Funkcialaj Ekvacioj, **27** (1984), 261–271

Normal Forms for the General Equilibrium

By

K. R. MEYER*

(University of Cincinnati, U.S.A.)

§ 1. Introduction.

Poincaré [12] considered the problem of reducing a system of differential equations of the form

$$\dot{x}=Ax+f(x) \tag{1}$$

to the linear equation

$$\dot{y}=Ay \tag{2}$$

by a change of variables $y=\phi(x)=x+\cdots$. In the above $x, y \in R^n$ (or C^n), A is an $n\times n$ constant matrix, and f is a formal or convergent power series in x starting with terms of degree 2. Poincaré found that there was a formal solution of this problem provided the matrix A is diagonalizable and its eigenvalues $\lambda_1, \cdots, \lambda_n$ satisfy

$$\lambda_j \neq \sum k_i\lambda_i \tag{3}$$

for $j=1, \cdots, n$ and all integer vectors $k=(k_1, \cdots, k_n)$ with $k_i \geqslant 0$ and $|k|=k_1+\cdots+k_n \geqslant 2$. Furthermore, he proved that if in addition to the above, the eigenvalues lie strictly to one side of a line through the origin in the complex plane then the formal series is actually convergent. Since that time there have been a multitude of papers which give generalizations or variations of these two results. In general these papers concentrate either on the formal question or on the smoothness question. Hartman [3] gives a good summary of the literature on the existence of smooth linearizations and Sell [13] has some more recent references. This paper shall deal with the formal problem only.

When condition (3) does not hold one tries to reduce equation (1) to a different system

$$\dot{y}=Ay+g(y) \tag{4}$$

where g is simplier than f. See for example Lattes [7] and Sternberg [14]. This is the problem of finding the normal form for equation (1). Except for a few exceptional

* This research was supported by NSF grant MCS-83-01919.

cases, almost all of the literature still assumes that the matrix A is diagonalizable and these exceptions use ad hoc methods. The most notable exception is Liapunov's stability study when $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ see [8]. This example will be discussed in more detail in Section IV. The case when A is 4×4, has eigenvalues $\pm i$ of multiplicity two, is not diagonalizable and the system (1) is Hamiltonian was treated briefly in Meyer and Schmidt [11] and will be discussed again in Section IV.

Recently, Kummer [5] has shown that the theory of Lie algebras is useful in studying normal forms in some special cases in celestial mechanics. Taking this as a lead Cushman, Deprit and Mosak [2] have taken some results from representation theory and given a complete discription of the normal form of a Hamiltonian system of two degrees of freedom when the matrix of the linear part is nilpotent. Although, they discuss one example it is clear that their procedure is quiet general and forms a basis for a complete theory of normal forms without the diagonalizable assumption.

This paper will use the general method given by Cushman, Deprit and Mosak to develop the general normal form for equation (1). This new normal form will be used to discuss some of the stability questions addressed by Liapunov and to discuss the existence of periodic solutions in a resonance problem.

Since the general procedure is given completely in [2], I will simply summarize the necessary results from representation theory and will be content with pedestrian matrix proofs. Thus even though a previous reading of [2] is advised, it is not necessary.

§ 2. Background results from linear algebra.

Throughout this section let V denote a finite dimensional vector space over the reals R or the complex numbers C. Let $\boldsymbol{L}=\boldsymbol{L}(V)$ denote the space of linear transformation of V into itself. Given P and $Q\in \boldsymbol{L}$ we define the Lie bracket $[P, Q]\in \boldsymbol{L}$ by the usual formula $[P, Q]=P\circ Q-Q\circ P$. Thus $\boldsymbol{L}$ becomes a Lie algebra. Note, that P and Q commute if and only if $[P, Q]=0$. The Jordan decomposition theorem states that for any $A\in L$ there exists $S, N\in \boldsymbol{L}$ such that $A=S+N$, S is diagonalizable, N is nilpotent and $[S, N]=0$. For now, fix A and let S and N be as given by Jordan's theorem. Let $\boldsymbol{R}$ denote the range of S and $\boldsymbol{K}$ the kernel of S. The fundamental property which is used over and over again in normal form theory is the fact that for a diagonalizable matrix S, the space $\boldsymbol{R}$ and $\boldsymbol{K}$ are invariant under S, i.e. S: $\boldsymbol{R}\to\boldsymbol{R}$ and S: $\boldsymbol{K}\to\boldsymbol{K}$, and V is the direct sum of $\boldsymbol{R}$ and $\boldsymbol{K}$, i.e. $V=\boldsymbol{R}\oplus\boldsymbol{K}$, Thus $S|\boldsymbol{R}$, the restriction of S to its range, is an invertable map. Since S and N commute, N also maps $\boldsymbol{R}$ into $\boldsymbol{R}$ and $\boldsymbol{K}$ into $\boldsymbol{K}$ and so does $A=S+N$. Since $S|\boldsymbol{R}$ is invertable and N is nilpotent the map $A|\boldsymbol{R}=(S+N)|\boldsymbol{R}$ is invertable.

