THE GLOBAL PHASE STRUCTURE OF THE THREE DIMENSIONAL ISOSCELES THREE BODY PROBLEM WITH ZERO ENERGY

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Abstract. We study the global flow defined by the three-dimensional isosceles three-body problem with zero energy. A new set of coordinates and a scaled time are introduced which alow the phase space to be compactified by adding boundary manifolds. Geometric arguments gives an almost complete sketch of the global phase portrait of this gravational system.

1. Introduction

In this paper, we apply the method developed in [1] to study the flow on the collision manifold of the three-dimensional isosceles three-body problem. The isosceles three-body problem has played an important role in the study of the Newtonian gravitational system for last twenty years [2, 3, 4, 5]. Our goal here is a clear picture of the phase structure of this flow.

We assume that three particles with masses $m_1 = m_2 = 1/2, m_3 = \beta$ move in Euclian 3-space with coordinates (x, y, z_2), $(-x, -y, z_2)$, $(0, 0, z_1)$ and initial velocities $(dx/dt, dy/dt, dz_2/dt)$, $(-dx/dt, -dy/dt, dz_2/dt)$, $(0, 0, dz_1/dt)$ respectively. Th symmetries of the motion will be maintained forever.

Use x, y and $z = z_1 - z_2$ as the coordinates of positions. Fix the center of mass at the origin, $\beta z_1 + z_2 = 0$, and therefore linear momentum is zero, $\beta dz_1/dt + dz_2/dt = 0$. The kinetic energy of this system is

$$\begin{aligned} T &= \frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2 + \beta\left(\frac{dz_1}{dt}\right)^2\right\} \\ &= \frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{\beta}{1+\beta}\right)\left(\frac{dz}{dt}\right)^2\right\} \end{aligned}$$

and the potential function expressed in x, y, z is

$$U = \frac{1}{8}\frac{1}{(x^2+y^2)^{1/2}} + \beta\frac{1}{(x^2+y^2+z^2)^{1/2}}.$$

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Let $q = (x, y, z)^T$ and $p = M(dx/dt, dy/dt, dx/dt)^T$ where M is the 3×3 matrix $M = \mathrm{diag}(1, 1, \epsilon)$ and $\epsilon = \beta/(1+\beta)$. The equations of motion of the system are

$$\frac{dq}{dt} = M^{-1}p, \qquad \frac{dp}{dt} = \nabla U(q).$$

Total energy $T - U$ is an integral and so equal to a constant h and the z-component of angular monentun is an integral so $k \cdot (q \times p) = c$ is constant.

In the next section we will introduce a sequence of changes of variables which are used in Section 3 to define the the collision manifold. This sequence of transformations will make the phase space a compact cube in three dimensional space with a non-trivial fictitious flows on the boundaries. In Section 4 we will discuss the final evolution of the solutions. Finally in the last section a sketch of the phase portrait will be given.

2. Changing Variables

In order to attach a boundary manifold to the phase space we shall preform a series of changes of variables. First, we scale the position vector, p, the momentum vector, q, and time, t, by the potential U to introduce new variables u, F, G and a new time τ by

$$\begin{aligned} &u = (2U(q))^{-1}, \quad d\tau = u^{-3/2}dt, \\ &F = u^{-1}q, \qquad\quad G = u^{1/2}p \end{aligned} \tag{1}$$

Note that F has only the dimension of mass and measures the geometry of the system, whereas u has the dimension of length/mass and so measures the size of the system. The variable u plays the role of I, the moment of inertia, of McGehee's scaling. The equations of u, F, G, are

$$\begin{aligned} \frac{du}{d\tau} &= -2(M^{-1}G, \nabla U(F))u, \\ \frac{dF}{d\tau} &= M^{-1}G + 2(M^{-1}G, \nabla U(F))F, \\ \frac{dG}{d\tau} &= \nabla U(F) - (M^{-1}G, \nabla U(F))G, \end{aligned} \tag{2}$$

where

$$G^T M^{-1} G = 1, \qquad U(F) = 1/2 - uh \tag{3}$$

and

$$U(F) = \frac{1}{8}\frac{1}{(F_1^2 + F_2^2)^{1/2}} + \beta\frac{1}{(F_1^2 + F_2^2 + F_3^2)^{1/2}}.$$

In these coordinates the conservation of angular mometum yields: $F_1G_2 - F_2G_1 = cu^{-1/2}$.

Next let

$$\begin{aligned} F_1 &= r\sin\phi \quad G_1 = R\sin\vartheta \\ F_2 &= r\cos\phi \quad G_2 = R\cos\vartheta. \end{aligned} \tag{4}$$

The equations for (r, R, ϕ, ϑ) are

$$
\begin{aligned}
\frac{dr}{d\tau} &= R\cos(\vartheta - \phi) + 2(M^{-1}G, \nabla U(F))r \\
\frac{dR}{d\tau} &= -(1/8r^2 + \beta r/(r^2 + F^3)^{3/2})\cos(\vartheta - \phi) - (M^{-1}G, \nabla U(F))R \\
\frac{d\phi}{d\tau} &= \frac{R\sin(\vartheta - \phi)}{r} \\
\frac{d\vartheta}{d\tau} &= \frac{[(1/8r^2 + \beta r/(r^2 + F^3)^{3/2})\sin(\vartheta - \phi)]}{R}
\end{aligned}
\tag{5}
$$

Note that, for the complete set of equations of motion, we also need the equations for F_3, G_3 and u given in (2).

