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Integral manifolds of the restricted three-body problem

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Abstract. We compute the homology of the integral manifolds of the restricted three-body problem—planar and spatial, unregularized and regularized. Holding the Jacobi constant fixed defines a three-dimensional algebraic set in the planar case and a five dimensional algebraic set in the spatial case (the integral manifolds). The singularities of the restricted problem due to collusions are removable, which defines the regularized problem.

There are five positive critical values of the Jacobi constant: one is due to a critical point at infinity, another is due to the Lagrangian critical points and three are due to the Eulerian critical points. The critical point at infinity occurs only in spatial problems. We compute the homology of the integral manifold for each regular value of the Jacobi constant. These computations show that at each critical value the integral manifolds undergo a bifurcation in their topology. The bifurcation due to a critical point at infinity shows that Birkhoff's conjecture is false even in the restricted problem.

Birkhoff also asked if the planar problem is the boundary of a cross section for the spatial problem. Our computations and homological criteria show that this can never happen in the restricted problem, but may be possible in the regularized problem for some values of the Jacobi constant. We also investigate the existence of global cross sections in each of the problems.

1. *Introduction*

We study the topology and bifurcations of the integral manifolds of the restricted three-body problem—planar and spatial, unregularized and regularized. The restricted problem is a Hamiltonian system with one integral—the Hamiltonian or the Jacobi constant. Holding this integral fixed defines a three-dimensional algebraic set $\mathfrak{m}$ in the planar case and a five-dimensional algebraic set $\mathfrak{M}$ in the spatial case. We will refer to these as the integral manifolds.

The restricted problem has two singularities corresponding to the collision of the infinitesimal with the primaries. These singularities are removable by a process known as regularization, which is discussed in §3. The unregularized system will be called the *restricted problem* to differentiate it from the *regularized problem*. The integral manifolds of the planar and spatial regularized problem will be denoted by $\mathfrak{r}$ and $\mathfrak{R}$, respectively.

There are five positive critical values of the Jacobi constant, $\mu(1-\mu) < 3 < c_1 \leq c_2 < c_3$, where $\mu(1-\mu)$ is due to a critical point at infinity, 3 is due to the Lagrangian equilateral triangular point and c_1, c_2 and c_3 are due to the Eulerian collinear critical points. $\mu(1-\mu)$ is a critical value only in the spatial problem. We compute the homology of the integral manifold for each regular value of the Jacobi constant for the planar and spatial, unregularized and regularized problems. From these computations we will show that at the critical values the integral manifolds undergo bifurcations in their topology.

In his discussion of the integral manifolds of the full three-body problem Birkhoff [**4**] stated that the only bifurcations of the integral manifolds are due to the critical points that correspond to relative equilibrium solutions. Although this is true in the planar case [**9, 29, 30**] it is false in the spatial problem [**21**]. Our computations show that the same conclusions hold in the simpler restricted and regularized problems, i.e. 'Birkhoff's conjecture' is true in the planar problems and false in the spatial problems.

In the same discussion Birkhoff observes that the integral manifold for the planar three-body problem is a codimension two invariant subset of the integral manifold of the spatial three-body problem. He then asks if the planar problem is the boundary of a cross section in the spatial problem. In [**20**] we develop some homological criteria for an invariant set of a flow to be the boundary of a cross section and answer Birkhoff's question in the negative.

As the planar restricted and regularized manifolds are also closed invariant codimension two subspaces of the spatial restricted and regularized manifolds, we can also ask if they could be the boundary of a cross section. By applying the homological criteria we will show that the planar restricted manifold can never be the boundary of a cross section of finite type in the spatial restricted manifold; but that for some energy levels, the planar regularized manifold may be the boundary of a cross section in the spatial regularized manifold.

As a separate issue, we also give in [**20**] some criteria for the existence of global cross sections to a flow. In the restricted problem there are no homological obstructions to the existence of a global cross section in the planar manifold; but there can never be a global cross section of finite type in the spatial manifold. The analysis for the regularized manifolds is less decisive—global cross sections are ruled out in some energy ranges and may exist in others.

1.1. *The integral manifolds.* The restricted three-body problem is defined by the Hamiltonian

$$H = \tfrac{1}{2}|y|^2 - x^T K y - U, \tag{1}$$

where U is the self-potential

$$U = \frac{\mu}{d_1} + \frac{1-\mu}{d_2}, \tag{2}$$

and $0 < \mu < 1$ is the mass ratio parameter, see [**1, 22, 26**]. The vector x is a Cartesian coordinate of an infinitesimal particle (the *satellite*) in a synodical coordinate system which leaves the two primaries of mass μ and $1-\mu$ fixed on the x_1 axis and y is the momentum conjugate to x. In the planar problem $x, y \in \mathbb{R}^2$, $d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2$, $d_2^2 = (x_1+\mu)^2 + x_2^2$ and in the spatial problem $x, y \in \mathbb{R}^3$, $d_1^2 = (x_1 - 1 + \mu)^2 + x_2^2 + x_3^2$, $d_2^2 = (x_1+\mu)^2 + x_2^2 + x_3^2$. The matrix K is

