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Entrainment Domains

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1. Introduction

There is a substantial literature on the existence and persistence of invariant tori for systems of ordinary differential equations [7, 8, 14, 15, 17]. The majority of this literature does not concern itself with the nature of the flow on the torus itself, since the works of Poincare and Denjoy have shown that these flows can be quite varied and complicated. However, by addressing himself to the flow on these invariant tori, Bushard [3, 4, 5] shed considerable light on the entrainment (or locking-in) of periodic solutions. This parper also studies the entrainment of periodic solutions on invariant tori. Whereas, Bushard considers a fairly general system and obtains qualitative information, we consider a more specific class of equations with the goal of devloping an effective procedure for obtaining quantitative information.

The prototype has been the forced van der Pol equation. Therefore, we illustrate our procedure by studying this equation and a system of two weakly coupled van der Pol equations introduced by Linkens [16] in a study of the electrical activity of the human gastrointestinal tract. For the forced van der Pol equation there are several studies on harmonic entrainment but few when the forcing frequency and natural frequency are considerably different. Hayashi [10] and others have considered the cases when the ratio of the natural and forcing frequency is near 1 to 2 and 1 to 3 in detail. He found that for a small range of detuning a periodic solution with frequency near the forcing frequency exists. Since other frequecy ratios require long computations they were not considered until now.

In order to fix some definitions, consider a system of equations of the form

$$\dot{x}=f(x, \lambda) \tag{1.1}$$

where f is a smooth function from $B^n \times B^k$ into R^n, B^n and B^k are open balls in R^n and R^k respectively, and $\dot{x}=dx/dt$. Suppose that for each $\lambda \in B^k$ the system (1.1) has a unique, smooth, two dimensional invariant torus $T_\lambda \subset B^n$ and that it varies smoothly with λ. Also let C_λ be a smooth closed curve on T_λ which is a global cross section for the flow on T_λ and that C_λ varies smoothly with λ. Thus to the

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flow on T_λ we may associate a real number $\rho(\lambda)$—the rotation number. Also $\rho(\lambda)$ is rational if and only if the flow on T_λ has a periodic solution. Let Γ_λ be another smooth, closed curve on T_λ which varies smoothly with λ and is such that C_λ and Γ_λ form a base for the first homology group of T_λ. If $\rho(\lambda)$ is rational, say $\rho(\lambda)=p/q$ where $(p, q)=1$, then the periodic solutions on T_λ are homologous to $pC_\lambda+q\Gamma_\lambda$. Thus a rational rotation number has a simple geometric interpretation: if the rotation number is p/q where $(p, q)=1$ then the periodic solutions of (1.1) on T_λ wind p times around C_λ and q times around Γ_λ before closing.

Following Bushard we define the p/q entrainment domain to be $A_{p/q}=\rho^{-1}(p/q)$. Since ρ is continuous, $A_{p/q}$ is closed in B^k. Clearly distinct rational numbers give rise to disjoint entrainment domains.

Even though the rotation number is continuous in λ it will not be differentiable in general. This is due to the "locking-in phenomenon" or the "entrainment of frequency phenomenon". Restricting our attention to the flow on the two dimensional torus a periodic solution has 2 characteristic multipliers 1 and μ, $\mu>0$. If $\mu\neq 1$ the periodic solution is called hyperbolic—a source if $\mu>1$ and a sink if $0<\mu<1$. If a periodic solution is hyperbolic, an easy application of the implicit function theorem implies that small perturbations of the equations have a periodic solution with the same rotation number. Thus if for $\lambda=\lambda_0$ equation (1.1) has a hyperbolic periodic solution with rotation number p/q then λ_0 is an interior point of $A_{p/q}$. In general one expects that most periodic solutions are hyperbolic. For generic one parameter families of flows on a torus, it is a consequence of the work of Sotomayor [20] that the entrainment domains are unions of nontrivial closed intervals and these intervals do not cluster. Thus generically the rotation number as a function of a single parameter has the essential qualitative features of the Cantor ternary function.

In view of the works of Sotomayor and Pugh [19] one would expect that the entrainment domains are k-manifolds with smooth boundaries and that their union is dense in B^k for most systems. We shall not pursue these general questions but concentrate on calculating the boundaries of the $A_{p/q}$ for a special class of equations by small parameter methods.

2. An illustrative example

In this section a special equation defined on the torus is presented to illustrate the entrainment phenomenon. This discussion also introduces some definitions and methods in a simple setting.

Consider the equation

$$\dot{\theta}=\omega-\varepsilon g(\theta, t) \tag{2.1}$$

where g is smooth and 1-periodic in θ and t. The two real parameters are ε and ω. As written (2.1) is a single non-autonomous equation, however, by considering the equivalent autonomous system

$$\begin{aligned}\dot{\theta}&=\omega-\varepsilon g(\theta, \tau)\\ \dot{\tau}&=1\end{aligned} \tag{2.2}$$

where θ and τ are defined mod 1 we obtain a dynamical system defined on the 2-torus $T^2=R^2/Z^2$. The closed curve $\tau=0$ mod 1 is a global cross section for the flow provided $\omega\neq 0$, ε is small and it together with the closed curve $\theta=0$ mod 1 form a base for the first homology group of T^2. A solution $\theta(t)$ of (2.1) which satisfies $\theta(t+q)=\theta(t)+p$ becomes a periodic orbit of (2.2) with rotation number p/q. Sometimes we shall refer to such a solution as a periodic solution of (2.1) with rotation number p/q. If $\theta(t)$ is such a solution then the initial conditions $\theta(0), \theta(1), \cdots, \theta(q-1)$ for $t=0$ also give rise to periodic solutions of (2.1) but all of these solutions give rise to the same closed orbit for (2.2). In view of this we shall identify these periodic solutions and refer to the whole class as a periodic orbit of (2.1) or (2.2). This last convention simplifies the counting of periodic solutions.

A solution $\theta(t)$ of (2.1) is a periodic solution (orbit) if and only if

$$\theta(q)-\theta(0)=p. \tag{2.3}$$

Thus the p/q entrainment domain, $A_{p/q}$, for (2.1) is the set of all (ε, ω) such that (2.1) has a solution $\theta(t)$ which satisfies (2.3). Clearly $(0, p/q)\in A_{p/q}$ and given any $\omega_0\neq p/q$ there is a neighborhood N of $(0, \omega_0)$ which does not meet $A_{p/q}$. Thus for small ε the only points in $A_{p/q}$ are near $(0, p/q)$.

Bushard [4] has shown that there are positive numbers ε_0, δ_0 and continuous functions a, b such that

$$a, b: I=[-\varepsilon_0, \varepsilon_0]\to J=[p/q-\delta_0, p/q+\delta_0], \qquad a(0)=b(0)=p/q,$$

a and b are differentiable at $\varepsilon=0$,

$$A_{p/q}\cap(I\times J)=\{(\varepsilon, \omega): 0\leq\varepsilon\leq\varepsilon_0 \quad\text{and}\quad \min(a(\varepsilon), b(\varepsilon))\leq\omega\leq\max(a(\varepsilon), b(\varepsilon))\}.$$

That is, close to $(0, p/q)$ the entrainment domain, $A_{p/q}$, is a sector bounded above and below by continuous curves which pass through $(0, p/q)$—see figure 1. The two curves a and b will be called the *local boundary curves* for $A_{p/q}$ and the set $A_{p/q}\cap(I\times J)$ will be called the *local sector* of $A_{p/q}$. The functions a and b may be equal (for example when $g\equiv 0$) and so the local sector may degenerate to a single curve. The example given below has only nondegenerate sectors.