We again have the following restrictions

$$
R^2 + G_3^2 = 1, \qquad 1/8r + \beta/(r^2 + F_3^2)^{3/2} = 1/2 - uh \tag{6}
$$

and the angular momentum integral

$$
u^{1/2}(F_1G_2 - F_2G_1) = c.
$$

The manifold defined by (6) with $u = 0$ is the celebrated collision manifold. The flow defined by (5) extends to this boundary and the boundary is invariant. The flow defined by (5) on the boundary is called the fictitious flow. It reflects the near-collision behavior of the gravitational system.

It is easy to see that setting $h = 0$ in (6) will give us the same set of equations as obtained by setting $u = 0$. So what has been found on the collision manifold is not that fictitious after all. It is exactly the flow of the original problem with zero energy. From now on, we will only discuss this flow. We see that with $u = 0$(or $h = 0$), (6) gives two constrains on variables $(r, R, \phi, \vartheta, F_3, G_3)$. Also notice that the equations in (5) are independent of u and ϕ and ϑ only as $\vartheta - \phi$. These observations will reduce the dimension of our problem to three.

Again, change variables by

$$
\begin{aligned}
&r = w\sin\xi, \quad R = \sin\eta, \\
&F_3 = w\cos\xi, \quad G_3 = \epsilon^{1/2}\cos\eta, \\
&\Phi = \vartheta - \phi.
\end{aligned}
$$

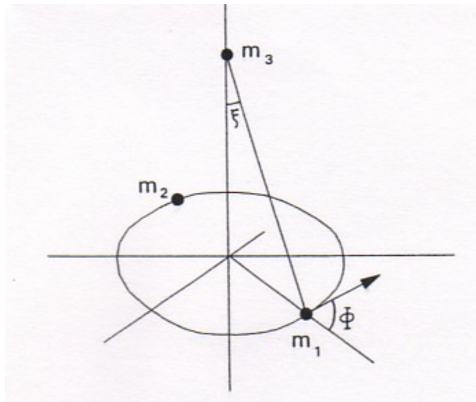


FIGURE 1. Coordinates for the isosceles problem in three-space.

We see that $w = (1 + 8\beta \sin\xi)/4\sin\xi$. The equations for (ξ, η, Φ) are

$$\begin{aligned} \frac{d\xi}{d\tau} &= \frac{4\sin\xi(\cos\xi\sin\eta\cos\Phi - \epsilon^{-1/2}\sin\xi\cos\eta)}{(1+8\beta\sin\xi)}, \\ \frac{d\eta}{d\tau} &= \frac{16[\beta\epsilon^{-1/2}\sin\eta\cos\xi\sin^2\xi - \cos\eta(1/8+\beta\sin^3\xi)\cos\Phi]}{(1+8\beta\sin\xi)^2}, \\ \frac{d\Phi}{d\tau} &= \frac{16\sin\Phi[(1/8+\beta\sin^3\xi) - (1/4+2\beta\sin\xi)\sin^2\eta]}{(1+8\beta\sin\xi)^2\sin\eta}. \end{aligned} \tag{7}$$

Let $\sin\eta d\tau' = d\tau$ and rewrite τ' as τ. The equations of motion become

$$\begin{aligned} \frac{d\xi}{d\tau} &= \frac{4\sin\xi\sin\eta(\cos\xi\sin\eta\cos\Phi - \epsilon^{-1/2}\sin\xi\cos\eta)}{(1+8\beta\sin\xi)}, \\ \frac{d\eta}{d\tau} &= \frac{16\sin\eta[\beta\epsilon^{-1/2}\sin\eta\cos\xi\sin^2\xi - \cos\eta(1/8+\beta\sin^3\xi)\cos\Phi]}{(1+8\beta\sin\xi)^2}, \\ \frac{d\Phi}{d\tau} &= \frac{16\sin\Phi[(1/8+\beta\sin^3\xi) - (1/4+2\beta\sin\xi)\sin^2\eta]}{(1+8\beta\sin\xi)^2}. \end{aligned} \tag{8}$$

The three quanities ξ, η, Φ are the coordinates we shall use to study this problem. The angle ξ is half the vertex angle at m_3 of the configuration triangle. The angle Φ is the angle between the radius vector and the velocity vector of particle m_1. The dimensionless quanity η measures the ratio of the velocity of particle m_3 and the radial velocity of particle m_1. Refer to Figure 2.

For any point of the original phase space D_0

$$D_0 = (0, \pi) \times (0, \pi) \times (0, \pi)$$

there is a solution of $h = 0$. However, (8) now has been defined on D:

$$D = [0, \pi] \times [0, \pi] \times [0, \pi],$$

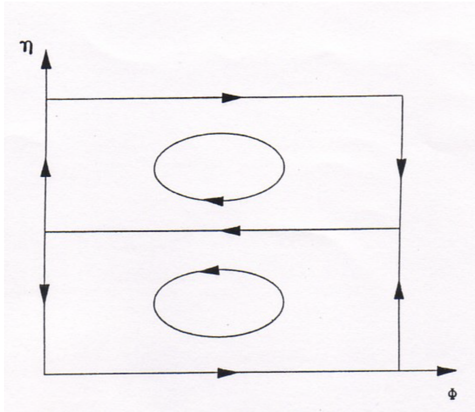


FIGURE 2. Escape boundary, $\xi = 0$ or π.

which is the closure of D_0. This is our compactified phase space.