Consider the standard problem of solving a system of linear equations of the form $Ax=y$ where y is given. Write $y=y_1+u$ where $y_1 \in \boldsymbol{R}$ and $u \in \boldsymbol{K}$. From the above there is an $x_1 \in \boldsymbol{R}$ such that $Ax_1=y_1$. Let $x=x_1+z$ where $z \in \boldsymbol{K}$, so the equation becomes $Ax=Ax_1+Az=y_1+u$ or $Az=u$. But since $z \in \boldsymbol{K}$ the problem reduces to: given $u \in k$ find a $z \in \boldsymbol{K}$ such taht $Nu=u$. Thus we have reduced the problem to a similar problem on a smaller space with nilpotent matrix of coefficients. For any particular N there are standard procedures available for deciding whether the problem has a solution and if so what the solution is.

However, in the theory of normal forms one is presented with a single matrix A as given in the introduction and this matrix is used to define a sequence of linear operators via Lie derivative operations. To handle this problem a solution of the solvability of $Nz=u$ in terms of Lie operations is needed. Here Cushman, Deprit and Mosak [2] invoke a theorem of Weyl which states in this special case that there exists $M, H \in \boldsymbol{L}$ such that

$$[H, N]=2H, \ [H, M]=-2M, \ [N, M]=H. \tag{1}$$

Notice that the above Lie bracket relationships are the same as the relationship for the three standard generators of $sl(2, R)$, namely

$$N=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Wely's theorem is much more general and the reader is referred to [4] for the general case and to [2] for the application of the general theorem to normal forms for Hamiltonian systems. For our purposes it is enough to display the solution. By the Jordan canonical form theorem it is enough to display a solution for a single nilpotent Jordan block. Thus if N is of the form

$$N=\begin{bmatrix} 0 & n & 0 & \cdots 0 & 0 \\ 0 & 0 & n-1 & \cdots 0 & 0 \\ 0 & 0 & 0 & \cdots 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots 0 & 1 \\ 0 & 0 & 0 & \cdots 0 & 0 \end{bmatrix}$$

one finds that

$$M=\begin{bmatrix} 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 \cdots & n-1 & 0 & 0 \\ 0 & 0 & 0 \cdots & 0 & n & 0 \end{bmatrix}$$

and

$$H=\begin{bmatrix} n & & & & & & \\ & n-2 & & & & & \\ & & n-4 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}$$

satisfy (1). The important properties of these operators are summarized in

Lemma. (a) *M and n are nilpotent.*
(b) *NM and MN are diagonalizable.*
(c) *kernel NM = kernel M, range NM = range N*
kernel MN = kernel N, range MN = range M.
(d) *V = kernel M ⊕ range N = kernel N ⊕ range M.*

The reader can verify this lemma from the matrix representations given above, but the lemma holds simply under the relations (1) see [2] for the general proof. Part (d) of the above lemma gives the natural replacement of the direct sum decomposition for diagonalizable matries.

Return now to the problem of solving $Nz=u$ where N is nilpotent. Given N then let M and H be as in the above. By the lemma the equation $Nz=u$ has a solution if and only if $Mu=0$. For a better formulation of this solution, let P be the projection of $V(=K)$ onto the subspace kernel M as given in the first direct sum decomposition in (d). Then for any $u \in V$ there exists a unique solution $z \in$ range M of the equation $Nz=u-Pu$.