Let $S=\{(\alpha, \beta): \alpha$ and β are relatively prime$\}$ and

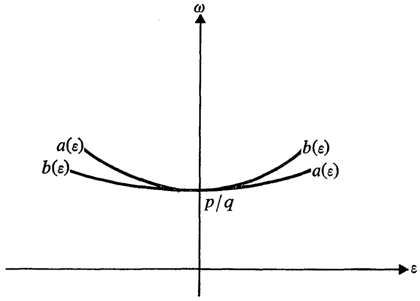

Figure 1: Local sector of entrainment domain $A_{p/q}$.

(2.4)
$$g(\theta, t)=\sum_S a_{\alpha\beta} \sin 2\pi(\alpha\theta-\beta t)$$

where the coefficients $a_{\alpha\beta}$, $(\alpha, \beta) \in S$ are positive with magnitudes so chosen that g is smooth. In order to study (2.1) in a neighborhood of $(0, p/q)$ we let $\omega=p/q+\varepsilon\Delta$. Thus arbitrarily small squares about $(0, p/q)$ can be given by fixing Δ in some finite range, say $-1\leq\Delta\leq 1$, and requiring ε to be small, say $|\varepsilon|\leq\varepsilon_0$. The variable Δ is classically called the detuning.

In order to find the periodic solutions of (2.1) of rotation number p/q we use the change of variables dictated by the method of averaging. The change of variables

(2.5)
$$\phi=\theta+\varepsilon u(\theta, t)$$

reduces (2.1) to

(2.6)
$$\dot{\phi}=p/q+\varepsilon\{\Delta+h(\theta, t)\}+0(\varepsilon^2)$$

where

(2.7)
$$h=-g+\left(\frac{p}{q}\right)\frac{\partial u}{\partial\theta}+\frac{\partial u}{\partial t}$$

—all functions evaluated at (θ, t). Thus if

$$u(\theta, t)=\sum_S b_{\alpha\beta} \cos 2\pi(\alpha\theta-\beta t)$$

where $b_{qp}=0$ and $b_{\alpha\beta}=qa_{\alpha\beta}/(\beta q-\alpha p)$ otherwise the equation (2.1) becomes

(2.8)
$$\dot{\phi}=p/q+\varepsilon\{\Delta-a_{qp} \sin 2\pi(q\phi-pt)\}+0(\varepsilon^2).$$

Note that u and hence h are independent of ε and Δ and thus (2.8) is a valid representation of (2.1) for ε sufficiently small. Equation (2.8) is easy to solve to first order in ε, indeed the solution of (2.8) satisfying $\phi(0)=\phi_0$ is given by

$$\phi(t, \phi_0, \varDelta, \varepsilon)=\phi_0+(p/q)t+\varepsilon t(\varDelta-a_{qp}\sin q2\pi\phi_0)+0(\varepsilon^2)$$

where $0(\varepsilon^2)$ is uniform for bounded t and $\varDelta$. From this last expression one computes

$$\phi(q, \phi_0, \varDelta, \varepsilon)-\phi_0-p=\varepsilon qr(\phi_0, \varDelta, \varepsilon) \tag{2.9}$$

where

$$r(\phi_0, \varDelta, \varepsilon)=\varDelta-a_{qp}\sin q2\pi\phi_0+0(\varepsilon). \tag{2.10}$$

From the above one sees that all solutions of (2.8) are periodic when $\varepsilon=0$ but the solution $\phi(t, \phi_0, \varDelta, \varepsilon)$ is periodic for $\varepsilon\neq0$ if and only if $r(\phi_0, \varDelta, \varepsilon)=0$. Thus we must investigate the zeros of r in more detail.

When $\varepsilon=0$ the equation $r=0$ can be solved for $\varDelta$ as a function of ϕ_0, i.e., $r(\phi_0, a_{qp}\sin q2\pi\phi_0, 0)\equiv0$. Moreover, $\dfrac{\partial r}{\partial\varDelta}\equiv1$. Thus by the implicit function theorem there is a function $d(\phi_0, \varepsilon)=a_{qp}\sin q2\pi\phi_0+0(\varepsilon)$ such that $r(\phi_0, d(\phi_0, \varepsilon), \varepsilon)\equiv0$. The function d is defined and smooth for all ϕ_0 and $|\varepsilon|<\varepsilon_2$ for some $\varepsilon_2>0$ and is 1-periodic in ϕ_0. It is important to note that we solved for $\varDelta$ as a function of ϕ_0 and ε instead of solving for ϕ_0 as a function of $\varDelta$ and ε as is customary in perturbation theory. As we shall see ϕ_0 is a multivalued function of $\varDelta$ and an attempt to solve for ϕ_0 leads to difficulties or partial results.

In the discussion which follows we wish to consider d as a function of ϕ_0 and ε as a parameter and so use the notation $d_\varepsilon(\phi_0)=d(\phi_0, \varepsilon)$. Also a periodic solution (orbit) will always be a periodic solution (orbit) with rotation number p/q. Since a periodic orbit gives rise to q periodic solutions the number of ϕ_0 mod 1 which give rise to periodic solutions must be a multiple of q. Thus the number of zeros of $\varDelta-d_\varepsilon(\phi_0)=0$ for fixed $\varDelta$ and ε must be a multiple of q. d_0 achieves its maximum value of a_{pq} at the points $(1+4i)/4q$, $i=0, \pm1, \pm2, \cdots$ and achieves its minimum value of $-a_{qp}$ at the points $(-1+4i)/4q$, $i=0, \pm1, \pm2, \cdots$. The derivative of d_0 at these points is zero and the second derivative is non-zero and so by the implicit function theorem these maxima and minima persist under perturbations. Since the number of zeros of $\varDelta-d_\varepsilon(\phi_0)=0$ must be a multiple of q in any interval of length 1 the maximum values of d_ε must all be the same. The same is true for minimum values. Thus for $|\varepsilon|\leq\varepsilon_2$, some $\varepsilon_2>0$, there are smooth functions

$$\begin{aligned}
&\psi_i(\varepsilon)=(-1+4i)/4q+0(\varepsilon), && i=0, \pm1, \pm2, \cdots\\
&\varPsi_i(\varepsilon)=(1+4i)/4q+0(\varepsilon), && i=0, \pm1, \pm2, \cdots\\
&M(\varepsilon)=a_{qp}+0(\varepsilon)\\
&m(\varepsilon)=-a_{qp}+0(\varepsilon)
\end{aligned}$$

such that d_ε achieves its maximum value of $M(\varepsilon)$ at $\varPsi_i(\varepsilon)$ and its minimum value of

$m(\varepsilon)$ at $\psi_i(\varepsilon)$. Moreover, the derivative of d_ε is positive for $\psi_i(\varepsilon) \le \phi_0 \le \Psi_i(\varepsilon)$ and negative for $\Psi_i(\varepsilon) \le \phi_0 < \psi_{i+1}(\varepsilon)$.

For $|\varepsilon| \le \varepsilon_2$ there are three important parameter ranges:
Case 1: $\Delta < m(\varepsilon)$ or $\Delta > M(\varepsilon)$
Case 2: $\Delta = m(\varepsilon)$ or $\Delta = M(\varepsilon)$
Case 3: $m(\varepsilon) < \Delta < M(\varepsilon)$.

In cases 1, 2 and 3 the number of zeros of $\Delta - d_\varepsilon(\phi_0) = 0$ in an interval of length 1 are 0, q and $2q$ respectively. So in cases 1, 2 and 3 equation (2.1) has 0, 1 and 2 periodic orbits respectively.

Thus the local boundary curves for $A_{p/q}$ are $a(\varepsilon) = p/q + \varepsilon M(\varepsilon)$, $b(\varepsilon) = p/q + \varepsilon m(\varepsilon)$ and they are clearly smooth.