3. The boundary manifolds

In this section we shall look at the flow on the various boundaries. First set $\xi = 0$ in (2.8) so the equations become

$$\frac{d\eta}{d\tau} = -2\sin\eta\cos\eta\cos\Phi = -\sin 2\eta\cos\Phi,$$
$$\frac{d\Phi}{d\tau} = (2 - 4\sin^2\eta)\sin\Phi = 2\cos 2\eta\sin\Phi.$$

These equations admit the integral $\sin 2\eta\sin\Phi = C$, so the phase portrait is as given in Figure 3. This is the flow in the limit as particle m_3 goes upward to infinity. The flow on the boundary $\xi = \pi$ is exactly the same.

Consider the boundary defined by setting $\Phi = 0$ in (2.8). The equations on this boundary are

$$\frac{d\xi}{d\tau} = \frac{4\sin\xi\sin\eta(\cos\xi\sin\eta - \epsilon^{-1/2}\sin\xi\cos\eta)}{(1+8\beta\sin\xi)}$$
$$\frac{d\eta}{d\tau} = \frac{16\sin\eta[\beta\epsilon^{-1/2}\sin\eta\cos\xi\sin^2\xi - \cos\eta(1/8+\beta\sin^3\xi)]}{(1+8\beta\sin\xi)^2}.$$

This is half of the triple collision manifold for the planar isosceles three-body problem. It corresponds to the collision part – see Figure 3.

The other half, the boundary $\Phi = \pi$, corresponds to the ejection part – see Figure 3.

Consider the boundary defined by $\eta = 0$ or π. When $\eta = 0$ or π the equations are

$$\frac{d\xi}{d\tau} = 0$$
$$\frac{d\Phi}{d\tau} = \frac{16\sin\Phi(1/8+\beta\sin^3\xi)}{(1+8\beta\sin\xi)^2}.$$

Figure 3 gives the phase portrait for these boundaries.

Putting all these pictures together, we have Figure 3.

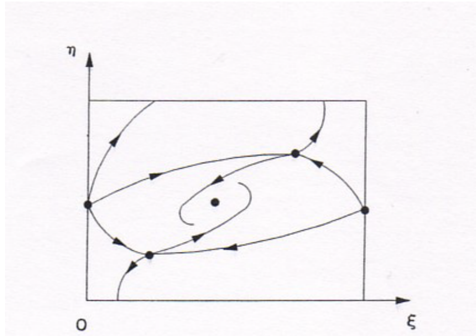


FIGURE 3. Collision boundary, $\Phi = 0$.

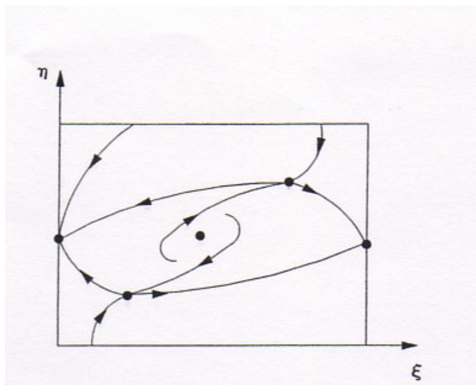


FIGURE 4. Ejection boundary, $\Phi = \pi$.

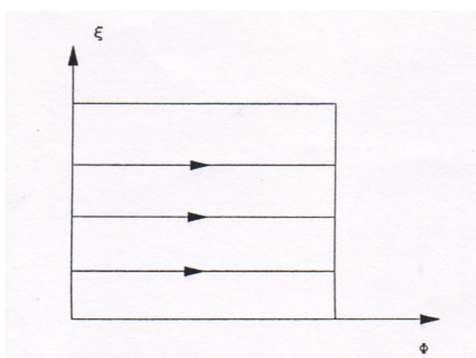


FIGURE 5. Boundary $\eta = 0$ or π.

4. The final evolution

First we claim that there is no rest point inside of the phase space D_0. By (2.8) an internal rest point would satisfy

$$\cos\xi \sin\eta \cos\Phi - \epsilon^{-1/2} \sin\xi \cos\eta = 0, \tag{1}$$

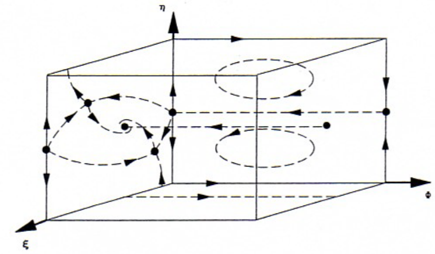


FIGURE 6. All boundraies.

$$\beta\epsilon^{-1/2}\sin\eta\cos\xi\sin^2\xi - \cos\eta(1/8+\beta\sin^3\xi)\cos\Phi = 0, \tag{2}$$

$$(1/8+\beta\sin^3\xi) - (1/4+2\beta\sin\xi)\sin^2\eta = 0. \tag{3}$$

From (1) and (2)

$$\beta\cos^2\xi\sin^2\eta\sin\xi = (1/8+\beta\sin^3\xi)\cos^2\eta. \tag{4}$$

Combining (3) and (4) gives

$$\beta\cos^2\xi\sin^2\eta\sin\xi = \sin^2\eta(1/4+2\beta\sin\xi)\cos^2\eta.$$

Therefore,

$$\beta\cos^2\xi\sin\xi = (1/4+2\beta\sin\xi)\cos^2\eta \tag{5}$$

Together (5) and (3) yield $1/8+\beta\sin\xi = 1/4+2\beta\sin\xi$, so $1/8+\beta\sin\xi = 0$. Since $\xi \in (0,\pi)$ and $\beta > 0$ this is a contradiction.

