THE INDEX SEQUENCE OF A FIXED POINT

KENNETH R. MEYER

Department of Mathematical Sciences
University of Cincinatti
Cincinatti, Ohio

INTRODUCTION

This note gives a set of examples of local diffeomorphisms whose fixed points satisfy the index restrictions given in Mallet-Paret and Yorke [1]. These examples cover every major case and in total show that the restrictions placed on the index sequence is sharp.

Let $f: R^n \to R^n$ be a continuous map which has the origin as an isolated fixed point, that is, $f(0) = 0$ and $f(x) \neq x$ for $0 < \|x\| \leq \varepsilon_1$ for some $\varepsilon_1 > 0$. Let $S^{n-1} = \{x \in R^n: \|x\| = 1\{$ be the unit $n - 1$ sphere in R^n and define

$$\tilde{f}: S^{n-1} \to S^{n-1}: x \to (\varepsilon_1 x - f(\varepsilon_1 x))/\|\varepsilon_1 x - f(\varepsilon_1 x)\|$$

The topological degree of $\tilde{f}$ is called the index of the fixed point and will be denoted by $\text{ind}(f,0)$. This notation emphasizes the fact that the index depends on the function and the fixed point. In general, $\text{ind}(f,x_0)$ will denote the index of the fixed point x_0 of f and can be calculated by translating this fixed point to the origin. The index is an invariant under changes of coordinates and is important in bifurcation analysis.

If the origin is an isolated fixed point of f^k, the kth iterate of f, for all $k \geq 1$ then define the index sequence $\{I_k\}$ by $I_k = \text{ind}(f^k,0)$. In general the index sequence $\{I_k\}$ is quite arbitrary, but Shub and Sullivan [2] have shown that if f is C^1 then the index sequence is bounded and more recently Mallet-Paret and Yorke [1] have shown that it is periodic. In fact the latter authors show that the spectrum of the Jacobian matrix of f at the origin places restrictions on the possible index sequences.

In order to understand the results of this note, it is necessary to review the results of [1] and introduce some notation. Let $\mathcal{B}^n$ denote the set of C^1 functions f defined in a neighborhood of the origin in R^n such that the origin is an isolated fixed point for f and *all* its iterates. Thus if $f \in \mathcal{B}^n$ then the index sequence is defined for f. Let D denote the derivative operator so that $Df(0)$ denotes the Jacobian matrix of f evaluated at the origin. For the rest of this introduction let $f \in \mathcal{B}^n$ and $Df(0) = A$.

If $I - A$ is nonsingular then

$$I_1 = \text{ind}(f,0) = \text{sgn}\det(I - A)$$

Let σ_+ (resp. σ_-) be the number of eigenvalues of A, counting multiplicity, in $(1,\infty)$ (resp. in $(-\infty,-1)$). In the generic case $I - A^k$ is nonsingular for all $k \geq 1$ and the index sequence is given by

$$I_k = \text{ind}(f^k,0) = \text{sgn}\det(I - A^k) = \begin{cases} (-1)^{\sigma_+} & k \text{ odd} \\ (-1)^{\sigma_+ + \sigma_-} & k \text{ even} \end{cases}$$

Thus in the generic case there are only four possible index sequences $I_k = 1$, $I_k = -1$, $I_k = (-1)^k$, and $I_k = -(-1)^k$. Linear maps show that these index sequences are achievable. To describe the general case, let $I = (I_1, I_2, \ldots)$ be the infinite vector of indices and introduce the special vectors

$$L_k = (L_{k1}, L_{k2}, \ldots)$$

where

$$L_{k\ell} = \begin{cases} k & \text{if } k \text{ divides } \ell \\ 0 & \text{if } k \text{ does not divide } \ell \end{cases}$$

Thus L_k has the integer k in those positions which are a multiple of k. Call I the index vector of $f \in \mathcal{B}^n$ and L_k the kth basic vector. Let

$$M = \{m \geq 1: \text{there is a } y \in R^n \text{ with } y, Ay, \ldots, A^{m-1}y \text{ distinct and } A^m y = y\}$$

If A has an eigenvalue which is an mth root of unity and another eigenvalue which is an ℓth root of unity, then clearly m, ℓ and $m\ell \in M$.