The information obtained so far is sufficient to calculate the characteristic multipliers of these periodic orbits. If $\phi(t, \phi_0, \Delta, \varepsilon)$ is a periodic solution of (2.1) then its characteristic multiplier is $\mu = \frac{\partial \phi}{\partial \phi_0}(q, \phi_0, \Delta, \varepsilon)$. Considering this solution as a periodic orbit for (2.2) then its characteristic multipliers are 1 and μ. Referring to (2.9) one computes

$$\begin{aligned} \frac{\partial \phi}{\partial \phi_0}(q, \phi_0, d(\phi_0, \varepsilon), \varepsilon) &= 1 + \varepsilon q \frac{\partial r}{\partial \Delta} \frac{\partial d}{\partial \phi_0} \\ &= 1 + \varepsilon q \frac{d d_\varepsilon}{d\phi_0}. \end{aligned}$$

Thus when $\frac{d d_\varepsilon}{d\phi_0} > 0$ (resp. < 0) the characteristic multiplier is greater (resp. less) than one and the periodic orbit is unstable (resp. stable). When the derivative of d_ε is zero the priodic orbit is degenerate.

In summary:
The local boundary curves for $A_{p/q}$ are smooth and of the form

$$\begin{aligned} a(\varepsilon) &= p/q + \varepsilon a_{qp} + O(\varepsilon^2) \\ b(\varepsilon) &= p/q - \varepsilon a_{qp} + O(\varepsilon^2). \end{aligned}$$

For points on these local boundary curves (except for $\varepsilon = 0, \omega = p/q$) equation (2.1) has one degenerate periodic orbit. For the interior points of the local sector the equation (2.1) has 2 periodic orbits one stable and one unstable.

3. Normalization via Lie transforms

There are a multitude of methods for establishing the existence of periodic solutions. An effective and standard procedure is to successively transform the equations into a simplier normal form and then use the implicit function theorem. The evolution of this method can be seen by referring to Poincare [18], Birkhoff [1], Krylov and Bogoliubov [14], Bogoliubov and Mitropolskii [2] and Diliberto [7]. A significant recent advance in this method is an algorithm given by Hori [13] to construct the transformation by Lie series. By modifying Hori's algorithm Deprit [6] and Henrard [11] developed a recursive algorithm. Since a recursive algorithm easily lends itself to inductive proofs and computer programs we use the Lie transform method of Deprit and Henrard. The first theorem and its corollary in this section are essentially the same as found in [7] and are present here to clarify the method and to help establish the subsequent new results.

Since the method of Lie transforms will be used repeatedly in subsequent arguments, a brief summary will be given. Consider a system of equations of the form

$$\dot{z}=Z_*(z,\varepsilon) \tag{3.1}$$

where Z_* has a formal expansion in ε of the form

$$Z_*(z,\varepsilon)=\sum_{j=0}^{\infty}(\varepsilon^j/j!)Z_j^0(z). \tag{3.2}$$

A change of variables $z=z(\zeta,\varepsilon)$ is constructed as a formal solution of a system of equations

$$\frac{dz}{d\varepsilon}=W(z,\varepsilon),\qquad z(0)=\zeta \tag{3.3}$$

where W has a formal expansion of the form

$$W(z,\varepsilon)=\sum_{j=0}^{\infty}(\varepsilon^j/j!)W_{j+1}(z). \tag{3.4}$$

Equation (3.1) in new coordinates becomes

$$\dot{\zeta}=Z^*(\zeta,\varepsilon) \tag{3.5}$$

where Z^* has the formal expansion

$$Z^*(\zeta,\varepsilon)=\sum_{j=0}^{\infty}(\varepsilon^j/j!)Z_0^j(\zeta). \tag{3.6}$$

The method of Lie transforms introduces a double indexed array of functions $\{Z_j^i\}$ which agree with the previous definitions when either i or j is zero and are related by the recursive formula

(3.7) $$Z_j^i = Z_{j+1}^{i-1} + \sum_{k=1}^{j} \binom{j}{k} [Z_{j-k}^{i-1}, W_{k+1}]$$

where [,] is the Lie bracket operator defined by

(3.8) $$[A, B] = \frac{\partial A}{\partial z} B - \frac{\partial B}{\partial z} A.$$

The interdependence of the functions $\{Z_j^i\}$ can be easily understood by considering the Lie triangle

$$\begin{array}{ccccc} Z_0^0 & & & & \\ \downarrow & & & & \\ Z_1^0 & \longrightarrow & Z_0^1 & & \\ \downarrow & & \downarrow & & \\ Z_2^0 & \longrightarrow & Z_1^1 & \longrightarrow & Z_0^2 \\ \downarrow & & \downarrow & & \downarrow . \end{array}$$

The coefficients of the expansion of Z_* are in the left column and of the expansion of Z^* are on the diagonal. From (3.7) one calculates an entry in the array by using the functions in the column one step to the left and up. The derivation and a discussion of these formulas are found in [11].

In the applications to follow the expansion for Z_* is given and the expansion for W is to be found so that the expansion for Z^* is in a specific normal form. In order to summarize the normalization procedure we introduce three sequences of vector spaces of functions, one for the coefficients of Z_*, one for the coefficients of Z^* and one for the coefficients of W. These sequences of functions can be used to state the following general theorem on normalization.

Theorem 1. *Let $\{P_i\}_{i=0}^{\infty}$, $\{Q_i\}_{i=1}^{\infty}$ and $\{R_i\}_{i=1}^{\infty}$ be sequences of vector spaces of functions (from n-space to n-space) with the following properties:*

i) $Q_i \subset P_i$, $i=1, 2, \cdots$

ii) $Z_i \in P_i$, $i=0, 1, 2, \cdots$

iii) $[R_i, P_j] \subset P_{i+j}$, $i, j=0, 1, 2, \cdots$

iv) *for any $A \in P_i$, and $i=1, 2, \cdots$, there exists $B \in Q_i$ and $C \in R_i$ such that*

$$B = A + [C, Z_0^0].$$

Then there exists a W with an expansion of the form (3.4) *with $W_i \in R_i$ which transforms* (3.1) *to* (3.5) *where Z^* has an expansion of the form* (3.6) *and $Z_0^i \in Q_i$,*

$i=1, 2, \cdots$.

Proof. Use induction on the rows of the Lie triangle. Induction Hypothesis I_n: Let $Z_j^i \in P_{i+j}$ for $0 \leq i+j \leq n$ and $W_i \in R_i$, $Z_0^i \in Q_i$ for $1 \leq i \leq n$.

I_0 is true by assumption and so assume I_{n-1}. By (3.7)

$$Z_{n-1}^1 = Z_n^0 + \sum_{k=0}^{n-2} \binom{n-1}{k} [W_{k+1}, Z_{n-1-k}^0] + [W_n, Z_0^0].$$

The last term is singled out since it is the only term not covered by the induction hypothesis or the hypothesis of the theorem. All the other terms are in P_n by I_{n-1} and iii). Thus

$$Z_{n-1}^1 = K^1 + [W_n, Z_0^0]$$

where $K^1 \in P_n$. A simple induction on the columns using (3.7) shows that

$$Z_{n-l}^l = K^l + [W_n, Z_0^0]$$

where $K^l \in P_n$ for $l=1, \cdots, n$ and so

$$Z_0^n = K^n + [W_n, Z_0^0].$$

By iv) there are solutions $W_n \in R_i$ and $Z_0^n \in Q_i$. Thus I_n is true.