Next we claim that the flow is gradient-like. Define

$$P = \frac{(F,G)}{(F^TMF)^{-1/4}}.$$

From (2.5) we have

$$\frac{dP}{d\tau} = \frac{[1-(F,G)^2/(F^TMF)]}{(F^TMF)^{-1/4}}$$

or

$$\frac{dP}{d\tau'} = \frac{\sin\eta[1-(F,G)^2/(F^TMF)]}{(F^TMF)^{-1/4}}.$$

Let $W = M^{1/2}F$ and $Z = M^{-1/2}G$. Since $G^TMG = 1$ we have $\| Z \| = 1$, $| (F,G)^2/(F^TMF) | = | (W/\| W \|, Z) | \leq 1$. So $dP/d\tau \geq 0$. Thus the flow is gradient-like.

Note that

$$P = \chi(\xi)(\sin\xi\sin\eta\cos\Phi + \epsilon^{1/2}\cos\xi\cos\eta)\sin^{-1/2}\xi$$

where

$$\chi(\xi) = \frac{(1+8\beta\sin\xi)^{1/2}}{2(\sin^2\xi+\epsilon\cos^2\xi)^{1/4}}.$$

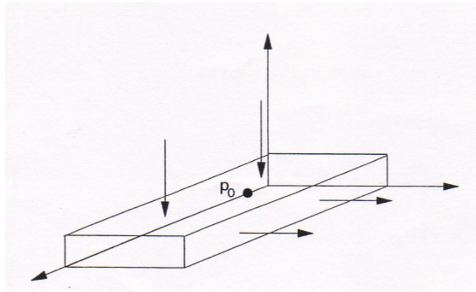


FIGURE 7. Flow near P_0.

Now what are the α- and ω-limit set of the solutions? Since the flow is gradient-like and there is no rest point inside of the phase space, the α- and ω-limit sets of the solutions will be in the boundaries of the cube.

All the points on the lines $\eta = 0, \pi$; $\Phi = 0, \pi$; $\xi \in (0, \pi)$ are rest points. Denote the collection of these lines by L.

Proposition 4.1. *No solution in D_0 has its α- or ω-limit set in the set L.*

Proof. We will prove the statement for the line $\eta = 0, \Phi = 0, \xi \in (0, \pi)$. The proof is similar for the other cases. Take any surface of the form

$$S_{\eta_0} = \{(\xi, \eta, \Phi)\eta = \eta_0, \Phi < \Phi_0, \xi \in (0, \pi)\}$$

where $\eta_0 < \epsilon, \Phi_0 < \epsilon$ for sufficiently small $\epsilon > 0$. See Figure 4. According to (2.8), we can take ϵ so small that $d\eta/d\tau < 0$ for any point on S_{η_0}.
Similarly, take

$$S_{\Phi_0} = \{(\xi, \eta, \Phi)\Phi = \Phi_0, \eta < \eta_0, \xi \in (0, \pi)\}$$

where $\eta_0 < \epsilon, \Phi_0 < \epsilon$. We can again make ϵ so small that for any point on $S_{\Phi_0}, d\Phi/d\tau > 0$. So η is decreasing and Φ is increasing inside the interior of the box with boundaries S_{η_0} and S_{Φ_0} and a solution entering this box must on S_{η_0} must leave it on S_{Φ_0}.

Now assume that a solution takes point p_0 on the line as an ω-limit point. This solution will intersect S_{η_0} infinitely many times, therefore it will have an ω-limit point on the surface $\Phi = 0$ with $\eta = \eta_0$. Because P increases along any non-trivial orbit on $\Phi = 0$(the flow is gradient-like), this is impossible.

Proposition 4.2. *The α- or ω-limit set of a solution inside of D_0 is one of the following:*

i) One of the rest point inside of the boundaries $\Phi = 0$ or $\Phi = \pi$.

ii) On the boundary $\xi = 0$.

iii) On the boundary $\xi = \pi$.

The proof of this proposition follows from the discussion above.

Look at the hyperbolic final evolution of the system. Assume that the ω-limit set of a solution is on the boundary $\xi = 0$. We see that as $t \to \infty$, the particle m_3 goes upward forever. So $z(t)$ increases and $dz(t)/dt > 0$ decreases monotonically. Also $P(t) \to \infty$ as $t \to \infty$ for this solution.

On the boundary $\xi = 0$, we have a special collection of solutions. It is the lines $\Phi = 0, \eta \in (0, \pi/2)$; $\Phi = \pi$, $\eta \in (0, \pi/2)$; $\eta = 0$ and $\pi/2$, $\Phi \in (0, \pi)$ together with four rest points $(0, \pi/2, 0)$; $(0, 0, 0)$; $(0, \pi/2, \pi)$; $(0, 0, \pi)$. Denote it as ℓ. See Figure 3 where ℓ is the lower loop made up of the four rest points and the four trajectories connecting them. We claim that if one point of ℓ is a limit point of a solution, so will be all the other points of ℓ .

Now assume that such a solution exists. $P \to \infty$ and $\xi \to 0$ as $\tau \to \infty$. By the definition of P we see $\cos\eta/(\tan\xi)^{1/2} \to \infty$.