With this notation the central result of [1] is

THEOREM (Mallet-Paret and Yorke). Let $f \in \mathcal{B}^n$ then the index vector I of f has the form

$$I = \begin{cases} \sum_{m \in M} c_m L_m & \sigma_+ \text{ even} \\ \sum_{m \in M} c_m L_m + \sum_{m \in M} c_m (L_m - L_{2m}) & \sigma_- \text{ odd} \end{cases} \quad m \text{ even}$$

where the coefficients c_n are integers. Furthermore c_1 and c_2 are subject to the additional restrictions

(i) $c_1 = (-1)^{\sigma_+}$ if $I - A$ is nonsingular

(ii) $c_1 \in \{-1, 0, 1\}$ if $I - A$ has one dimensional kernel

(iii) $c_2 \in \{0, (-1)^{\sigma_+ + \sigma_- + 1}\}$ if $I - A$ is nonsingular and $I - A^2$ has one dimensional kernel.

These authors conjecture that this theorem is sharp in the sense that for any given A and I permitted by this theorem there is a map $f \in \mathcal{B}^n$ with $Df(0) = A$ and with index vector I. This note verifies this conjecture in most cases by constructing specific examples. Even though the most general case is not considered in this note, the most interesting cases are considered. It should be clear to the reader that the methods given here can be used to construct the general example.

Moreover the examples constructed below have unfoldings with generic fixed points--in fact this is the method used to compute the index sequence. That is we give functions $f\mu$ depending on a small real parameter μ with the following properties. First $f = f_0 \in \mathcal{B}^n$ is the desired example satisfying the restrictions of the Mallet-Paret and Yorke Theorem. For μ small and nonzero $f\mu$ and all its iterates have only generic fixed points and all these fixed points tend to the origin as $\mu \to 0$.

Examples: Here we give a sequence of examples which illustrate the various cases covered by the Mallet-Paret and Yorke Theorem. The examples are ordered so that the complexity grows gradually.

EXAMPLE 1. Let $\alpha = -1$, 0 or 1, then there is an $f \in \mathcal{B}^1$ such that $Df(0) = (1)$ and with index vector αL_1.

By the Mallet-Paret and Yorke Theorem the index sequence for any $f \in \mathcal{B}^1$ with $Df(0) = (1)$ is either L_1, $0L_1$ or $-L_1$ and so it is enough to give and $f \in \mathcal{B}^1$ with $Df(0) = (1)$ such that

$$L_{11} = \text{ind}(f,0) = 1, 0 \text{ or } -1$$

Consider

$$f_\mu(x) = x + \mu x - x^2$$

where $x, \mu \in R^1$ and μ is considered as a small parameter. We shall show that $f_0 \in \mathcal{B}^1$, $Df_0(0) = (1)$ and $\text{ind}(f_0,0) = 0$. The fixed point equation is

$$0 = x - f_\mu(x) = x(x - \mu)$$

and so for $\mu \neq 0$ there are two fixed points namely $x = 0$ and $x = \mu$. For $\mu = 0$ there is a unique fixed point. Since $(d/dx)(x - f_\mu(x)) = 2x - \mu$ the index of the fixed point at the origin is -1 for $\mu > 0$ and the index of the fixed point at $x = \mu$ is +1 for $\mu > 0$. The unique fixed point at the origin is degenerate (i.e., not generic) when $\mu = 0$ and so cannot be calculated from the derivate alone. As $\mu \to 0^+$ the two fixed points converge to the unique fixed point at the origin. Since the index of the fixed point at the origin for $\mu = 0$ must be the sum of the indices of all the fixed points that converge to it, we see that the index of the isolated fixed point at the origin is 0.

The jth iterate of f_0 is given by

$$f_0^j(x) = x - jx^2 + 0(x^4)$$

and so $x - f_0^j(x) = jx^2 + 0(x^4)$ is positive for small x. Thus the origin is an isolated fixed point for all iterates of f_0 and so $f \in \mathcal{B}^1$.