Remark. Usually the function Z_* is given and so the spaces $\{P_i\}$ are defined naturally. The spaces $\{Q_i\}$ and $\{R_i\}$ are defined by the operator $L: C \rightarrow [C, Z_0^0]$. If for example the operator L is such that the spaces P_i are the direct sum of the range and the kernel of L then the natural choice for Q_i is the range of L and the one for R_i is the kernel of L.

The following well known theorem is a consequence of this theorem.

Corollary. *Consider the system*

(3.9) $$\dot{x} = Ax + f_*(x, \varepsilon)$$

where A is diagonalizable, $f_*(x, 0)=0$ and f_* has a formal expansion in ε with coefficients which are polynomials in x. Then there exists a formal change of variables $x=x(\xi, \varepsilon)$, $x(\xi, 0)=\xi$, which reduces (3.9) *to*

(3.10) $$\dot{\xi} = A\xi + f^*(\xi, \varepsilon)$$

where f^* has the property

(3.11) $$e^{-At} f^*(e^{At}\xi, \varepsilon) = f^*(\xi, \varepsilon).$$

Proof. Use coordinates such that A is diagonal and let P_i be the vector space of polynomials in these coordinates. It is easy to see that P_i is the direct sum of the range of L and the kernel of L as in the remark above. It is also easy to check that each term in the kernel of L satisfies (3.11).

If f_* is analytic in ε then a convergent transformation can be constructed by truncation and so (3.9) can be reduced to

$$\dot{\xi}=A\xi+f_N^*(\xi,\varepsilon)+0(\varepsilon^{N+1}) \tag{3.12}$$

where f_N^* satisfies (3.11) and N is any fixed positive integer. The equation $\dot{\xi}=A\xi+f_N^*(\xi,\varepsilon)$ will be called the Nth average of (3.9).

Equation (3.10) is easy to analyze. The function $e^{\lambda At}u_0$, λ and u_0 constant, is a solution of (3.10) if and only if

$$0=(1-\lambda)Au_0+f^*(u_0,\varepsilon). \tag{3.13}$$

Thus if the linearized system is T-periodic, i.e., $e^{AT}=I$, then (3.10) has a periodic solution of the form $e^{\lambda At}u_0$ if and only if (3.13). Equations (3.13) are the bifurcation equations. Similarly for (3.12), if $e^{AT}=I$ then a periodic solution of (3.12) of the form $(u_0+0(\varepsilon^N))e^{(\lambda+0(\varepsilon^N))At}$ exists if and only if

$$0=(1-\lambda)Au_0+f^*(u_0,\varepsilon)+0(\varepsilon^N). \tag{3.14}$$

With A. Deprit we have written a $P1/I$ program which preforms the algebraic manipulations to effect the normalization described in the corollary to Theorem 1. The abundance of examples given in section 5 illustrates the effectiveness of the program and the method.

We wish to consider two systems. The first is the standard forced van der Pol equation

$$\ddot{u}+\varepsilon(u^2-1)u+\omega_1^2u=\varepsilon A\cos\omega_2 t. \tag{3.15}$$

In view of the example we should look for periodic solutions when the ratio of the natural frequency ω_1 to the forcing frequency ω_2 is nearly rational. With this in mind, replace ω_1^2 by $p^2+\varepsilon^2 p\Delta$ and ω_2 by q. Then the above system is equivalent to

$$\begin{aligned}\dot{u}_1&=pu_3\\ \dot{u}_2&=qu_4\\ \dot{u}_3&=-pu_1+\varepsilon\{(1-u_1^2)u_3+u_2\}-\varepsilon^2\Delta u_1\\ \dot{u}_4&=-qu_2.\end{aligned} \tag{3.16}$$

The second is the system of two weakly coupled van der Pol equations (see [16])

$$\begin{aligned}\ddot{u}_1+\varepsilon\{(u_1+\varepsilon\lambda u_2)^2-1\}\dot{u}_1+\omega_1^2(u_1+\varepsilon\lambda u_2)=0\\ \ddot{u}_2+\varepsilon\{(u_2+\varepsilon\lambda u_1)^2-1\}\dot{u}_2+\omega_2^2(u_2+\varepsilon\lambda u_1)=0.\end{aligned} \tag{3.17}$$

By letting $\omega_1^2=p^2+\varepsilon^2 p\Delta$ and $\omega_2^2=q^2$ we obtain the equivalent system

$$\begin{aligned}&\dot{u}_1=pu_3\\ &\dot{u}_2=qu_4\\ &\dot{u}_3=-pu_1+\varepsilon\{(1-u_1^2)u_3-\lambda pu_2\}+\varepsilon^2\{-2\lambda u_1u_2u_3-\Delta u_1\}-\varepsilon^3\{\lambda^2u_2^2u_3+\lambda\Delta u_2\}\\ &\dot{u}_4=-qu_2+\varepsilon\{(1-u_2^2)u_4-\lambda qu_1\}-2\varepsilon^2\lambda u_1u_2u_4-\varepsilon^3\lambda^2u_1^2u_4.\end{aligned} \tag{3.18}$$

In order to simplify the calculations we change to complex coordinates in (3.16) and (3.18) by letting

$$\begin{aligned}y_1&=u_1-iu_3\\ y_2&=u_2-iu_4.\end{aligned} \tag{3.19}$$

In these variables the forced van der Pol equation (3.16) becomes

$$\begin{aligned}\dot{y}_1=&\,ipy_1+(\varepsilon/8)\{4(y_1-\bar{y}_1)+4\Delta i(y_1+\bar{y}_1)\\ &-4i(y_2-\bar{y}_2)-y_1^3-y_1^2\bar{y}_1+y_1\bar{y}_1^2+\bar{y}_1^3\}\\ \dot{y}_2=&\,iqy_2.\end{aligned} \tag{3.20}$$

If the ratio p/q is not 1 then the first average is easy to compute by using the intrinsic condition (3.11). Look at the right hand side of the first equation in (3.20) term by term. Consider a term of the form

$$cy_1^{\alpha_1}y_2^{\alpha_2}\bar{y}_1^{\beta_1}\bar{y}_2^{\beta_2}. \tag{3.21}$$

Since the solutions of the linear system (i.e., when $\varepsilon=0$) are $y_1(t)=a_1e^{ipt}$, $y_2(t)=a_2e^{iqt}$ such a term satisfies (3.11) if and only if

$$(\alpha_1-\beta_1-1)p+(\alpha_2-\beta_2)q=0. \tag{3.22}$$

Clearly, any term of the form

$$(y_1\bar{y}_1)^r(y_2\bar{y}_2)^s y_1 \tag{3.23}$$

satisfies this condition. However, when the ratio p/q is rational there may be other solutions of (3.22). The terms of the form (3.21) where the exponents satisfy (3.22) but are not of the form (3.23) will be called *resonance terms*. As we shall see these terms are important in determining the entrainment domains. Thus by looking at the terms in (3.10) which satisfy (3.22) the first average of (3.20) when $p/q\neq 1$ is

$$\begin{aligned}\dot{y}_1&=\{ip+(\varepsilon/8)(4-y_1\bar{y}_1)\}y_1\\ \dot{y}_2&=iqy_2.\end{aligned} \tag{3.24}$$

These averaged equations are easily analyzed in the polar coordinates $r_1e^{i\theta_1}=y_1$, $r_2e^{i\theta_2}=y_2$. In these coordinates (3.24) becomes

$$\begin{aligned}\dot{r}_1&=(\varepsilon/8)(4-r_1^2)r_1\\ \dot{\theta}_1&=p\\ \dot{r}_2&=0\\ \dot{\theta}_2&=q.\end{aligned} \tag{3.25}$$

Similarly the first average of (3.18) in complex coordinates is

$$\begin{aligned}\dot{y}_1&=\{ip+(\varepsilon/8)(4-y_1\bar{y}_1)\}y_1\\ \dot{y}_2&=\{iq+(\varepsilon/8)(4-y_2\bar{y}_2)\}y_2\end{aligned} \tag{3.26}$$

and in polar coordinates is

$$\begin{aligned}\dot{r}_1&=(\varepsilon/8)(4-r_1^2)r_1\\ \dot{\theta}_1&=p\\ \dot{r}_2&=(\varepsilon/8)(4-r_2^2)r_2\\ \dot{\theta}_2&=q.\end{aligned} \tag{3.27}$$

Both of these averaged equations admit invariant tori. For (3.25) the torus $r_1=2$, $r_2=A/p$ (A constant) is invariant and for (3.27) the torus $r_1=r_2=2$ is invariant. The flows on these invariant tori are linear. A classical theorem [8, 9, 15] asserts that these tori persist under small perturbations and so equations (3.16) and (3.17) admit invariant tori for small ε.

