DOUBLY-SYMMETRIC PERIODIC SOLUTIONS OF HILL'S LUNAR PROBLEM

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Abstract. The existence of a new family of periodic solutions to the spatial Hill's lunar problem is established. These solutions have large inclinations and are symmetric with respect to two coordinate planes. In this family the infinitesimal particle is very close to the primary.

1. Introduction

This paper[1] establishes the existence of a new family of periodic solutions to the spatial Hill's lunar problem. The periodic solutions of this family have large inclinations and are symmetric with respect to two coordinate planes — hence the name doubly-symmetric periodic solutions. In this family the infinitesimal particle (the *moon*) is very close to the primary (the *earth*). These periodic solutions are perturbations of circular solutions of the Kepler problem. By the Kepler problem we mean the spatial central force problem with the inverse square law of attraction.

A related paper of the authors [7] established the existence of two new families of periodic solutions to the spatial restricted three-body problem by Poincaré's continuation method. These families exist for all values of the mass ratio parameter μ and have large inclinations. In one of the family the infinitesimal particle is far from the primaries and in the other case the infinitesimal is very close to a primary. In this note we will indicate that the latter family exists in Hill's lunar problem also. This is reasonable since Hill's lunar problem is a limit of restricted problem developed to study the motion of the moon [5].

The small parameter ε will be introduced as a scale parameter in such a way that ε small means the infinitesimal is close to the primary. The perturbation problem is very degenerate. First of all, even to the second approximation the characteristic multipliers are all $+1$, and second, the periodic solutions that we establish are undefined when $\varepsilon = 0$. These difficulties are overcome by exploiting the symmetries of the problem and using the implicit function theorem of Arenstorf [1].

In 1965, Jeffreys [8] showed that there exist doubly symmetric, periodic solutions to the three dimensional restricted three-body problem. His method of the proof depends heavily on a symmetry argument, together with a standard perturbation method applied to the mass

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ratio μ of the restricted problem There is no natural parameter in Hill's lunar problem corresponding to μ and so a scaling parameter is introduced. The problem becomes considerable more difficult.

2. Hill's lunar problem

One of Hill's major contributions to celestial mechanics was his reformulation of the main problem of lunar theory: he gave a new definition for the equations of the first approximation for the motion of the moon [5]. Since his equations of the first approximation contained more terms than the older first approximations, the perturbations were smaller and he was able to obtain series representations for the position of the moon that converge more rapidly than the previously obtained series. Indeed, for many years lunar ephemerides were computed from the series developed by Brown, who used the main problem as defined by Hill. Even today, most of the searchers for more accurate series solutions for the motion of the moon use Hill's definition of the main problem [4].

Before Hill, the main problem consisted of two Kepler problems — one describing the motion of the earth and moon about their center of mass, and the other describing the motion of the sun and the center of mass of the earth-moon system. The coupling terms between the two Kepler problems are neglected at the first approximation. Delaunay used this definition of the main problem for his solution of the lunar problem, but after twenty years of computation was unable to meet the observational accuracy of his time.

In Hill's definition of the main problem, the sun and the center of mass of the earth-moon system still satisfy a Kepler problem, but the motion of the moon is described by a different system of equations known as Hill's lunar equations. Using heuristic arguments about the relative sizes of various physical constants, he concluded that certain other terms were sufficiently large that they should be incorporated into the main problem.

In a popular description of Hill's lunar equations, one is asked to consider the motion of an infinitesimal body (the moon) which is attracted to a body (the earth) fixed at the origin. The infinitesimal body moves in a coordinate system rotating so that the positive x axis points to an infinite body (the sun) infinitely far away. The ratio of the two infinite quantities is taken so that the gravitational attraction of the sun on the moon is finite.

