = \text{const}$. By considering the Poincaré map in an energy surface, the last two $+1$ multipliers disappear. The fixed points of Σ correspond to periodic solutions.

First look at the form of the Poincaré map up to order ϵ^5. To that order the period is 2π. Dropping the terms of order ϵ^6 and higher leaves just the Hamiltonian of the N-body problem in fixed coordinates since the rotation terms are at order ϵ^6. In the energy surface through the relative equilibrium there is a two-dimensional surface filled with the elliptic periodic solutions discussed in Section 3. These periodic solutions will all have period 2π also and so the period map fixes points on this two-dimensional surface. Also up to that order the variables ρ and P are fixed. Thus there is a four-dimensional manifold which is fixed under the period map to order ϵ^5. Let σ be a local coordinate in this manifold and τ the complementary coordinate in the energy surface. The point $\sigma = 0$, $\tau = 0$ corresponds to the fixed point up to order ϵ^6. Thus the Poincaré map is of the form Σ: $(\sigma, \tau) \to (\sigma', \tau')$ where

$$\begin{aligned} \sigma' &= \sigma + \epsilon^6(E_1\sigma + E_2\tau + S(\sigma, r)) + \mathrm{O}(\epsilon^7), \\ \tau' &= A_4\tau + \epsilon^6(E_3\sigma + E_4\tau + T(\sigma, \tau)) + \mathrm{O}(\epsilon^7), \end{aligned} \tag{8.1}$$

where A_4, E_1, E_2, E_3 and E_4 are constant matrices of the appropriate size, S and T are smooth functions with $S(0, 0) = T(0, 0) = 0$. From the discussion of the multipliers, the eigenvalues of A_4 are $\delta_j = \exp(2\pi\lambda_j) \neq +1$, $j = 5,\ldots,4N-4$, and the eigenvalues of E_1 are $\pm 2\pi \mathrm{i}$.

To find a fixed point of Σ, one must solve

$$0 = E_1\sigma + E_2\tau + S(\sigma, \tau) + \mathrm{O}(\epsilon), \qquad 0 = (A_4 - I)\tau + \epsilon^6(E_3\sigma + E_4\tau + T(\sigma, \tau)) + \mathrm{O}(\epsilon^7). \tag{8.2}$$

A direct application of the implicit function theorem gives a solution of (8.2) of the form $\sigma = \sigma^*(\epsilon) = \mathrm{O}(\epsilon^7)$, $\tau = \tau^*(\epsilon) = \mathrm{O}(\epsilon^7)$ where σ^* and τ^* are smooth functions of ϵ for small ϵ and $\sigma^*(0) = \tau^*(0) = 0$.

In summary: *Take any nonresonant relative equilibrium solution of the N-body problem. Then there is a periodic solution of the $(N+1)$-body problem on the reduced space where N of the* *particles remain close to the relative equilibrium solution and the remaining particle is close to a circular orbit of the Kepler problem encircling the center of mass of the N-particle system.*

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