On the formal side a further normalization can be made. Consider the formal system

$$\begin{aligned}\dot{r}&=R_*(r,\theta,\varepsilon)=\sum_{j=1}^{\infty}(\varepsilon^j/j!)R_j^0(r,\theta)\\ \dot{\theta}&=\theta_*(r,\theta,\varepsilon)=\sum_{j=0}^{\infty}(\varepsilon^j/j!)\theta_j^0(r,\theta)\end{aligned}$$

where r is a k vector, θ is an l vector, and R_j^0, θ_j^0 have finite Fourier expansions with smooth coefficients.

Theorem 2. *Assume that* $R^0(r,\theta)=\rho(r)$, $\rho(r_0)=0$, $\dfrac{\partial\rho}{\partial r}(r_0)$ *has no pure imaginary eigenvalues and* $\theta_0^0(r,\theta)=\omega$ *is a constant vector. Then there exists a formal* *change of variables* $(r, \theta) \rightarrow (\bar{r}, \bar{\theta})$ *such that the above equations become*

$$\dot{\bar{r}} = R^*(\bar{r}, \bar{\theta}, \varepsilon)$$
$$\dot{\bar{\theta}} = \theta^*(\bar{r}, \bar{\theta}, \varepsilon)$$

where R^* *and* θ^* *are like* R_* *and* θ_* *but have the additional property that*

$$R^*(\bar{r}_0, \bar{\theta}, \varepsilon) \equiv 0, \qquad \theta^*(\bar{r}_0, \bar{\theta} + \omega t, \varepsilon) \equiv \theta^*(\bar{r}_0, \bar{\theta}, \varepsilon).$$

Proof. See [2] or [7]. A constructive proof based on Lie transformations can be given, but this proof is essentially the same as the proof of Theorem 1.

Theorem 2 applies directly to the weakly coupled van der Pol equations after the first average transformation has been made. Due to the special nature of the second set of equations for the forced van der Pol equation the above theorem or its proof can be made to apply to these equations also.

We want to analyze the flow on the invariant tori by searching for periodic solutions. Therefore we write down the usual bifurcation equations, but we find that up to order ε the bifurcation equations are dependent and that we cannot call on the implicit function theorem to show that the full bifurcation equations can be solved. For this reason we will go to higher averaged equations until the bifurcation equations become independent. This occurs with the appearance of the first nonzero resonance term.

Condition (3.22) and the fact that our equations contain only terms of odd degree determine that the resonance term of smallest degree in the differential equation for y_1 is $\bar{y}_1^{q-1} y_2^p$ in case p and q are odd and it is $\bar{y}_1^{2q-1} y_2^{2p}$ in case p and q have different parities.

Our goal is to predict at which order of ε these resonance terms can occur for the first time in the averaged equations. We will restrict the discussion to the weakly coupled van der Pol equations (3.18) as the result for the forced van der Pol equation follows easily afterwards. The equations (3.18) written in the complex coordinates (3.20) have the following general form

$$\text{(3.28)} \qquad \begin{aligned} \dot{y}_1 &= ipy_1 + \varepsilon f_1^0 + \frac{\varepsilon^2}{2} f_2^0 + \cdots \\ \dot{y}_2 &= iqy_2 + \varepsilon g_1^0 + \frac{\varepsilon^2}{2} g_2^0 + \cdots \end{aligned}$$

plus 2 conjugate complex equations for $\bar{y}_1$ and $\bar{y}_2$. The notation for the functions was chosen to coincide with the one used in the Lie transformation. We will show that all functions in an entire row of the Lie triangle lie in certain subspaces of the vector space of homogeneous polynomials. The functions f_i^0 and g_i^0 contain terms of the form (3.21). Let $\gamma_1 = \alpha_1 + \beta_1$ and $\gamma_2 = \alpha_2 + \beta_2$ so that (γ_1, γ_2) indicates the order of a term in the two sets of conjugate variables. All polynomials whose terms lead to the same values of γ_1 and γ_2 define a subspace in the vector space of homogeneous polynomials. Instead of listing all possible pairs (γ_1, γ_2) which occur in a row of the Lie triangle we will specify γ_2 but only the maximal value for γ_1. Thus (γ_1, γ_2) defines a subspace which has the following basis vectors $y_1^{\alpha_1}y_2^{\alpha_2}\bar{y}_1^{\beta_1}\bar{y}_2^{\beta_2}$ with $0 \leq \alpha_1+\beta_1 \leq \gamma_1$ and $\alpha_2+\beta_2=\gamma_2$. Later on it will be convenient to allow γ_1 or γ_2 to be negative and we adopt the convention that the corresponding subspace consists only of the zero vector.

In accordance with (3.7) and (3.4) we set $Z_j^i=(f_j^i, g_j^i, \bar{f}_j^i, \bar{g}_j^i)^T$ and $W_m=(u_m, v_m, \bar{u}_m, \bar{v}_m)^T$. We are now in a position to describe the terms which appear in the $(m+1)^{st}$ row of the Lie triangle, $i+j=m$ and in the transforming function W_m. For example the terms of f_1^0 lie in subspaces described by $(3, 0)$ and $(0, 1)$ whereas those for g_1^0 are given by $(0, 3)$ and $(1, 0)$. Due to the symmetry between the functions f_i^0 and g_i^0 it will be sufficient to list the subspaces for the first component in Z_i^j and W_m only.

Lemma. *For the weakly coupled van der Pol equation, terms in f_j^i with $i+j=m$ and in u_m lie in subspaces described by $(2m+1, 0)$ or by $(2m-2-\rho-(-1)^\rho, \rho)$ $\rho=1, 2, \cdots$.*

Proof. We use the result and notation of Theorem 1. We have $P_m=R_m$ and the first component of this vectorspace has a basis of the form $y_1^{\alpha_1}y_2^{\alpha_2}\bar{y}_1^{\beta_1}\bar{y}_2^{\beta_2}$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1+\beta_1 \leq \gamma_1$, $\alpha_2+\beta_2=\gamma_2$ and the possible pairs of (γ_1, γ_2) are given by $(2m+1, 0)$ or $(2m-2-\rho-(-1)^\rho, \rho)$. One verifies that $Z_m^0 \in P_m$, that is the terms in f_m^0 have the proper form. For example, P_1 is specified by (3,0) (0,1) which agrees with the bounds on the exponents given for f_1^0 earlier.

Next we have to verify $[P_i, P_j] \subset P_{i+j}$, $i, j=0, 1, \cdots$. Because of the bilinearity of the Lie bracket it will be sufficient to verify it for a basis only. Let $Z_k=(f_k, g_k, f_k, g_k)^T \in P_k$, $k=i$ or j, be basis vectors and it suffices to verify our bound on the order of the terms for the expression

$$\frac{\partial f_i}{\partial y_1} f_j + \frac{\partial f_i}{\partial y_2} g_j. \tag{3.29}$$

Both products give terms whose exponents are bounded by the expression given in the lemma with $m=i+j$. The statement of the lemma follows now from Theorem 1.