The Hamiltonian of the three-dimensional Hill's lunar problem is

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) - x_1 y_2 + x_2 y_1 - \frac{1}{2}(x_1^2 - x_2^2 - x_3^2) - \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \tag{1}$$

(see [9]).[2]

3. Symmetries and Special Coordinates

The Hamiltonian (1) is invariant under the two anti-symplectic reflections:

$$\begin{aligned} \mathcal{R}_1 &: (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (x_1, -x_2, -x_3, -y_1, y_2, y_3), \\ \mathcal{R}_2 &: (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (x_1, -x_2, x_3, -y_1, y_2, -y_3). \end{aligned} \tag{2}$$

These are time-reversing symmetries, so if $(x_1(t), x_2(t), x_3(t),\ y_1(t), y_2(t), y_3(t))$ is a solution, then so are $(x_1(-t), -x_2(-t), \pm x_3(-t),\ -y_1(-t), y_2(-t), \mp y_3(-t))$. The fixed set of

[2]These is a typographical error in the Hamiltonian of Hill's lunar problem found in [10].

these two symmetries are Lagrangian subplanes, i.e.

$$\mathcal{L}_1 = \{(x_1, 0, 0, 0, y_2, y_3)\}, \qquad \mathcal{L}_2 = \{(x_1, 0, x_3, 0, y_2, 0)\},$$

are fixed by the symmetries $\mathcal{R}_1, \mathcal{R}_2$. If a solution starts in one of these Lagrangian planes at time $t = 0$ and hits the other at a later time $t = T$ then the solution is $4T$-periodic and the orbit of this solution is carried into itself by both symmetries. We shall call such a periodic solution *doubly-symmetric*. Geometrically, an orbit intersects $\mathcal{L}_1$ if it hits the x_1-axis perpendicularly and it intersects $\mathcal{L}_2$ if it hits the x_1, x_3-plane perpendicularly.

To be more specific, let

$$(X_1(t,\alpha,\beta,\gamma), X_2(t,\alpha,\beta,\gamma), X_3(t,\alpha,\beta,\gamma), Y_1(t,\alpha,\beta,\gamma), Y_2(t,\alpha,\beta,\gamma), Y_3(t,\alpha,\beta,\gamma)), \tag{3}$$

be a solution which starts at $(\alpha, 0, 0, 0, \beta, \gamma) \in \mathcal{L}_1$ when $t = 0$, i.e.

$$\begin{aligned} &X_1(0,\alpha,\beta,\gamma) = \alpha, \qquad X_2(0,\alpha,\beta,\gamma) = 0, \qquad X_3(0,\alpha,\beta,\gamma) = 0, \\ &Y_1(0,\alpha,\beta,\gamma) = 0, \qquad Y_2(0,\alpha,\beta,\gamma) = \beta, \qquad Y_3(0,\alpha,\beta,\gamma) = \gamma. \end{aligned} \tag{4}$$

The solution with $\alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0$ will be doubly-symmetric periodic with period $4T$ if it hits the $\mathcal{L}_2$ plane after a time T, i.e.

$$X_2(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad Y_1(T, \alpha_0, \beta_0, \gamma_0) = 0, \quad Y_3(T, \alpha_0, \beta_0, \gamma_0) = 0. \tag{5}$$

This solution will be a *nondegenerate doubly–symmetric periodic solution* if the Jacobian

$$\frac{\partial(X_2, Y_1, Y_3)}{\partial(t, \alpha, \beta, \gamma)}(T, \alpha_0, \beta_0, \gamma_0) \tag{6}$$

has rank three.

It follows from the Implicit Function Theorem that nondegenerate doubly-symmetric periodic solutions can be continued under a small conservative perturbation which preserves the symmetries. In general, a nondegenerate doubly-symmetric periodic solution may not be nondegenerate in the classical sense, i.e. a nondegenerate doubly-symmetric periodic solution may have all its multipliers equal to one.

Jefferys [8] proved the existence of nondegenerate doubly-symmetric periodic solutions of the spatial restricted three-body problem by first setting the mass ratio parameter μ equal to zero to get the Kepler problem in rotating coordinates. He then showed that some of the circular solutions of the Kepler problem where nondegenerate doubly symmetric periodic solutions. Thus, by the above remarks these solutions can be continued into the restricted problem for small μ. Since there is no natural parameter like μ we will introduce a scale parameter ε. This makes the analysis much more delicate.