Go back to the original definition of η and ξ in Section 2.. We will have $z^{1/2}dz/dt \to \infty$. So for any $C > 0, z(t) > Ct^{2/3}$ for $t > T$ sufficiently large. This can happen only when the motion of m_3 is hyperbolic. See [6]. Therefore $dz(\infty)/dt = lim_{t\to\infty}(dz/dt) > 0$ for this solution.

But we will have $\tau_n \to \infty$, such that $\eta(\tau_n) \to \pi/2$. This simply means $(dx/dt)^2 + (dy/dt)^2 \mid_{\tau_n} \to \infty$. But this is impossible and so the points of ℓ are not limit points.

Let $Q = \sin 2\eta \sin \Phi$. From (2.8)

$$dQ/d\tau = 16\beta \sin^2\eta \sin\Phi \sin 2\xi(\epsilon^{-1/2}\cos 2\eta \sin\xi - \cos\xi \sin 2\eta \sin\Phi)/(1 + 8\beta \sin\xi)^2.$$

Proposition 4.3. *If the ω-limit of a solution is on the boundary $\xi = 0$, then $lim_{\tau\to\infty}Q(\tau) = Q_0$ exists and $0 < Q_0 < 1$.*

Proof. This solution can not take ℓ as its limit. So there is an $\epsilon' > 0$, such that $\xi(\tau) \to 0, \Phi(\tau) > \epsilon'$ and $\eta(\tau) > \epsilon'$ for $\tau \in [0, \infty).P(\tau) \to \infty$ as $\tau \to \infty$. Set $U = 1/P$ so

$$\frac{dU}{dt} = -P^{-2}\frac{dP}{dt} = \frac{-[1 - (F, G)^2/(F^T MF)]}{P^2(F^T MF)^{1/4}}.$$

Since $1 - (F, G)^2/(F^T MF) = 1 - (\sin\xi \sin\eta \cos\Phi + \epsilon^{-1/2}\cos\xi \cos\eta)^2/(\sin^2\xi + \epsilon \cos^2\xi)$ there is $aM(\epsilon') > 0$, such that $\mid 1 - (F, G)^2/(F^T MF) \mid > M(\epsilon')$. It will turn out that $\mid dQ/dU \mid < M'(\epsilon')$. Where $M'(\epsilon')$ is another constant related to $M(\epsilon')$. According to this inequality, the limit of Q exists as $\tau \to \infty$ $(U \to 0)$.

Q_0 can not be 1 since nearby $Q = 1, dQ/d\tau < 0$.

By this proposition, a solution takes either a rest point on the boundary $\Phi = 0$(the parabolic case) or a periodic solution on the boundary $\xi = 0$(or π)(the hyperbolic case) as its α- or ω-limit set.

The boundary manifolds $\eta = 0$ and π are rather artificial. They appear when we break the collision manifold of the planar problem into two pieces and paste them on the boundaries $\Phi = 0$ and $\Phi = \pi$. Taking the surface, say, $\eta = 0$ as the singular element, we can use the conception of block regularization to simplify the picture. This means we can identify the lines $\eta = 0, \Phi = 0$ and $\eta = 0, \Phi = \pi$ and ignore the surface $\eta = 0$. The phase space now is, topologically, $D^2 \times I$ and the flow on its boundary $S^1 \times I$ will be exactly the flow on the collision manifold of the planar problem. Here I = [0, 1], and D^2 is the two dimensional disk.

5. The McGehee manifold

Set $\Phi = 0$ in (2.7). The unbroken collision manifold is $S^1 \times I$, and the fictitious flow on it is

$$\frac{d\xi}{d\tau} = \frac{4\sin\xi(\cos\xi \sin\eta - \epsilon^{-1/2}\sin\xi \cos\eta)}{(1 + 8\beta \sin\xi)}$$

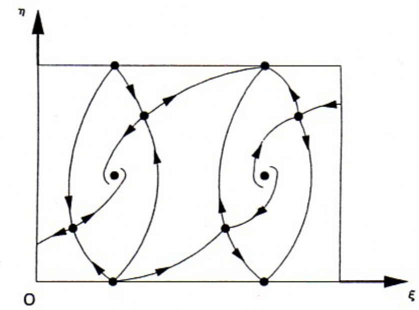


FIGURE 8. Flow on total collision manifold.

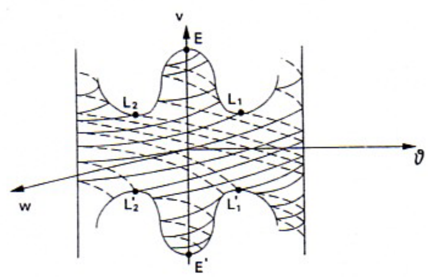


FIGURE 9. Flow on McGehee manifold.

$$\frac{d\eta}{d\tau} = \frac{16[\beta\epsilon^{-1/2}\sin\eta\cos\xi\sin^2\xi - \cos\eta(1/8+\beta\sin^3\xi)]}{(1+8\beta\sin\xi)^2}$$

The phase picture of this flow is in Figure 5.

Figure 5 looks different from the familiar picture of McGehee's collision manifold for the planar problem. The difference is caused by the different treatments of binary collision. McGehee regularizes binary collision. While we have a sub-boundary of binary collisions.

We will need more information about the flow in which binary collision is regularized. So we are going to study the picture on McGehee's manifold first, then come back to that of ours later.

The flow depends on β. Refer to [5]. Its picture is depicted in Figure 5 for small β. We will concentrate only on this picture from now on. For the case of larger β, the picture is different but the analysis is similar.