Theorem 3. *Consider equation* (3.18) *written in complex coordinates. In case p and q are both odd a resonance term of the form $y_2^p\bar{y}_1^{q-1}$ can occur for the first time at order $\varepsilon^{(p+q)/2}$ in the normalized equation for y_1. In case p and q have* *different parities the resonance term* $y_2^{2p}\bar{y}_1^{2q-1}$ *will be of order* ε^{p+q+1}.

Proof. In case p and q are odd we need a term of order $(q-1, p)$. From the previous lemma $\rho=p$ and $q-1\leq 2m-2-\rho-(-1)^\rho$. The last inequality reduces to $p+q\leq 2m$. Since $p+q$ is even the smallest integer which satisfies the inequality is $m=(p+q)/2$.

In case $p+q$ is odd we need $\rho=2p$ and $2q-1\leq 2m-2-\rho-(-1)^\rho$ which is satisfied by $m=p+q+1$.

Similar results can be found for the forced van der Pol equation.

Theorem 4. *Consider equations* (3.20). *In case p and q are both odd a resonance term can occur for the first time at order* $\varepsilon^{(3p+q-2)/2}$ *and in case p and q have different parities it will at order* ε^{3p+q-1}.

Proof. Although the symmetry of the equations is destroyed the proof is very similar to the previous one and in some respects even easier since the equation for y_2 is already in normal form. One shows that the functions in the $(m+1)^{st}$ row of the Lie triangle (order ε^m) lie in a vectorspace described by $(2m-3\rho+1, \rho)$ where $\rho=0, 1, 2, \cdots$. From this bound on the order of the terms the statement of the theorem follows at once.

Table 1 gives the ratios $p:q$ which produce resonance terms of low order as predicted by our theorems. It has to be pointed out that our theorems only provide a lower bound as the coefficient of the resonance term could happen to be zero.

As remarked before we have written a *PL/I* program which performs the normalization described in theorem 1 and we have used this program to check all cases in table 1 up to order 8 for the forced van der Pol equation and up to order 6 for the weakly coupled van der Pol equations.

Table 1. Ratios of $p:q$ which result in resonance terms at order ε^m

m	$p:q$ for 3.16 or 3.20	$p:q$ and $q:p$ for 3.18
2	1:3	1:3
3	1:5	1:5
4	1:7, 1:2, 3:1	1:7, 3:5, 1:2
5	1:9	1:9, 3:7
6	1:11, 1:4, 3:5, 2:1	1:11, 5:7, 1:4, 2:3
7	1:13 3:7 5:1	1:13, 3:11, 5:9
8	1:15, 1:6 2:3, 5:3	1:15, 3:13, 5:11, 7:9, 1:6, 2:5, 3:4
9	1:17 3:11	1:17, 5:13, 7:11
10	1:19, 1:8, 3:13, 2:5, 5:7, 3:2, 7:1	1:19, 13:17, 7:13, 9:11, 1:8, 2:7, 4:5

Many of the computations for the forced van der Pol equation were checked against a similar Fortran-Assembler program written by J. Henrard. Except for the 1:7

resonance case for the forced van der Pol equation the resonance term did appear as predicted. We checked this case by hand and found that the coefficient of the resonance term $\bar{y}_1^6 y_2$ was zero at order ε^4. At the next order a resonance term was found. As we shall see in the next section the order at which a resonance term appears (with nonzero coefficient) has a direct bearing on the width of the entrainment domain.

4. Computing the boundary curves

In this section we shall compute the boundary curves for the entrainment domains from the averaged equations. Due to the symmetry of the forced van der Pol equation (see [12]) there are three cases for the averaged equations depending on the parities of p and q. Recall that the forced van der Pol equation is

$$\ddot{u}+\omega_1^2 u=\varepsilon\{(1-u^2)\dot{u}+A\cos\omega_2 t\} \tag{4.1}$$

and that we have set $\omega_2=q$ and $\omega_1^2=p^2+\varepsilon^2 p\Delta$. The averaged equations in polar coordinates are as follows:

Case 1: $p+q$ odd. Then $m=3p+q-1$ is even and the averaged equations have the form

$$\begin{aligned}\dot{r}&=\{(\varepsilon/8)(4-r^2)+\varepsilon^3\xi_3(r,A,\Delta)+\cdots+\varepsilon^{m-1}\xi_{m-1}(r,A,\Delta)\}r\\&\quad+\varepsilon^m C r^{2q-1}A^{2p}\sin 2q(\theta-pt)\\\dot{\theta}&=p+\varepsilon^2 n_2(r,A,\Delta)+\cdots+\varepsilon^m n_m(r,A,\Delta)\\&\quad+\varepsilon^m C r^{2q-2}A^{2p}\cos 2q(\theta-pt).\end{aligned} \tag{4.2}$$

Case 2: $p+q$ even and $(p-q)/2$ even. Then $m=(3p+q-2)/2$ is odd and the averaged equations have the fom

$$\begin{aligned}\dot{r}&=\{(\varepsilon/8)(4-r^2)+\cdots+\varepsilon^m\xi_m(r,A,\Delta)\}r\\&\quad+\varepsilon^m C r^{q-1}A^p\sin q(\theta-pt)\\\dot{\theta}&=p+\cdots+\varepsilon^{m-1}\eta_{m-1}(r,A,\Delta)\\&\quad+\varepsilon^m C r^{q-1}A^p\cos q(\theta-pt).\end{aligned} \tag{4.3}$$

Case 3: $p+q$ even and $(p-q)/2$ odd. Then $m=(3p+q-2)/2$ is even and the averaged equations have the form

$$\begin{aligned}\dot{r}&=\{(\varepsilon/8)(4-r^2)+\cdots+\varepsilon^{m-1}\xi_{m-1}(r,A,\Delta)\}r\\&\quad+\varepsilon^m C r^{q-1}A^p\cos q(\theta-pt)\\\dot{\theta}&=p+\cdots+\varepsilon^m\eta_m(r,A,\Delta)\\&\quad-\varepsilon^m C r^{q-1}A^p\sin q(\theta-pt).\end{aligned} \tag{4.4}$$

In the above $\xi_3, \xi_5, \cdots$ and $\eta_2, \eta_4, \cdots$ are polynomials in r, A and Δ with $\xi_3(2, A, \Delta) = \xi_5(2, A, \Delta) = \cdots = 0$ and $\eta_2(2, A, \Delta) = \Delta/2 - \eta_2^*(A)$. In order to find the boundary curves we must expand the frequency ratio in a series in ε. Let $\Delta = \Delta_2 + \Delta_3\varepsilon + \cdots$ so that $\omega_1^2 = p(p + \Delta_2\varepsilon^2 + \Delta_3\varepsilon^3 + \cdots)$ and $\omega_2 = q$. The coefficients Δ_2, $\Delta_3, \cdots$ are to be determined so that the above equations have a periodic orbit on the torus with rotation number p/q.