Now let us collect everything. Given a linear operator A find its Jordan decomposition $A=S+N$. For the given N find M and H as above. Define $A^*=S+M$. Then for any $y \in V$ there exists unique $w \in$ range A^* and $v \in$ kernel A^* such that $Aw=y-v$.

§ 3. The normal form.

Assume that f in (1.1) is a formal power series in x which starts with terms of degree at least two. Scale by $x \to \varepsilon x$ where ε is to be a small parameter. Equation (1.1) becomes

$$(1) \qquad \dot{x}=Ax+F_*(x, \varepsilon)$$

where

(2) $$F_*(x, \varepsilon)=\sum_{i=1}^{\infty}\left(\frac{\varepsilon^i}{i!}\right)F_i^0(x)$$

and F_i^0 is an n-vector, each component of which is a homogeneous polynomial of degree $i+1$. Sometime we shall let $F_0^0(x)=Ax$. In normal form theory a formal change of variables $x=x(\xi, \varepsilon)=\xi+\cdots$ is constructed which reduces (1) to

(3) $$\dot{\xi}=A\xi+F^*(\xi, \varepsilon)$$

where

(4) $$F^*(\xi, \varepsilon)=\sum_{j=1}^{\infty}\left(\frac{\varepsilon j}{j!}\right)F_0^j(\xi).$$

The general theorem on normal forms is

Theorem. *Let $\{P_i\}_{i=0}^{\infty}$, $\{Q_i\}_{i=1}^{\infty}$ and $\{R_i\}_{i=1}^{\infty}$ be sequences of vector spaces of smooth vector fields on a common domain with the following properties*
(a) $Q_i\subset P_i$; $i=1, 2, \cdots$
(b) $F_i^0\in P_i$; $i=0, 1, 2, \cdots$
(c) $[R_i, P_j]\subset P_{i+j}$, $i,j=0, 1, 2, \cdots$
(d) *for any $U\in P_i$, $i=1, 2, \cdots$ there is a $v\in Q_i$ and a $w\in R_i$ such that*

$$v=u+[w, F_0^0].$$

Then there exists a formal transformation $x=x(\xi, \varepsilon)=\xi+\cdots$ which transforms (1) *into* (2) *and where $F_0\in Q_i$ for $i=1, 2, 3, \cdots$.*

A proof of this general result using the method of Lie transforms can be found in [10]. In the above $[\cdot, \cdot]$ is the Lie bracket of vector fields. Since equation (1) is given and each F_i^0 is an n-vector of homogeneous polynomials of degree $i+1$, it is natural to take P_i as the vector space of all n-vectors of homogeneous polynomials of degree $i+1$. Furthermore, if Q_i and R_i are taken as subspaces of P_i then conditions (a), (b) and (c) of the theorem are automatically satisfied. Thus, it remains to investigate the condition (d). Since $F_0^0(x)=Ax$ is given and the Lie bracket is bilinear, the condition (d) is a statement about the solvability of an infinite set of linear equations.

For the matrix A let S, N, M, H and A^* be as given in Section 2. For any one of these matrices, say A, define the linear operator

$$L_A:\ P_i\to P_i:\ W\to[W, Ax].$$

Since these are linear maps the Lie bracket is well defined on them and by Jacobi's identity

$$[L_A, L_B]=L_{[A,B]}.$$

Thus, the same commutator relations hold for L_A, L_S, L_N, etc. as hold for A, S, N, etc.. Since the equation in part (d) is simply

$$L_A W + U = V$$

the theory from the previous section assures the existence of a solution pair v and w where $L_{A^*} v = 0$ and W is in the range of L_{A^*}. Thus, if we define R_i = range L_{A^*} and Q_i = kernel L_{A^*} where L_{A^*} is considered as a map of P_i into itself then all the conditions of the above theorem hold. Therefore, we have:

Theorem. *Let A, A^*, P_i, Q_i, R_i be as defined above. Then there exists a formal transformation x: $x(\xi, \varepsilon) = \xi + \cdots$ which transforms* (1) *to* (3) *where each F_0 satisfies* $[F_0, A^* x] = 0$.