Now consider

$$f_\mu(x) = x \pm (\mu x + x^3)$$

The fixed point equation is

$$0 = x - f_\mu(x) = \mp x(\mu + x^2)$$

which has a unique solution $x = 0$ for $\mu > 0$. For $\mu > 0$ this fixed point is generic with index ∓ 1 and so for $\mu = 0$ the index is ∓ 1 by continuity of the index. Clearly $Df_0(0) = (1)$. The jth iterate of f_0 is given by

$$f_0^j(x) = x \pm jx^3 + 0(x^6)$$

and so the fixed point equation is

$$x - f_0^j(x) = \mp x(jx^2 + 0(x^5))$$

Again we see that the origin is an isolated fixed point of all iterates of f_0 and so $f_0 \in \mathcal{B}^1$.

Let $J(\lambda,h)$ denote the $k \times k$ Jordan block matrix

$$J(\lambda,k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

where λ is any complex number, let d_k denote the k-dimensional row vector $(1,0,\ldots,0)$ and let e_k denote the k-dimensional column vector $(0,\ldots,0,1)^T$. In many cases we shall write $J(\lambda)$ for $J(\lambda,k)$, d for d_k and e for e_k when the dimension is clear from the context.

EXAMPLE 2. Let $\alpha = -1,0,1$, then there exists an $f \in \mathcal{B}^n$ such that $Df(0) = J(1,n)$ and with index vector αL_1.

Let $\alpha = -1,0,1$ be given. In the previous example we gave a scalar function $f\mu(x) = x + h(x,\mu)$ which had generic fixed points for $\mu \neq 0$ such that the sum of the indices of these fixed points is α. Moreover as $\mu \to 0$ these generic fixed points tend to the origin and so for $\mu = 0$ the origin is a fixed point of index α.

Let y be an n-vector and consider

$$g_\mu(y) = J(1)y + eh(\mu,dy)$$

The fixed point equation for this function is

$$0 = y - g_u(y)$$

or equivalently

$$\left.\begin{aligned} 0 &= -y_2 \\ 0 &= -y_3 \\ &\vdots \\ 0 &= -y_n \end{aligned}\right\}$$
$$0 = -h(\mu,y_1)$$

Thus if $\xi(\mu)$ is a fixed point of f_μ then $h(\mu,\xi(\mu)) = 0$ and so $y_2 = \ldots = y_n = 0$, $y_1 = \xi(\mu)$ is a fixed point of $g_\mu(y)$. If $\xi(\mu)$ is a generic fixed point of index $+1$ (resp. -1) then $h_x(\mu,\xi(\mu)) > 0$ (resp. < 0). Now

$$\det\begin{vmatrix} 0 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & -1 \\ -h_x(\mu,\xi) & 0 & 0 & & 0 & 0 \end{vmatrix} = -h_x(\mu,\xi)$$

Thus if $\xi(\mu)$ is a generic fixed point of index $+1$ (resp. -1) of f_μ then $y_2 = \ldots = y_n = 0$, $y_1 = \xi(\mu)$ is a generic fixed point of g_μ of index $+1$ (resp. -1). Thus for $\mu \neq 0$ the functions f_μ and g_μ have the same number of generic fixed points with the same index and for both functions the fixed points tend to the origin as $\mu \to 0$. Thus the index of the origin for both f_μ and g_μ are the same or g_μ has index vector αL_1.

EXAMPLE 3. Let $\alpha = 0$ or -1, then there is an $f \in \mathcal{B}^1$ such that $Df(0) = (-1)$ and with index vector $L_1 + \alpha L_2$.

If $Df(0) = (-1)$ then $\mathrm{ind}(f,0) = 1$ and so the examples given below must have $\mathrm{ind}(f^2,0) = 1$ or -1.