Consider case 1—the other cases are similar. Since $\xi_3(2, A, \Delta) = \cdots = \xi_{m-1}(2, A, \Delta) = 0$ the invariant torus is of the form $r = \rho(\theta, \varepsilon) = 2 + 0(\varepsilon^{m-1})$. Thus the equation for the flow on this torus has the form

$$\dot{\theta} = p + \varepsilon^2\eta_2(2, A, \Delta) + \cdots + \varepsilon^m\eta_m(2, A, \Delta)$$
$$+ \varepsilon^m C 2^{2q-2} A^{2p} \cos 2q(\theta - pt) + 0(\varepsilon^{m+1}).$$

This equation will not have a periodic solution for small ε unless Δ is chosen so that the $\eta_2, \cdots, \eta_{m-1}$ are zero. Using the expansion for Δ yields $\eta_k(2, A, \Delta) = \Delta_k/2 - \eta_k^*(A, \Delta_2, \cdots, \Delta_{k-1})$ for $k = 2, 3, \cdots$ and $\eta_k^* = 0$ if k is odd. Thus set the odd terms to zero and choose the even terms by recursion. That is, choose

$$\begin{aligned} \Delta_2 &= 2\eta_2^*(A) \\ \Delta_3 &= 0 \\ \Delta_4 &= 2\eta_4^*(A, \Delta_2, \Delta_3) \\ &\vdots \\ \Delta_{m-1} &= 0. \end{aligned} \tag{4.5}$$

Also define

$$\Delta_m^* = \eta_m^*(A, \Delta_2, \cdots, \Delta_{m-1}) \quad \text{and} \quad C^* = 2^{2q-2} A^p C. \tag{4.6}$$

Now the flow on the torus is

$$\dot{\theta} = p + \varepsilon^m\{\Delta_m - \Delta_m^* + C^* \cos 2q(\theta - pt)\} + 0(\varepsilon^{m+1}). \tag{4.7}$$

This equation is almost of the same form as the example treated in section 2 and can be studied in the same manner. In particular the boundary curves are

$$\omega_1^2/\omega_2^2 = \frac{p}{q^2}(p + \Delta_2\varepsilon^2 + \cdots + \Delta_{m-2}\varepsilon^{m-2} + (\Delta_m^* \pm C^*)\varepsilon^m\} + 0(\varepsilon^{m+1})$$

where the coefficients are given by (4.5) and (4.6). Proceeding as in section 2 with this case and the others yields:

Theorem 5. *For the forced van der Pol equation the boundary curves have order of contact at least equal to* $m-1$.

If for a particular p and q the resonance term appears, i.e. $C\neq 0$, then the boundary curves are analytic in ε and have an order of contact equal to $m-1$. They can be computed by the scheme given above. For the interior points of the local section defined by these boundary curves the equation has two (resp. one) stable and two (resp.one) unstable periodic orbits with rotation number p/q in case 1 (resp. cases 2 and 3).

For the weakly coupled van der Pol equations (3.17) the averaged equations and analysis is similar. The algebra is slightly more complicated but straightforward and so only a summary will be given. There are three cases depending on the parity of p and q.

Case 1: $p+q$ odd.

$$(4.8)\quad \begin{aligned} \dot r_1&=r_1\{(\varepsilon/8)(4-r_1^2)+\varepsilon^3P_3+\cdots+\varepsilon^{m-1}P_{m-1}\}-\varepsilon^m c r_1^{2q-1}r_2^{2p}\sin 2(p\theta_2-q\theta_1)\\ \dot r_2&=r_2\{(\varepsilon/8)(4-r_2^2)+\varepsilon^3Q_3+\cdots+\varepsilon^{m-1}Q_{m-1}\}-\varepsilon^m d r_1^{2q}r_2^{2p-1}\sin 2(p\theta_2-q\theta_1)\\ \dot\theta_1&=p+\varepsilon^2P_2+\cdots+\varepsilon^mP_m+\varepsilon^m c r_1^{2q-2}r_2^{2p}\cos 2(p\theta_2-q\theta_1)\\ \dot\theta_2&=q+\varepsilon^2Q_2+\cdots+\varepsilon^mQ_m+\varepsilon^m d r_1^{2q}r_2^{2p-2}\cos 2(p\theta_2-q\theta_1). \end{aligned}$$

In these equations P_i and Q_i are polynomials in $(r_1, r_2, \Delta, \lambda)$. Moreover, $P_2(2, 2, \Delta, \lambda)=\Delta/2-P_2^*(\lambda)$ and $P_i(2, 2, \Delta, \lambda)=Q_i(2, 2, \Delta, \lambda)=0$ for odd i.

Case 2: $p+q$ even and $m=(p+q)/2$ even. In this case the resonance term for the r_1 equation has the form $\varepsilon^m c r_1^{q-1}r_2^p\cos(p\theta_2-q\theta_1)$. The other resonance terms are similar.

Case 3: $p+q$ even and $m=(p+q)/2$ even. In this case the resonance term for the r_1 equation has the form $-\varepsilon^m c r_1^{q-1}r_2^p\sin(p\theta_2-q\theta_1)$.

For all these cases the invariant torus is $r_1=2+0(\varepsilon^{m+1})$, $r_2=2+0(\varepsilon^{m+1})$ and so the equation to order m for the flow on the invariant torus is obtained by setting $r_1=r_2=2$ in the θ_1 and θ_2 equations and then computing the equation for $d\theta_1/d\theta_2$. In case 1 the equation has the form

$$(4.9)\quad \frac{d\theta_1}{d\theta_2}=\frac{p}{q}+\varepsilon^2R_2+\cdots+\varepsilon^m(R_m+C^*\cos 2(q\theta_1-p\theta_2))$$

where R_j, $j=2, \cdots, m$ is a polynomial in Δ and λ and C^* is a constant. Proceeding as before expand Δ in a series in order to find periodic solutions on the invariant torus. In the next section we list this expansion in the form

$$(4.10)\quad \frac{\omega_1^2}{\omega_2^2}=\frac{p}{q^2}\{p+\varepsilon^2\Delta_2+\cdots+\varepsilon^m\Delta_m\}$$

by listing the Δ_i. In summary:

Theorem 6. *For the weakly coupled van der Pol equations the boundary curves have order of contact at least equal to* $m-1$.

If for a particular p *and* q *the resonance term appears, i.e.,* $C^* \neq 0$, *then the boundary curves are analytic in* ε *and have order of contact equal to* $m-1$. *For the interior points of the local sectors defined by these boundary curves the equations have two (resp. one) stable and two (resp. one) unstable periodic orbits with rotation number* p/q *in case* 1 *(resp. cases* 2 *and* 3).

5. Results

The previous section showed that the boundary curves for the local sectors are of the form

$$\frac{\omega_1^2}{\omega_2^2} = \frac{p}{q^2}(p + \varepsilon^2 \Delta_2 + \varepsilon^4 \Delta_4 + \cdots + \varepsilon^m \Delta_m) + 0(\varepsilon^{m+1}) \tag{5.1}$$

for both equations (3.15) and (3.17). The term Δ_m has the form $\Delta_m^* \pm C^*$ and the terms Δ_k depend only on the parameter λ of the given differential equations. In the forced van der Pol equation (3.15) the parameter λ was set to be $\lambda = \frac{A}{2p}$.

The tables 2 and 3 contain the values of the series (5.1) for the equations (3.15) and (3.17) respectively. We list the terms which are contained in Δ_2 to Δ_m for different values of p and q. The first column gives the subscript of Δ_k that is the order in ε. The second column gives the power with which λ appears in Δ_k and the third column its coefficient. The prefix $+-$ distinguishes the term C^* in Δ_m.