The 2-manifold in Figure 5 is defined in (ϑ, w, v) -space by equation

$$(v^2\cos^2\vartheta + w^2)/2 + U(\vartheta)\cos^2\vartheta = 0.$$

See [2] for the details of this equation.

There are six rest points in Figure 5. Those labeled, E, E' are the Eulerian points, and L_1, L_1', L_2, L_2' are the Langrangian points. E is a sink, E' is a source and L_1, L_1', L_2, L_2' are all saddles. Two curves $\gamma^+(L_1)$ and $\gamma^-(L_1)$ in Figure 5 form the stable manifold of L_1.

Denote by K as the curve $\vartheta = 0$ on this manifold and

$K^+ =$ all the points of K with $w > 0$,

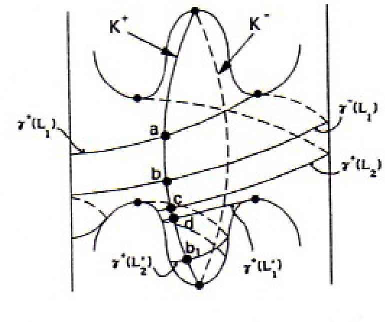


FIGURE 10. Stable and unstable manifolds with key points.

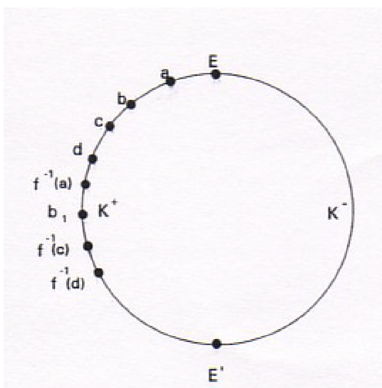


Figure 11. Boundary of Poincare section

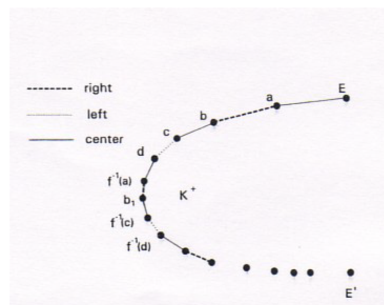


Figure 12. Orbits on Poincare section.

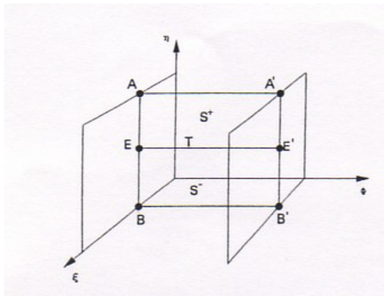
K^- = all the points of K with $w < 0$,
a = the last intersection of $\gamma^+(L_1)$ with K^+,
b = the last intersection of $\gamma^-(L_1)$ with K^+.
Similarly, let $\gamma^+(L_2)$ be the stable curve of L_2 depicted in Figure 5 and
c = the last intersection of $\gamma^+(L_2)$ with K^+.
We still need two more crucial points:
d = the last intersection of $\gamma^+(L_1')$ with K^+,
b_1 = the last intersection of $\gamma^+(L_2')$ with K^+.
Where, as depicted in Figure 5, $\gamma(L_1')$ is the stable curve of L_1'. $\gamma(L_2')$ is that of L_2'.
Let $F : K^+ \to K^+$ be the flow-defined Poincare map, and

$$\Gamma = \{a, b, c, d, b_1\} \cup \{F^{-n}(a), F^{-n}(c), F^{-n}(d), F^{-n}(b_1) : n \in Z^+\}.$$

In there natural order from E to E', the points of Γ can be listed as

$$E, a, b, c, d; f^{-1}(a), b_1, f^{-1}(c), f^{-1}(d); \ldots;$$
$$f^{-n}(a), f^{-n+1}(b_1), f^{-n}(c), f^{-n}(d); \ldots E'.$$

We also have a similar sequence on K^-. See Figure 5.



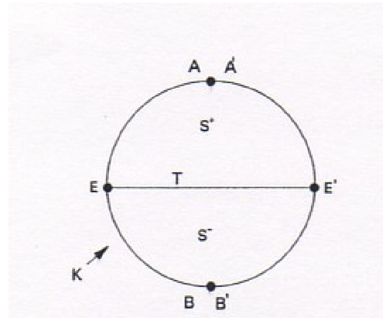


Figure 13. Poincare Section   Figure 14. Boundary of Poincare Section.

The points on $K^+ \backslash \Gamma$ have three possible destinations as $\tau \to \infty$ namely the right arm, the left arm or the Center. The set Γ divides K^+ into countable many intervals (see Figure 5). The destiny of these intervals are in the order of

$$center, right, center, left, center, right, center, left \ldots$$

points on the intervals $(f^{-n+1}(b_1),\ f^{-n}(c))$ and $(f^{-n}(d),\ f^{-n-1}(a))$ will go to the center, $(f^{-n}(a),\ f^{-n+1}(b_1))$ to the right and $(f^{-n}(c),\ f^{-n}(d))$ to the left.