Consider $f_\mu(x) = -x + \mu^2 x \pm x^3$. In order to compute the iterates of f_μ it is convenient to scale the variables by making the change of variables $x = \mu y$ and $f_\mu = \mu g_\mu$ so that

$$g_\mu(y) = -y + \mu^2 y(1 \pm y^2)$$

and

$$g_{\mu^2}(y) = y - 2\mu^2 y(1 \pm y^2) + 0(\mu^4)$$

Thus the fixed point equation is

$$h_\pm(y,\mu) = (y - g_\mu^2(y))/2\mu^2 = y(1 \pm y^2) + 0(\mu)$$

Clearly $h_\pm(0,0) = 0$, $h_-(\pm 1,0) = 0$, $(\partial h/\partial y) \pm (0,0) = 1$ and $(\partial h/\partial y) -(\pm 1,0) = -1$. Thus the implicit function theorem yields functions $\xi_0 = 0$ and $\xi_\pm(\mu) = \pm 1 + 0(\mu)$ which satisfy $h_\pm(\xi_0, \ ) = h_-(\xi_\pm(\mu),\mu) = 0$ for small μ. Thus g_μ^2 has one or three fixed points for small $\mu \neq 0$ depending on whether the $+$ or $-$ sign is taken. The sum of the indices of these fixed points is $+1$ or -1, again depending on the sign taken. Thus f_μ^2 has either $\xi_0 = 0$ or $\xi_0 = 0$ and $\mu\xi_\pm(\mu)$ as fixed points with total index either $+1$ or -1. These fixed points of f_μ^2 tend to the origin as $\mu \to 0$ and so $\mathrm{ind}(f_0^2,0) = \pm 1$. As in Example 1 it is not hard to show that $f_0 \in \mathcal{B}^1$ and so the index vector for f_0 is $L_1 + \alpha L_2$ where $\alpha = 0$ or -1.

Implicit in all scale arguments is the assertion that all solutions have been found. This assertion is based on the Newton polygon method which gives rise to the scaling.

EXAMPLE 4. Let $\alpha = 0$ or -1, then there is an $f \in {}^n$ such that $Df(0) = J(-1,n)$ and with index vector $L_1 + \alpha L_2$.

Here we make a slight change in the function of the previous example so that the scaling works correctly. Let $h(\mu,x) = \mu^{2n}x \pm x^{2n+1}$ and consider

$$f_\mu(y) = J(-1)y + eh(\mu,dy)$$

Scale by $y_1 \to \mu y_1$, $y_2 \to \mu^2 y_2, \ldots, y_n \to \mu^n y_n$ (here we do not introduce new notation for the scaled variables as in Example 2) so that

$$f_\mu(y) = \begin{pmatrix} -y_1 + \mu y_2 \\ \vdots \\ -y_{n-1} + \mu y_n \\ -y_n + \mu^{n+1} y_1(1 \pm y_1^{2n}) \end{pmatrix}$$

and

$$f_\mu^2(y) = \begin{bmatrix} y_1 - 2\mu y_2 + O(\mu^2) \\ \vdots \\ y_{n-1} - 2\mu y_n + O(\mu^2) \\ y_n - 2\mu^{n+1} y_1(1 \pm y_1^{2n}) + O(\mu^{n+2}) \end{bmatrix}$$

The fixed point equations become

$$0 = (f_\mu^2 - y_1)/2\mu \quad = y_2 + O(\mu)$$
$$\vdots$$
$$0 = (f_{\mu n-1}^2 - y_{n-1})/2\mu = y_n + O(\mu)$$
$$0 = (f_{\mu n}^2 - y_n)/2\mu^{n+1} = y_1(1 \pm y_1^{2n}) + O(\mu)$$

for $\mu = 0$ these equations have the solutions $y_1 = y_2 = \ldots = y_n = 0$ and $y_1 = \pm 1$, $y_2 = \ldots = y_n = 0$ if the minus sign is taken. The fixed point at the origin has index $(-1)^{n-1}$ while the fixed points at ± 1 have index $(-1)^n$. Proceed now as in the previous examples to show that the unscaled f_0^2 has an isolated fixed point at the origin with index ± 1 and so the index vector of f_0 is $L_1 + \alpha L_2$ where $\alpha = 0$ or -1.