Of course an analytic transformation can be obtained by truncating the formal transformation. In this case the condition $[F_0^i, A^* x] = 0$ would only hold for a finite range of i's.

§ 4. Example and applications.

A) Stability Theory: Liapunov left an unfinished manuscript which investigated the stability of a critical point when the matrix of a linearized equations have the eigenvalue zero of multiplicity two. This work was completed in [8]. This work contains a multitude of special cases, some of which can be codefied using a suitable normal form. A complete treatment will not be given here since [9] and [8] contain complete discussions of this problem. Here we shall just given an illustrative example.

Consider the system (1.1), i.e.

$$\dot{x} = Ax + f(x) \tag{1}$$

where now $x \in R^2$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $A^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By the results of the last section this equation is in normal form (relative to this choice of A^*) if $[f(x), A^* x] = 0$. Let $x = (u, v)^T$ and $f = (g, h)^T$, so the conditions to be satisfied are

$$\begin{aligned} u g_v &= 0 \\ u h_v - g &= 0 \end{aligned} \tag{2}$$

the first condition implies that g is a function of u only and since g is of second order at least we may set $g(u) = \alpha(u) u$. The second condition in (2) becomes $h_v = \alpha(u)$ and so $h(u, v) = \beta(u) + \alpha(u) v$. Thus the normal form for this equation is

$$\begin{aligned} \dot{u} &= \alpha(u)u + v \\ \dot{v} &= \beta(u) + \alpha(u)v \end{aligned} \tag{3}$$

where α and β are arbitrary series in u.

Assume that the normalization transformation converges to avoid the tedium of working with remainder terms. Consider the Liapunov function

$$V = \frac{1}{2} v^2 - \int_0^u \beta(\tau) d\tau \tag{4}$$

whose derivative along the trajectories of (3) is

$$\dot{V} = \alpha(u)v^2 - u\beta(u)\alpha(u). \tag{5}$$

If $u\beta(u) < 0$ for small non-zero u then V is positive definite with respect to the origin. If furthermore, $\alpha(u) < 0$ for small non-zero u, then $\dot{V} < 0$ for u small and $\dot{V} = 0$ only when $u = 0$. Thus by LaSalle's theorem [6] the origin is asymptotically stable. Many of the other cases can be treated with equal ease, but this normal form does not avoid the classical problem of the center. See [8, 9] for a further discussion.

B) Bifurcation Theory: Consider now the case when the matrix A in equation (1) has a pair of double pure imaginary eigenvalues, say $\pm i$, and A is not diagonalizable. By a complex linear change of variables A can be reduced to

$$A = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{bmatrix}. \tag{6}$$

The system (1) is no longer real, but now satisfies certain reality conditions, namely $Pf(Px) = \overline{f(x)}$ where P is the 4×4 matrix which interchanges rows 1 and 3 and rows 2 and 4. To reflect this, introduce $y = (x_1, x_2)^T$, $\bar{y} = (x_3, x_4)^T$, $g = (f_1, f_2)^T$, $\bar{g} = (f_3, f_4)^T$ and $B = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$. In this notation equation (1) becomes

$$\begin{aligned} \dot{y} &= By + g(y, \bar{y}) \\ \dot{\bar{y}} &= B\bar{y} + \bar{g}(y, \bar{y}). \end{aligned} \tag{7}$$

Since the second equation in (7) is just the conjugate of the first it will be ignored in the subsequent discussion.

As in the previous example $B^* = B^T = S + M$ where $S = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The Lie bracket operator takes on a slightly different form in these coordinates, namely

$$[g(y,\bar{y}),\, Cy]=\frac{\partial g}{\partial y}\,Cy+\frac{\partial g}{\partial y}\bar{C}\bar{y}-Cg \tag{8}$$

for any 2×2 matrix C. The equation (7) are in normal form as defined in the previous section if $[g, B^*y]=0$. In Section 2, it was shown that the kernel of $[\cdot, B^*y]$ is the kernel of $[\cdot, My]$ restricted to the kernel of $[\cdot, Sy]$. The kernel of $[\cdot, Sy]$ is well known and easy to calculate; it consists of series which contain only terms of the form

$$\begin{gathered} b\, y_1^{a_1} y_2^{a_2} \bar{y}_1^{a_3} \bar{y}_2^{a_3} \\ a_1+a_2=a_3+a_4+1. \end{gathered} \tag{9}$$