We follow Jefferys by using a variation of the Poincaré-Delaunay elements. First, the Delaunay elements (ℓ, g, k, L, G, K) are a coordinates on the elliptic domain of the Kepler problem. The elliptic domain is the open set in $\mathbb{R}^6$ which is filled with the elliptic solutions of the Kepler problem. The elements are: ℓ the mean anomaly measured from perigee, g the argument of the perigee measured from the ascending node, k the longitude of the ascending node measured from the x_1 axis, L semi-major axis of the ellipse, G total angular momentum, K the component of angular momentum about the x_3-axis. ℓ, g, and k are angular variables defined modulo 2π, and L, G and K are radial variables. If i is the inclination of the orbital plane to the x_1, x_2 reference plane, then $K = \pm G \cos i$, and so an orbit is in the x_1, x_2-plane when $K = G$. (Often, k and K are denoted by h and H, but we are Hamiltonophiles.)

An orbit hits $\mathcal{L}_1$ at time $t = 0$ if it is perpendicular to the x_1-axis. So its orbital plane must be through the x_1-axis or $k \equiv 0 \mod \pi$, its perigee must be on the x_1-axis or $g \equiv 0 \mod \pi$, and it must be at perigee (apogee) or $\ell \equiv 0 \mod \pi$. Thus, $\mathcal{L}_1$ in Delaunay elements is defined by $\ell \equiv g \equiv k \equiv 0 \mod \pi$.

An orbit hits $\mathcal{L}_2$ at time $t = T$ if it is perpendicular to the x_1, x_3-plane. So its orbital plane must be perpendicular to the x_1, x_3-plane or $k \equiv \pi/2 \mod \pi$, its perigee must be in the x_1, x_3-plane or $g \equiv \pi/2 \mod \pi$, and it must be at perigee (apogee) or $\ell \equiv 0 \mod \pi$. Thus, $\mathcal{L}_2$ in Delaunay elements is defined by $\ell \equiv 0$, $g \equiv k \equiv \pi/2 \mod \pi$.

Since these coordinates are not valid in a neighborhood of the circular orbits of the Kepler problem, we change to Poincaré elements as follows: first make the symplectic linear change of variables

$$\begin{aligned} q_1 &= l + g + k, & p_1 &= L - G + K, \\ q_2 &= -k - g, & p_2 &= L - G, \\ q_3 &= l + g, & p_3 &= G - K, \end{aligned}$$

and now apply the symplectic change of variables defined by the generating function

$$W(q, P) = q_1 P_1 + \frac{P_2^2}{2} \tan q_2 + P_3 q_3$$

so that $P_2 = \sqrt{2p_2} \cos q_2$ and $Q_2 = \sqrt{2p_2} \sin q_2$. This combination of variable changes gives the new variables:

$$\begin{aligned} Q_1 &= q_1 = l + g + k, & P_1 &= p_1 = L - G + K, \\ Q_2 &= -\sqrt{2(L-G)} \sin(k + g), & P_2 &= \sqrt{2(L-G)} \cos(k + g), \\ Q_3 &= q_3 = l + g, & P_3 &= p_3 = G - K. \end{aligned} \tag{7}$$

These variables are valid on circular orbits which occur at $L = G$ (see [6, 12]). The circular orbits with $L = G$ correspond to $Q_2 = P_2 = 0$.

Thus, $\mathcal{L}_1$ in Poincaré elements is defined by $Q_2 = 0$, $Q_1 \equiv Q_3 \equiv 0 \mod \pi$, and $\mathcal{L}_2$ in Poincaré elements is defined by $Q_2 = 0$, $Q_1 \equiv 0 \mod \pi$, $Q_3 \equiv \pi/2 \mod \pi$.

4. Approximate Solutions

Move the infinitesimal mass close to the origin by scaling the variables: $x \to \varepsilon^2 x$, $y \to \varepsilon^{-1} y$, which is symplectic with multiplier ε^{-1}. Letting $H \to \varepsilon^{-1} H$, expanding the potential in ε, and by dropping the constant terms, the Hamiltonian becomes

$$H = \varepsilon^{-3} \left\{ \frac{|y|^2}{2} - \frac{1}{|x|} \right\} - (x_1 y_2 - x_2 y_1) + \varepsilon^3 H^\dagger(x, y, \varepsilon), \tag{8}$$

where $H^\dagger$ is analytic and order 1 in ε.