EXAMPLE 5. Let α and m be integers with $m \geq 3$ and let λ be an mth root of unity. Let A be a 2×2 matrix with eigenvalues $\lambda, \bar{\lambda}$. Then there exists an $f \in \mathcal{B}^2$ such that $Df(0) = A$ and with index vector $L_1 + \alpha L_m$.

Case 1. $\alpha \geq 1$. Let z be a complex variable and consider $f_\mu(z) = \lambda z + \mu^{\alpha m} z - z^{\alpha m+1}$ as a function from $C = R^1 \times R^1$ into itself. Scale by $z \to \mu z$ and so $f_\mu(z) = \lambda z + \mu^{\alpha m} z(1 - z^{\alpha m})$ and $f_\mu^m(z) = z + m\lambda^{m-1}\mu^{\alpha m} z(1 - z^{\alpha m}) + O(\mu^{\alpha m+1})$. Thus the fixed point equation is

$$(\partial(u,v)/\partial(x,y)) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x = u_x^2 + u_y^2 > 0$$

by virtue of the Cauchy-Riemann equations. Thus in the unscaled variables these are $m+1$ fixed points of f_μ^m of index $+1$ for $\mu > 0$ which tend to the origin as $\mu \to 0$. Thus $\operatorname{ind}(f_0^m, 0) = \alpha m + 1$ or the index vector of f_0 is $L_1 + \alpha L_m$.

Case 2. $\alpha \leq 1$. Consider $f_\mu(z) = \lambda z + z(\mu^{-\alpha m} - \bar{z}^{1-\alpha m})$. As above scale by $z \to \mu z$ and compute the fixed point equation

$$0 = (z - f_\mu^m(z))/m\lambda^{m-1}\mu^{-\alpha m} = -z(1 - \bar{z}^{-\alpha m}) + O(\mu)$$

This equation has the origin and the $-m$ roots of unity $\xi_1, \dots, \xi_{-\alpha m}$ as solutions when $\mu = 0$ as in the above example. This function is not analytic and so we must compute the Jacobian. Instead of using $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ as coordinates, use z and $\bar{z}$. If $h(z,\bar{z}) = -z(1 - \bar{z}^{-\alpha m})$ then

$$(\partial(h,\bar{h})/\partial(z,\bar{z})) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \quad \text{when } z = \bar{z} = 0$$

$$= \begin{vmatrix} 0 & -\alpha m \xi \bar{\xi}^{-\alpha m - 1} \\ -\alpha m \bar{\xi} \xi^{-\alpha m - 1} & 0 \end{vmatrix} = -\alpha^2 m^2$$

$$\text{when } z = \xi, \ \bar{z} = \bar{\xi} \text{ an } -\alpha m \text{ root of unity}$$

Then the fixed point at the origin has index +1 and the index of the fixed points at the roots of unity are -1. Thus as before the unscaled f_0 has index vector $L_1 + \alpha L_m$.

Case 3. $\alpha = 0$. Consider $f_\mu(z) = \lambda z + \mu^2 z + z^2 \bar{z}$. Scale $z \to \mu z$ and compute $0 = (z - f_\mu^n(z))/m\lambda^{m-1}\mu^2 = -z(1 + |z|^2) + O(\mu)$. Thus there is only one fixed point at the origin and it has index +1. Therefore the index vector of f_0 is L_1.

EXAMPLE 6. Let α and $m \geq 3$ be integers and λ an mth root of unity. Let A be a $2k \times 2k$ real matrix which is similar to $\operatorname{diag}(J(\lambda,k), J(\bar{\lambda},k))$ over the complex numbers. Then there exists an $f \in \mathcal{B}^{2k}$ such that $Df(0) = A$ and with index vector $L_1 + \alpha L_m$.