For g to be in the kernel of $[\cdot, My]$ it must satisfy

$$\begin{gathered} \frac{\partial g_1}{\partial y_2}\,y_1+\frac{\partial g_1}{\partial y_2}y_1=0 \\ \frac{\partial g_2}{\partial y_2}\,y_1+\frac{\partial g_1}{\partial y_2}y_1=g_1. \end{gathered} \tag{10}$$

The effect of the first operator in (10) on a term of the form (9) is to generate two similar terms but with the exponent sum a_2+a_4 reduced by 1. From this observation it is easy to see that g_1 must be of the form $y_1h(y_1\bar{y}_1)$ where h is an arbitrary series in the product $y_1\bar{y}_1$. If g_1 has this form then a particular solution of the second equation is $y_2h(y_1\bar{y}_1)$. The general solution of the second equation in (10) is obtained by adding the general solution of the homogeneous equation and so is $g_2=y_1k(y_1\bar{y}_1)+y_2h(y_1, \bar{y}_2)$ where k like h is an arbitrary series in the product $y_1\bar{y}_1$. Thus the normal form is

$$\begin{gathered} \dot{y}_1=(i+h(y_1\bar{y}_1))y_1+y_2 \\ \dot{y}_2=k(y_1\bar{y}_1)y_1+(i+h(y_1\bar{y}_1))y_2. \end{gathered} \tag{11}$$

The linear system, when $k\equiv h\equiv 0$, has many special properties and a natural question to ask is which of these properties persist in the non-linear system. The linear system has two independent integrals, namely $y_2\bar{y}_2$ and $y_1\bar{y}_2-\bar{y}_1y_2$, and a surface filled with periodic solutions, namely the surface $y_2\equiv\bar{y}_2\equiv 0$. Neither of these properties are preserved in general since a nonlinear term can make the origin asymptotically stable. (For example take $k\equiv 0$ and $h(y_1\bar{y}_1)=-y_1\bar{y}_1$). However, it will be shown that often the existence of an integral for the non-linear problem implies the existence of a surface of periodic solutions.

First consider the special case when $h\equiv 0$ and $k(y_1\bar{y})=\alpha(y_1\bar{y}_1)$, where α is a constant, so that the system is

$$y_1 = iy_1 + y_2 \tag{12}$$
$$y_2 = \alpha y_1 \bar{y}_1 + iy_2.$$

This system admits the integral $y_1\bar{y}_2 - \bar{y}_1 y_2$ provided the constant α is real which will henceforth be assumed. In order to find periodic solutions of (12) use the time honored guess method and try $y_1 = p_1 \exp i(1+\beta)t$, $y_2 = p_2 \exp i(1+\beta)t$ where p_1, p_2 and β are constants to be determined. Since β is a frequence correction term it must be real. Putting this guess into (12) yields

$$i\beta p_1 = p_2 \tag{13}$$
$$i\beta p_2 = \alpha p_1 \bar{p}_1.$$

Solving the first equation for p_2, substituting into the second and dividing by p_1 (a non-trivial solution is sought) yields $-\beta^2 = \alpha(p_1\bar{p}_1)$. Thus if $\alpha < 0$ there are non-trivial solutions; in fact two surfaces of periodic solutions which eminate form the origin. Thus suggests that a generalization of the classical result of Buchanan [1] can be obtained using normal forms.

All the previous discussion has delt with formal series and in general one expects these series to diverge. However, a truncation of the normalization transformation will converge and yield a partially normalized equation. Let us assume that a system of the form (1) is given where A is as (6). Furthermore, assume that your favoriate algebraic manipulator has normalized the equations through the third order to yield a system

$$\dot{y}_1 = (i + \beta(y_1\bar{y}_1))y_1 + y_2 + E_1(y_1, y_2, \bar{y}_1, \bar{y}_2) \tag{14}$$
$$\dot{y}_2 = \alpha y_1 \bar{y}_1 + (i + \beta(y_1\bar{y}))y_2 + E_2(y_1, y_2, \bar{y}_1, \bar{y}_2)$$

where α and β are constants, $\alpha < 0$, and E_1 and E_2 are convergent power series in the displayed arguments beginning with terms of degree at least 4. Furthermore, assume that the full system admits an analytic integral $I(y_1, y_2, y_1, y_2)$ which has a convergent series representation which begins with terms to degree two. Assume that the quadratic terms in the expansion of I are a non-degenerate quadratic form.