The solutions that will be establish will have the new x, y coordinates of order 1 in ε and so the original x will be order ε^2 and the original y will be order ε^{-1}. We will not scale time and the solutions we establish will have periods which are order 1 in ε. Note that as $\varepsilon \to 0$ the Hamiltonian tends to infinity, thus we can not just set $\varepsilon = 0$. We will need approximate solutions to the equations and good estimates. (Scaling time does not remove the difficulties of the problem. If we scale time so that the Hamiltonian becomes order 1 in ε then the new periods will tend to infinity.)

In Delaunay elements, the Hamiltonian becomes

$$H = \frac{-\varepsilon^{-3}}{2L^2} - K + \varepsilon^3 H^\dagger(\ell, g, k, L, G, K, \varepsilon). \tag{9}$$

Since these coordinates are not valid in a neighborhood of the circular orbits, we change to Poincaré elements (7) and the Hamiltonian becomes

$$H = \frac{-\varepsilon^{-3}}{2\left(P_1 + P_3\right)^2} - P_1 + \frac{1}{2}\left(P_2^2 + Q_2^2\right) + \varepsilon^3 H^\dagger(Q_1, Q_2, Q_3, P_1, P_2, P_3, \varepsilon). \tag{10}$$

Thus the equations of motion are $\dot{Q} = H_P$, $\dot{P} = -H_Q$ or

$$\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^{-3}}{\left(P_1 + P_3\right)^3} - 1 + \varepsilon^3 f_1, & \dot{P}_1 &= 0 + \varepsilon^3 f_4, \\
\dot{Q}_2 &= P_2 + \varepsilon^3 f_2, & \dot{P}_2 &= -Q_2 + \varepsilon^3 f_5, \\
\dot{Q}_3 &= \frac{\varepsilon^{-3}}{\left(P_1 + P_3\right)^3} + \varepsilon^3 f_3, & \dot{P}_3 &= 0 + \varepsilon^3 f_6,
\end{aligned} \tag{11}$$

where the f_i are the appropriate partials of $H^\dagger$.

First let us consider the approximate equations in order to find the correct approximate periodic solutions. Consider the approximate equations

$$\begin{aligned}
\dot{Q}_1 &= \frac{\varepsilon^{-3}}{\left(P_1 + P_3\right)^3} - 1, & \dot{P}_1 &= 0, \\
\dot{Q}_2 &= P_2, & \dot{P}_2 &= -Q_2, \\
\dot{Q}_3 &= \frac{\varepsilon^{-3}}{\left(P_1 + P_3\right)^3}, & \dot{P}_3 &= 0.
\end{aligned} \tag{12}$$

These are of course, the equations of motion for the Kepler problem in scaled, rotating Poincaré elements. The solution of equation (12) are

$$\begin{aligned}
Q_1(t) &= \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} - 1\right) t + q_1, & P_1(t) &= p_1, \\
Q_2(t) &= q_2 \cos t + p_2 \sin t, & P_2(t) &= -q_2 \sin t + p_2 \cos t, \\
Q_3(t) &= \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3}\right) t + q_3, & P_3(t) &= p_3,
\end{aligned} \tag{13}$$

for initial conditions $(q_1, q_2, q_3, p_1, p_2, p_3)$ at $t = 0$.

The periodicity conditions are the same as those in the Section 3. That is, at $t = 0$; $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$ and at $t = T$; $Q_1 = (i + k)\pi$, $Q_2 = 0$, $Q_3 = (j + m + 1/2)\pi$ where i and j are 0 or 1, and k, and m are arbitrary integers. To satisfy these symmetry condition at $t = 0$ and at $t = T$ we have so solve the equations

$$
\begin{aligned}
Q_1(T) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1 \right) T + i\pi = (i+k)\,\pi, \\
Q_2(T) &= p_2 \sin T = 0, \qquad (14) \\
Q_3(T) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} \right) T + j\pi = (j+m+1/2)\pi,
\end{aligned}
$$

The second equation is solved by taking $p_2 = 0$, thus selecting a circular orbit of the Kepler problem. The difference between the first and third equation has a solution with $T = (m-k+1/2)\pi$. It remains to solve the third equation. With this choice of T it becomes

$$
(p_1+p_3)^3 = \frac{\varepsilon^{-3}\left(m-k+\frac{1}{2}\right)}{\left(m+\frac{1}{2}\right)}. \qquad (15)
$$

Recall that $P_1 + P_3 = L$ which is the semi-major axis in the Kepler problem and that we want solutions which are order 1 in L.