Any two points on the same interval have the same intersection number (See its definition at the beginning of next segment), and the intersection numbers of these intervals are in the order of

$$\infty, 0, \infty, 0; \infty, 1, \infty, 1; \ldots; \infty, n, \infty, n; \infty, n+1, \infty, n+1; \ldots$$

6. The Global Picture

We now go back to the three-dimensional picture the phase space (ξ, η, Φ) in Figure 3 and define the intersection number. For a point p in D_0, let

$$\begin{aligned} p &= (\xi(p), \eta(p), \Phi(p)), \\ O(\tau, p) &= \text{ the solution at time } \tau \text{ which starts from p at time } \tau = 0, \\ O(p) &= \{O(\tau, p) : 0 < \tau < \infty\}, \\ S &= \{p : p \in D_0 \text{ and } \xi(p) = \pi/2\}. \end{aligned}$$

S will serve as a global Poincare section in our discussion. From now on, by a S-neighborhood U of p, we will mean that $p \in S$ and U is a neighborhood of p in S. Divide S into three parts by

$$\begin{aligned} S^+ &= \{p : p \in S, \eta(p) > \pi/2\}, \\ S^- &= \{p : p \in S, \eta(p) < \pi/2\}, \\ T &= \{p : p \in S, \eta(p) = \pi/2\}. \end{aligned}$$

We see that $S = S^+ \cup S^- \cup T$, and T itself is a solution of (2.8). See Figure 6.

Definition 6.1. *For a given point $p \in S$, let $N(p)$ = cardinality of $\{O(p) \cap S\}$. $N(p)$ is the intersection number of p.*

For the six rest points, easy calculation shows that E is a sink, E' is a source, L_1, L_2 are hyperbolic points with 2-dimensional stable manifold and 1-dimensional unstable manifold and L_1', L_2' are that with 1-dimensional stable manifold and 2-dimensional unstable manifold.

Denote the stable manifolds of L_1 and L_2 as $W^s(L_1)$ and $W^s(L_2)$ and $\Lambda = W^s(L_1) \cup W^s(L_2)$. In $D, W^s(L_1)(W^s(L_2))$ is a manifold with $\gamma^+(L_1)$ and $\gamma^-(L_1)(\gamma^+(L_2)$ and $\gamma^-(L_2))$ as its boundary. Clearly we have:

Proposition 6.1. *For any $p \in S, p \notin \Lambda$, there is a S-neighborhood U of p, such that for any $p' \in U, N(p') = N(p)$.*

The solutions which tend to the boundaries $\xi = 0$ form an open set in the phase space. So do the solutions which tend to the boundary $\xi = \pi$ and to the Eulerian rest points. So the intersection number changes only on $\Lambda \cap S$.

Since the vector field is transversal to S^+ and S^-, and Λ is an embedded two dimensional manifold inside of the phase space, we have:

Proposition 6.2. *$\Lambda \cap S^+$ and $\Lambda \cap S^-$ are one-dimensional embedded manifold in S^+ and S^- respectively.*

As we noted at the end of Section 2, the surfaces $\eta = 0$ and $\eta = \pi$ can be ignored and the phase space could be regarded, topologically, as $D^2 \times$I. Then S becomes the disk in Figure 6. Its boundary is exactly the circle K on McGehee's manifold we discussed in Section 5. We see that EAA'E' $= K^+$, EBB'E' $= K^-$ as well.

Now let us look to the global structure of the flow. For a given C^∞-curve c in S, $c(t) : (-\infty, \infty) \to S$, the α- and ω- limit set of $c(t)$ are defined by

$$\omega(c) = \{p : p \in \bar{S};\ \text{there is } t_n \to \infty \text{ such that } c(t_n) \to p\},$$

$$\alpha(c) = \{p : p \in \bar{S};\ \text{there is } t_n \to -\infty \text{ such that } c(t_n) \to p\},$$

where $\bar{S}$ is the closure of S. We also say that $c(t)$ is a curve connecting $\alpha(c)$ and $\omega(c)$.

Since $\Lambda \cap S = (\Lambda \cap S^+) \cup (\Lambda \cap S^-) \cup (\Lambda \cap T)$ and $\Lambda \cap T = \emptyset$, we see that $\Lambda \cap S = (\Lambda \cap S^+) \cup (\Lambda \cap S^-)$. Take B as a connected branch of $\Lambda \cap S$, either $B \subset \Lambda \cap S^+$ or $B \subset \Lambda \cap S^-$. Without loss of generality, we always assume $B \subset \Lambda \cap S^+ . B$ is an one-dimensional embedded manifold of S^+ by Proposition 5.

Proposition 6.3. *B has no limit point on T.*

Remember that T is a solution connecting E and E'. For any $q \in T$, there is a S-neighborhood U_q of q, such that all the points $q' \in U_q$ approch E. Therefore q can not be a limit point of B. Also, we have:

Proposition 6.4. *For two given points $p_1, p_2 \in B; N(p_1) = N(p_2)$.*

For a given point $p \in B$, take a small S-neighborhood U_p of p. For any $p \in U_p \cap B, O(p')$ will stay close to $O(p)$ therefore end up with L_1(or L_2) by the same intersection number. This means $N(p)$ can not be changed at any point of B. So we must have $N(p_1) = N(p_2)$.

Proposition 6.5. *B does not take E or E' as its limit point.*

Proof. Since the flow is gradient like and $P(E) > P(L_{1,2})$, E can not be a limit point of B.

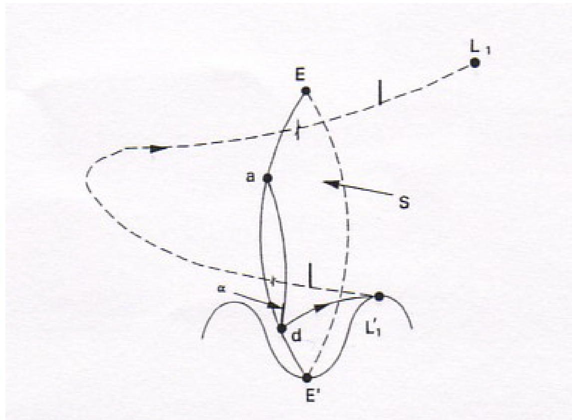


Figure 15. Connection between points a and d.