Let z be a complex k-vector and $f_\mu: C^k = R^{2k} \to C^k = R^{2k}$ given by

Case 1. $\alpha \geq 1$: $f_\mu(z) = J(\lambda)z + e(\mu^{\alpha m+2k}dz - (dz)^{\alpha m+1}(dzd\bar{z})^k)$

Case 2. $\alpha \leq 1$: $f_\mu(z) = J(\lambda)z + e(\mu^{2k-\alpha m}d\bar{z} - (d\bar{z})^{1-\alpha m}(dzd\bar{z})^k)$

Case 3. $\alpha = 0$: $f_\mu(z) = J(\lambda)z + e(\mu^{2k}dz + dz(dzd\bar{z})^k)$

Scale by $z_i \to \mu^i z_i$ and proceed as in the previous examples.

EXAMPLE 7. Let α be an integer, $n = 2$ and $A = I$ the 2×2 identity matrix. Then there exists an $f \in \mathcal{B}^2$ such that $Df(0) = A$ and with index vector αL_1.

Let z be a complex variable and $f_\mu: C = R^2 \to C = R^2$ be given by

Case 1. $\alpha \geq 1$: $f_\mu(z) = z + \mu z + z^\alpha(z\bar{z})$

Case 2. $\alpha \leq -1$: $f_\mu(z) = z + \mu\bar{z} + \bar{z}^{-\alpha}(z\bar{z})$

Case 3. $\alpha = 0$: $f_\mu(z) = z + \mu + z\bar{z}$

EXAMPLE 8. Let k,ℓ and α be integers with $k,\ell \geq 1$ and let $A = \mathrm{diag}(J(1,k),J(1,\ell))$. Then there exists an $f \in \mathcal{B}^{k+\ell}$ such that $Df(0) = A$ and the index vector of f is αL_1.

Let the f_μ of the previous example be of the form

$$f_\mu(\xi + i\eta) = (\xi + a(\mu,\xi,\eta)) + i(n + b(\mu,\xi,\eta))$$

Now let $x \in R^k$ and $y \in R^\ell$ and consider the map

$$g_\mu(x,y) = \begin{pmatrix} J(\lambda,k)x + e_k a(\mu,d_k x,d_\ell y) \\ J(\lambda,\ell)y + e_\ell b(\mu,d_k x,d_\ell y) \end{pmatrix}$$

The map $g_0 \in \mathcal{B}^{k+\ell}$, $Dg_0(0) = A$ and the index vector of G_0 is αL_1.

EXAMPLE 9. Let α be an integer and $A = -I$. Then there exists an $f \in \mathcal{B}^2$ such that $Df(0) = A$ and with index vector $L_1 + \alpha L_2$.

Let z be a complex variable and f_μ defined by

Case 1. $\alpha \geq 1$: $f_\mu(z) = -z + \mu z + z^{\alpha+1}$

Case 2. $\alpha \leq -1$: $f_\mu(z) = -z + \mu\bar{z} + \bar{z}^{1-\alpha}$

Case 3. $\alpha = 0$: $f_\mu(z) = -z + \mu + (z\bar{z})z$

EXAMPLE 10. Let k,ℓ and α be integers with $k,\ell \geq 1$ and let $A = \mathrm{diag}(J(-1,k),J(-1,\ell))$. Then there exists an $f \in \mathcal{B}^{\overline{\ell+k}}$ such that $Df(0) = A$ and with index vector $L_1 + \alpha L_2$.

Proceed as in Example 8 using the functions Example 9.

EXAMPLE 11. Let α,β,γ,ℓ and m be integers with $\ell,m \geq 3$ and $(\ell,m) = 1$. Let λ be an ℓth root of unity, ν an mth root of unity and A similar to $\mathrm{diag}(\lambda,\bar{\lambda},\nu,\bar{\nu})$ over the complex numbers. Then there exists an $f \in \mathcal{B}^4$ with $Df(0) = A$ and whose index vector is $L_1 + \alpha L_\ell + \beta L_m + \gamma L_{\ell m}$.