Let n be a fixed small integer. To solve (15) set $m + 1/2 = \varepsilon^{-3}$, $k = m - n$, and $(p_1+p_3)^3 = (n+1/2)$. With this choice the period becomes $T = n + 1/2$

We shall indicate in Section 5 that these solutions of the approximate equations (12) are actually approximations of actual doubly-symmetric periodic solutions of the true equations (11). Thus, our main theorem is

Theorem 4.1. *There exist doubly-symmetric periodic solutions of the spatial Hill's lunar problems with large inclination which are arbitrarily close to the primary.*

5. Outline of the proof

Here we will outline the proof and refer the reader to [7] for a more detailed account of a similar result. Consider the equations (11). The periodicity conditions remain: at $t = 0$; $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$ and at $t = T$; $Q_1 = (i+k)\pi$, $Q_2 = 0$, $Q_3 = \left(j+m+\frac{1}{2}\right)\pi$ where i, j, are 0 or 1 and k, and m are arbitrary integers. By as standard but lengthy Grownwall argument, the solution to this system of differential equations (11) is of the form:

$$
\begin{aligned}
Q_1(t) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} - 1 \right) t + q_1 + \varepsilon^3 g_1, & P_1(t) &= p_1 + \varepsilon^3 g_4, \\
Q_2(t) &= q_2 \cos t + p_2 \sin t + \varepsilon^3 g_2, & P_2(t) &= p_2 \cos t - q_2 \sin t + \varepsilon^3 g_5, \\
Q_3(t) &= \left(\frac{\varepsilon^{-3}}{(p_1+p_3)^3} \right) t + q_3 + \varepsilon^3 g_3, & P_3(t) &= p_3 + \varepsilon^3 g_6,
\end{aligned}
$$

for initial conditions $(q_1, q_2, q_3, p_1, p_2, p_3)$ and for time $t\epsilon[0, \gamma]$, where $g_i = g_i(t, q_1, q_2, q_3, p_1, p_2, p_3)$.

To satisfy the symmetry condition at $t = 0$ we have the solution

$$Q_1(t) = \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} - 1 \right) t + i\pi + \varepsilon^3 g_1,$$

$$Q_2(t) = p_2 \sin t + \varepsilon^3 g_2,$$

$$Q_3(t) = \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} \right) t + j\pi + \varepsilon^3 g_3.$$

Next we need to solve this for the symmetry condition at $t = T$ which are now:

$$Q_1(t) = \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} - 1 \right) t + i\pi + \varepsilon^3 g_1 = (i + k)\,\pi,$$

$$Q_2(t) = p_2 \sin t + \varepsilon^3 g_2 = 0,$$

$$Q_3(t) = \left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} \right) t + j\pi + \varepsilon^3 g_3 = \left(j + m + \tfrac{1}{2}\right) \pi,$$

or

$$\left(\frac{\varepsilon^{-3}}{(p_1 + p_3)^3} - 1 \right) t - k\pi + \varepsilon^3 g_1 = 0,$$

$$p_2 \sin t + \varepsilon^3 g_2 = 0,$$

$$\left(\frac{\varepsilon^{-3}}{(P_1 + P_3)^3} - 1 \right) t - (m + \tfrac{1}{2})\pi + \varepsilon^3 g_3 = 0.$$

This is done by applying the Arenstorf implicit function theorem twice. Roughly speaking Arenstorf's theorem applies to situations where the problem is undefined when $\varepsilon = 0$, all that is required is that the perturbation and the derivatives of the perturbation are sufficiently small. See [1, 7] for details.