Table 2. Local sectors for the forced van der Pol equation (3.15) for different resonance cases

order ε	order λ	coefficient
$P=1,\ Q=3$		
2	0	1.250000000000E−01
2	1	+−1.250000000000E−01
$P=1,\ Q=5$		
2	0	1.250000000000E−01
3	1	+−5.208333333333E−03
$P=1,\ Q=7$		
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−3.002025462963E−04
5	1	+−4.973234953714E−05
$P=1,\ Q=2$		
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	3.703703703703E−03
4	2	+−2.777777777777E−02

order ε	order λ	coefficient
$P=3,\ Q=1$		
2	0	4.166666666666E−02
4	0	−5.545910493831E−04
4	2	1.136718749999E−01
4	3	+−1.757812500000E−02
$P=1,\ Q=9$		
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−9.505208333334E−05
5	1	+−2.712673611093E−06
$P=1,\ Q=11$		
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−3.959986772486E−05
6	0	2.484356915491E−03
6	1	+−2.938729745382E−07
6	2	8.517445721442E−06
6	4	3.068622317823E−09

$P=1$, $Q=4$

order ε	λ	coefficient
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−6.624338624338E−03
6	0	2.484356915491E−03
6	2	3.357591632772E−03
6	2	+−1.113040123456E−03
6	4	4.834803057025E−05

$P=3$, $Q=5$

order ε	λ	coefficient
2	0	4.166666666666E−02
4	0	−5.545910493831E−04
4	2	−7.742745535714E−03
6	0	1.022369101022E−05
6	2	1.581745747033E−03
6	3	+−1.322428385415E−05
6	4	2.225467136927E−04

$P=2$, $Q=1$

order ε	λ	coefficient
2	0	6.250000000000E−02
4	0	−1.871744791666E−03
4	2	6.338624338624E−01
6	0	7.763615360970E−05
6	2	−6.283028204107E−01
6	4	−5.469723691945E−01
6	4	+−7.901234567901E−02

$P=1$, $Q=13$

order ε	λ	coefficient
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−1.946587166612E−05
6	0	2.484356915491E−03
6	2	4.302956813584E−06
6	4	7.479760098577E−10
7	1	+−3.754146512604E−08

$P=3$, $Q=7$

order ε	λ	coefficient
2	0	4.166666666666E−02
4	0	−5.545910493831E−04
4	2	1.382812499999E−03
6	0	1.022369101022E−05
6	2	−1.994781894878E−04
6	4	−2.201660156250E−05
7	3	+−9.773254394531E−06

$P=5$, $Q=1$

order ε	λ	coefficient
2	0	2.499999999999E−02
4	0	−1.197916666665E−04
4	2	1.955708498677E−02
6	0	7.949942129454E−07
6	2	−1.382577699047E−03
6	4	−1.689814455995E−03
7	5	+−2.452866040139E−05

$P=1$, $Q=15$

order ε	λ	coefficient
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−1.069647539075E−05
6	0	2.484356915491E−03
6	2	2.408644265599E−06
6	4	2.268722305866E−10
8	0	−3.705793938550E−04
8	1	+−5.384691687817E−09
8	2	−6.085753311916E−07
8	4	−1.432160915784E−10

$P=1$, $Q=6$

order ε	λ	coefficient
2	0	1.250000000000E−01
4	0	−1.497395833333E−02
4	2	−6.362163913184E−04
6	0	2.484356915491E−03
6	2	1.171239151354E−04
6	4	6.960062936741E−07
8	0	−3.705793938550E−04
8	2	−3.847043128384E−05
8	2	+−5.680006377530E−07
8	4	−2.729921717543E−07

$P=2$, $Q=3$

order ε	λ	coefficient
2	0	6.250000000000E−02
4	0	−1.871744791666E−03
4	2	−9.688888888888E−02
6	0	7.763615360970E−05
6	2	8.934309633095E−02
6	4	1.948444444444E−02
8	0	−2.895151515248E−06
8	2	−8.584718717207E−02
8	4	−6.401513371620E−02
8	4	+−3.448223733935E−04

$P=5$, $Q=3$

order ε	λ	coefficient
2	0	2.499999999999E−02
4	0	−1.197916666665E−04
4	2	6.483289930555E−02
6	0	7.949942129454E−07
6	2	−1.599023603644E−02
6	4	−1.212623384263E−02
8	0	−4.743416244089E−09
8	2	3.908820568173E−03
8	4	7.861452323055E−03
8	5	+−1.537421393016E−05

Table 3. Local sectors for the coupled van der Pol equation (3.17) for different resonance cases

order ε	order λ	coefficient
$P=1$, $Q=3$		
2	0	1.111111111111$E-01$
2	1	$+-$1.750000000000$E+00$
2	2	1.250000000000$E+00$
$P=1$, $Q=5$		
2	0	1.200000000000$E-01$
2	2	1.083333333333$E+00$
3	1	$+-$7.291666666666$E-01$
$P=1$, $Q=2$		
2	0	9.375000000000$E-02$
2	2	1.666666666666$E+00$
4	0	$-$1.403808593750$E-02$
4	2	1.791005291005$E+00$
4	2	$+-$9.527777777777$E-01$
4	4	2.962962962962$E+00$
$P=1$, $Q=7$		
2	0	1.224489795918$E-01$
2	2	1.041666666666$E+00$
4	0	$-$1.496772178259$E-02$
4	1	$+-$2.855902777777$E-01$
4	2	$-$8.077344179464$E-01$
4	4	1.085521556712$E+00$
$P=3$, $Q=5$		
2	0	2.666666666666$E-02$
2	2	6.375000000000$E+00$
4	0	$-$4.827160493832$E-04$
4	1	$+-$3.038194444444$E-03$
4	2	$-$5.262165178571$E-01$
4	4	1.556396484374$E+01$
$P=1$, $Q=9$		
2	0	1.234567901234$E-01$
2	2	1.025000000000$E+00$
4	0	$-$1.497167606564$E-02$
4	2	$-$6.151477305487$E-01$
4	4	1.050785156249$E+00$
5	1	$+-$1.022460937499$E-01$
$P=3$, $Q=7$		
2	0	3.401360544217$E-02$
2	2	4.350000000000$E+00$
4	0	$-$5.358813971548$E-04$
4	2	1.280080994897$E+00$
4	4	6.527718749999$E+00$
5	1	$+-$3.978587962963$E-04$

order ε	order λ	coefficient
$P=1$, $Q=4$		
2	0	1.171875000000$E-01$
2	2	1.133333333333$E+00$
4	0	$-$1.491546630859$E-02$
4	2	$-$2.843730537980$E+00$
4	4	1.289481481481$E+00$
6	0	2.367223302505$E-03$
6	2	7.634791879674$E+00$
6	2	$+-$1.288018284281$E+00$
6	4	2.640966588843$E+00$
6	6	1.472874403292$E+00$
$P=2$, $Q=3$		
2	0	3.472222222222$E-02$
2	2	5.199999999999$E+00$
4	0	$-$1.502017425411$E-03$
4	2	$-$3.472700577200$E+00$
4	4	1.684799999999$E+01$
6	0	4.995900647218$E-05$
6	2	$-$1.023487625528$E+00$
6	2	$+-$4.117748123514$E-02$
6	4	1.133690681096$E+01$
6	6	6.537023999999$E+01$
$P=1$, $Q=11$		
2	0	1.239669421487$E-01$
2	2	1.016666666666$E+00$
4	0	$-$1.497293559183$E-02$
4	2	$-$5.292743735981$E-01$
4	4	1.033681712962$E+00$
6	0	2.468887604467$E-03$
6	1	$+-$3.484099211516$E+02$
6	2	8.969081053884$E-02$
6	4	2.348613590037$E-01$
6	6	1.051053308416$E+00$
$P=5$, $Q=7$		
2	0	1.224489795918$E-02$
2	2	1.541666666666$E+01$
4	0	$-$8.860891295277$E-05$
4	2	$-$3.083627553828$E+00$
4	4	6.426287615740$E+01$
6	0	4.633676408758$E-07$
6	1	$+-$2.454323743386$E+06$
6	2	1.847531357120$E-01$
6	4	2.076539379226$E+01$
6	6	3.376032348130$E+02$

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