Let a,b,c be positive integers and z,w be complex variables. Consider the maps given below as maps of $R^4 = C^2$ into itself which depend on a small parameter μ.

$$f_\mu(z) = z\{\lambda + (\mu^{a\ell} - z^{a\ell})(4\mu^{cm} - w^{cm})$$

$$g_\mu(w) = w\{\nu + (\mu^{bm} - w^{bm})(4\mu^{\ell} - z^{\ell})$$

Scale by $z \to \mu z$ and $w \to \mu w$ to obtain

$$f_\mu(z) = \lambda z + \mu^{ab\ell m}(1 - z^{a\ell})(4 - w^{cm})$$

$$g_\mu(w) = \nu w + \mu^{b\ell m}(1 - w^{bm})(4 - z^{\ell})$$

The ℓth iterate of this map is

$$f_\mu^{\ell}(z) = z + \ell\mu^{ac\ell m}z(1 - z^{a\ell})(4 - w^{cm}) + \ldots$$

$$g_\mu^{\ell}(w) = \nu^{\ell}w + \ldots$$

where the omitted terms are higher order in μ than the displayed terms. Since $(\ell,m) = 1$ and ν is an mth root of unity $\nu^{\ell} \neq 1$. Thus the fixed point equations are

$$0 = (z - f_\mu^\ell(z))/\ell\mu^{ac\ell m} = -z(1 - z^{a\ell})(4 - w^{cm}) + \ldots$$

$$0 = (w - g_\mu^\ell(w)) = (1 - \nu^\ell)w + \ldots$$

and these equations have solutions $z = w = 0$ and $w = 0$, $z = \xi_1, \ldots, \xi_{a\ell}$ where ξ_j is an $(a\ell)$th root of unity when $\mu = 0$. As before it is easy to invoke the implicit function theorem to show that these equations have $1 + a\ell$ solutions for small μ. As before these fixed points have index +1. Thus the index of the unique fixed point at the origin of the unscaled map for $\mu = 0$ has index $1 + a\ell$.

Similarly the index of the unique fixed point at the origin for the mth iterate of the unscaled map for $\mu = 0$ has index $1 + bm$.

Now compute the (ℓm)th iterate and the fixed point equations become

$$0 = (z - f_\mu^{\ell m}(z))/\ell m\mu^{ac\ell m} = -z(1 - z^{a\ell})(4 - w^{cm}) + \ldots$$

$$0 = (w - g_\mu^{\ell m}(w))/\ell m\mu^{b\ell m} = -w(1 - w^{bm})(4 - z^\ell) + \ldots$$

where $\mu = 0$ this system of equations is equivalent to the systems

$$\begin{aligned} z = w &= 0 \\ z = 0, \quad w^{bm} &= 1 \\ z^{a\ell} = 1, \quad w &= 0 \\ z^{a\ell} = 1, \quad w^{bm} &= 1 \\ z^\ell = 4, \quad w^{cm} &= 4 \end{aligned}$$

and so there are a total of $1 + bm + a\ell + (ab + c)\ell m$ fixed points and they all have index +1. Using the now familiar argument the index of the isolated fixed point at the origin for the unscaled equation when $\mu = 0$ is $L_1 + aL\ell + bL_m + (ab + c)L_{\ell m}$.

Thus we have given the required example when $\alpha \geq 1$, $\beta \geq 1$ and $\gamma > \alpha\beta$. The other cases are similar. In order to achieve fixed points with negative index simply conjugate the appropriate term in the formulas for f_μ and g_μ. For example if α is negative simply replace $(\mu^{a\ell} - z^{a\ell})$ by $(\mu^{a\ell} - \bar{z}^{a\ell})$ in the definition of f_μ. This will yield a function in $\mathcal{B}^4$ with index vector $L_1 - aL_\ell + bL_m + (c - a\ell)L_{\ell m}$.

Even though the general example is not given in this note, enough cases are given to show all the technical difficulties that can arise.

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REFERENCES

1. J. Mallet-Paret and J. A. Yorke, Degenerate orbits as clusters of simple orbits, and a bifurcation invariant, preprint.

2. M. Shub and D. Sullivan, A remark on the Lefschetz fixed point formula for differentiable maps, *Topology 13*: 189-191 (1974).