First we consider the difference of the first and third equation, together with the second equation. This is the system of equations

$$t + \left(k - m - \tfrac{1}{2}\right) \pi + \varepsilon^3 g_3 - \varepsilon^3 g_1 = 0,$$

$$p_2 \sin t + \varepsilon^3 g_2 = 0.$$

This has solution $t = (m + 1/2 - k)\pi$, $p_2 = 0$ at $\varepsilon = 0$. Along this solution, the determinant of the derivative of the system with respect to t and p_2 is given by

$$\begin{vmatrix} 1 & 0 \\ 0 & \sin(m + 1/2 - k)\pi) \end{vmatrix} = \pm 1 \neq 0.$$

Thus by the Arensdorf implicit function theorem, there exists a neighborhood $\mathcal{N}$ of 0 and functions $T(p_1, p_3, \varepsilon)$ near $(m + 1/2 - k)\pi$ and $p_2(p_1, p_3, \varepsilon)$ near 0 for $\varepsilon \in \mathcal{N}$ and (p_1, p_3) arbitrary such that

$$
\begin{aligned}
T(p_1,p_3,\varepsilon)-(m+\tfrac{1}{2}-k)\pi+\varepsilon^3 g_3-\varepsilon^3 g_1 &= 0,\\
p_2(p_1,p_3,\varepsilon)\sin T(p_1,p_3,\varepsilon)+\varepsilon^3 g_2 &= 0.
\end{aligned}
$$

Putting this solution for T into the third equation, we need to solve $Q_3(T)-(m+\frac{1}{2})\pi=0$, or

$$
\frac{\varepsilon^{-3}}{(p_1+p_3)^3}\left[\left(m+\frac{1}{2}-k\right)\pi-\varepsilon^3 g_3+\varepsilon^3 g_1\right]-\left(m+\frac{1}{2}\right)\pi+\varepsilon^3 g_3=0,
$$

which is equivalent to solving

$$
\left(m+\frac{1}{2}-k\right)\pi-\varepsilon^3 g_3+\varepsilon^3 g_1-\left(\varepsilon^3\left(m+\frac{1}{2}\right)\pi-\varepsilon^6 g_3\right)(p_1+p_3)^3=0
$$

for $(p_1+p_3)^3$ whenever $\varepsilon\in\mathcal{N}-\{0\}$. Now the solution for T left both m and k arbitrary, so for the moment regard m and k as free variables. Then letting $m+\frac{1}{2}=\varepsilon^{-3}$ and letting $k=m-n$ for n a small integer, we need to solve

$$
R=\left(n+\frac{1}{2}\right)\pi-\varepsilon^3 g_3+\varepsilon^3 g_1-\left(\pi-\varepsilon^6 g_3\right)(p_1+p_3)^3=0.
$$

By this choice of m and k, T becomes $T(p_1,p_3,\varepsilon)=(n+1/2)\pi+\varepsilon^3 g_3-\varepsilon^3 g_1$ which is uniformly bounded as ε approaches zero. $\partial T/\partial p_1$ at $\varepsilon=0$ is given by $-4\partial(\varepsilon^3 g_6-\varepsilon^3 g_2)/\partial p_1$ where the partials of the g_i are evaluated along solutions; $t=(n+\frac{1}{2})\pi$, $p_2=0$, $\varepsilon=0$, (p_1,p_3) arbitrary. By the another Grownwall argument the partials of the g_i with respect to initial conditions are also uniformly bounded as ε approaches zero. Thus we can differentiate R along the solution $(p_1+p_3)^3=n+1/2$, $\varepsilon=0$ to get $\partial R/\partial p_1=-3\pi(n+1/2)^{2/3}\neq 0$.

Thus we have shown that there exists a deleted neighborhood $\mathcal{N}-\{0\}$ of 0 such whenever $\varepsilon\in\mathcal{N}-\{0\}$ and ε is of the form $(m+1/2)^{-1/3}$ for m an integer, the system has a periodic solution with period near $(4n+2)\pi$ for n a small integer. These solutions are doubly symmetric, and approximately circular orbits of small.

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