E' is a spiral-source. For any $n > 0$, there is a S-neighborhood U_n of E' such that $N(q) > n$ if $q \in U_n$. E' can not be a limit point of B because the intersection number of B is finite.

No limit points are located in S^+ since B is an embedded manifold in S^+. Also, B can not take any of the points of $K^+ \backslash \Gamma$ as its limit since the sets of hyperbolic solutions and going-to-center solutions are open.

But Γ is a countable set of discrete point. So We have the following for $\Lambda \cap S^+$

Proposition 6.6. *B is a curve connecting two points of* Γ*.*

From this proposition, B can only be one of the following:

(1) A curve which takes the same point of Γ as its α- and ω-limit set. We will call it a loop.

(2) A curve connecting two different points of Γ. We will call it a regular branch.

Proposition 6.7. *For any given point p of* Γ*, there exists at least one branch B which takes p as its limit point.*

In any given neighborhood of $q \in \Gamma$, there are at least two points with different destinations. Take a curve connecting these two points. There must be a point q' on this curve, such that $q' \in \Lambda$. So the B we claimed exists.

Proposition 6.8. *The limit points of B must have the same intersection number as that of the points on B.*

Now let us see the possible branches of Λ for the point a in Figure 5:

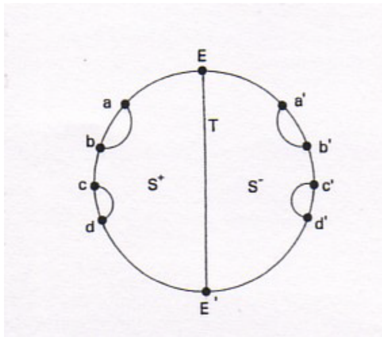
(1) No loops since a loop destroyes the hyperbolic structure of L_1.

(2) B can not connect a and c since they have different destinations.

(3) B can not connect a and d.

If it does. Take a segment α of B ended with d (Refer to Figure 6). Under the action of our flow, α will go forward with one of its ends on $\gamma^+(L_1')$. After reach L_1', it will go along with an orbit connecting L_1' and L_1. But there is no such a connection with intersection number zero. So the points of α will have intersection number at least 2. But the intersection number of a is zero.

(4) Also, B can not connect b_1 by the reason of different intersection numbers.



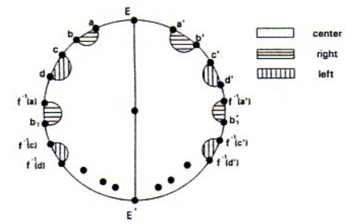


Figure 16. Connection of points Figure 17. Ideal phase structure.

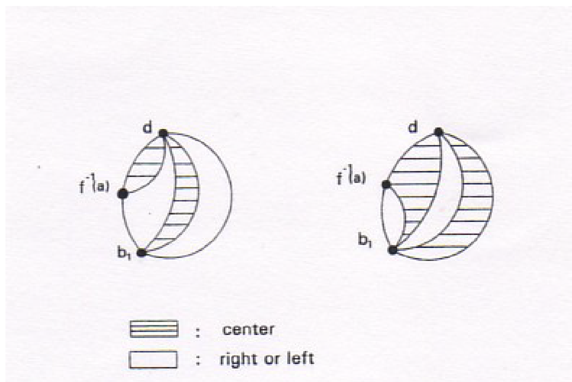


Figure 18. Other possibilities.

(5) Similarly other points of Γ, except b, can not be connected to a.

So we have:

Proposition 6.9. *There is only one regular branch connecting a and b, and it is the only branch for a and b.*

Similarly, there is only one branch of Λ which connects c and d. Now the picture looks like Figure 6. Notice we have the same thing on K^-.

What happens for d might be complicated. However, we see that

(1) The possible connections are between d and $f^{-1}(a)$, and b_1.

(2) The connection among $d, f^{-1}(a), b_1$ on K^+ forms the fundamental diagrams. Transfering this digrams by the flow defined Poincare map backward will give the whole picture of the $\Lambda \cap S$.

What about the connection among $d, f^{-1}(a), b_1$?

(1) The ideal picture is in Figure 6. There would be one connection between $f^{-1}(a)$ and b_1, no connection between d and $f^{-1}(a)$. It would be true if there were only two connecting orbits for L_1, L_2, L_1', L_2'. One joins L_1' and L_2 and another joins L_2' and L_1.

(2) Unfortunately, what really happens might not be so nice. We might have loops around d and b_1 if there are non-transversal intersections among $W^s(L_{1,2})$ and $W^u(L_{1,2}')$. The number of the regular curves connecting d and b_1 is determined by the number of transversal intersections between $W^s(L_{1,2})$ and $W^u(L_{1,2}')$. However, we know that there will be only one regular curve landing $f^{-1}(a)$. The destinations of the points on different sides of any branch

of $\Lambda \cap S$ will be different, and one should be the center. Refer to Figure 6. Since there is virtually no method available to analyze the intersections of $W^s(L_{1,2})$ and $W^u(L'_{1,2})$, these are all we can do.

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