Scale equations (14) by $y_1 \to \varepsilon y_1$, $y_2 \to \varepsilon^2 y_2$ where ε is a small parameter. Then equations (14) become

$$\dot{y}_1 = iy_1 + \varepsilon y_2 + 0(\varepsilon^2) \tag{15}$$
$$\dot{y}_2 = iy_2 + \varepsilon\alpha y_1 \bar{y}_1 + 0(\varepsilon^2).$$

(Note, the similarity with the example in (12).)

This system is in the form necessary to apply the lemma on page 100 of Meyer and Schmidt [11]. Calculating, the bifurcation equations

$$B(\beta, \xi)=\begin{pmatrix}\beta i\xi_1+2\pi\xi_2 \\ \beta i\xi_2+2\pi\alpha\xi_1\bar{\xi}_1\end{pmatrix}. \tag{16}$$

(Note the similarity with (13).) Solving $B=0$ yields $\beta^2+\alpha(2\pi)^2\xi_1\bar{\xi}_1=0$ which has a two one-parameter family of solutions with β real provided $\alpha<0$. It is not difficult to calculate that the rank of the Jacobian of B along these solutions is precisely 3 and so the lemma of Meyer and Schmidt applies. Thus under the above enunciated hypothesis there exists two families of periodic solutions eminating from the origin for equations (14).

The details of this argument were kept brief since they are essentially the same as found in [11]. This outline is given to illustrate that the Hamiltonian nature of the problem considered in Buchanan [1] and Meyer and Schmidt [11] was not that important. All that was needed was a non-degenerate integral which of course comes free in a Hamiltonian system and that one term in a normal form had a non-zero coefficient.

It goes without saying that if A depends on an additional parameter λ and $A(\lambda)$ has the form (6) when $\lambda=0$ then the antilog of the results in [11] can be carried out.

References

[1] Buchanan, D., Trojan Satellites—limiting case, Trans. Roy. Soc. Canada, **31** (1941), 9–25.
[2] Cushman, R., Deprit, A. and Mosak, R., Normal forms and representation theory, preprint.
[3] Hartman, P., *Ordinary Differential Equations,* John Wiley, New York, 1964.
[4] Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory,* Springer, Berlin, 1980.
[5] Kummer, M., On resonant non-linearly coupled oscillators with two equal frequences, Comm. Math. Phys., **48** (1976), 137–139.
[6] LaSalle, J. P. and Lefschetz, S., *Stability by Liapunov's Direct Method with Applications,* Academic Press, New York, 1961.
[7] Lattes, S., Sur les forms reduites des transformations ponctuelles dans le domaine d'un point double, Bull. Soc. Math. France, **8** (1911), 309–345.
[8] Liapunov, A. M., *Stability of Motion,* Academic Press, New York, 1966.
[9] Liu, J. C., Normal forms and stability via Lie transforms, Ph. D. thesis, University of Cincinnati, 1981.
[10] Meyer, K. R. and Schmidt, D. S., Entrainment domains, Funkcial. Ekvac., **20** (1977), 171–192.
[11] ——, Periodic orbits near L_4 for mass ratios near the critical mass ratio of Routh, Celestial Mech., **4** (1971), 99–109.
[12] Poincaré, H., Memoire sur les courbes definie por une equation differentielle IV, J. Math. Pures Appl., **1** (1885), 167–244.
[13] Sell, G., Smooth linarization near a fixed point, Institute for Mathematics and its Applications, preprint series #16.
[14] Sternberg, S., Local contractions and a theorem of Poincaré, Amer. J. Math., **79** (1957),

809–824.

nuna adreso:
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
U.S.A.

(Ricevita la 9-an de aŭgusto, 1983)