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{or} \quad K = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in the planar and spatial problems, respectively. These problems have two singularities at the primaries, i.e. the singular sets for the planar and spatial restricted problems are

$$\delta = \{(-\mu, 0), (1-\mu, 0)\} \times \mathbb{R}^2, \quad \Delta = \{(-\mu, 0, 0), (1-\mu, 0, 0)\} \times \mathbb{R}^3.$$

The equations of motion are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} = y + Kx \\ \dot{y} &= -\frac{\partial H}{\partial x} = Ky + \frac{\partial U}{\partial x}. \end{aligned}$$

The Hamiltonian H is the only known integral of these equations. Instead of $H(x, y)$ one often considers the Jacobi constant $C(x, \dot{x})$ as the integral of motion where $C = -2H + \mu(1-\mu)$, i.e.

$$C(x, \dot{x}) = V(x) - |\dot{x}|^2, \quad \text{where } V(x) = x_1^2 + x_2^2 + 2U + \mu(1-\mu),$$

is the amended potential.

The integral manifolds for the restricted problem are

$$\begin{aligned} \mathfrak{m}(c) &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \delta \mid C(x, y) = c\}, \\ \mathfrak{M}(c) &= \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \mid C(x, y) = c\}. \end{aligned}$$

The finite critical points for V are equilibrium points for the equations of motion. There are five equilibrium points for the restricted problem, denoted by $\mathcal{L}_i$, $i = 1, \ldots, 5$. $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ lie on the x_1 axis and are called the Eulerian collinear equilibrium points and $\mathcal{L}_4$ and $\mathcal{L}_5$ are at the vertices of an equilateral triangle with base vertices at the two primaries and are called the Lagrangian points. Note that in the spatial problem V has a critical point at infinity with critical value $\mu(1-\mu)$, since $\nabla V \to 0$ and $V \to \mu(1-\mu)$ as $x_3 \to \infty$. The critical value due to the Lagrangian points is 3. Let c_1, c_2 and c_3 denote the critical values due to the Eulerian points. One verifies that $\mu(1-\mu) < 3 < c_1 \le c_2 < c_3$. (The value c_3 corresponds to the critical point between the two primaries.) By classical Morse theory [**23**], the topology of the integral manifolds can only change as the parameter c passes through a critical value, i.e. if there are no critical values in $[a, b]$ then $\mathfrak{M}(a)$ is diffeomorphic to $\mathfrak{M}(b)$. Our homology computations show that each of the critical values gives rise to distinct topologies.

By the process called regularization, the singularities due to the collision of the satellite with the primaries can be removed. The analytic details of this process will be discussed in §3, but the geometry is easy to state. Let

$$\mathfrak{d} \cong (S^1 \times D^2) \cup (S^1 \times D^2), \quad \mathfrak{D} \cong (S^2 \times D^3) \cup (S^2 \times D^3),$$

where S^n is the n-sphere and D^n is the closed n-ball. There is a neighborhood $\mathfrak{N}$ of the singular set Δ in $\mathfrak{M}(c)$ and an embedding $\Sigma : \mathfrak{N} \to \mathfrak{D}$ whose image is $(S^2 \times (D^3 \setminus \{0\})) \cup (S^2 \times (D^3 \setminus \{0\}))$. Furthermore, there is a smooth non-singular flow χ on $\mathfrak{D}$, and Σ takes the orbits of the restricted problem to orbits of χ. After the flow on $\mathfrak{M}(c)$ is reparameterized the diffeomorphism Σ takes parameterized trajectories of the restricted problem to parameterized trajectories χ. Thus, there is a well defined non-singular flow defined on $\mathfrak{M}(c) \cup_\Sigma \mathfrak{D}$ which is an extension of the flow of the restricted problem. The planar problem is regularized in the same way. In fact, the sets $\mathfrak{n}$ and $\mathfrak{d}$ used to regularize the planar problem are simply the restriction of $\mathfrak{N}$ and $\mathfrak{D}$ to their planar subsets. Thus, the regularized manifolds are defined by

$$\mathfrak{r}(c) = \mathfrak{m}(c) \cup_\sigma \mathfrak{d}, \quad \mathfrak{R}(c) = \mathfrak{M}(c) \cup_\Sigma \mathfrak{D}.$$

1.2. *Summary of results.* The five critical values divide the real line into six intervals denoted by

$$\mathrm{I} = (-\infty, \mu(1-\mu)), \quad \mathrm{II} = (\mu(1-\mu), 3), \quad \mathrm{III} = (3, c_1),$$
$$\mathrm{IV} = (c_1, c_2), \quad \mathrm{V} = (c_2, c_3), \quad \mathrm{VI} = (c_3, \infty).$$

Our main results are summarized in the four tables of homology, Tables 1–4. The body of these tables give the integral homology groups and Euler–Poincaré characteristic χ in each range of regular values, for each of the four problems.

One feature reflected in the tables is that for large values of the Jacobi constant (cases V and VI), these manifolds are disconnected. In case V, each of the four manifolds splits into two components, one a bounded neighborhood of the primaries, the other unbounded. In case VI, the bounded component further decomposes into disjoint neighborhoods around the two primaries. In case V, the bounded component is denoted by the subscript b and the unbounded component by the subscript u (e.g. $\mathfrak{M}_b(c)$ or $\mathfrak{r}_u(c)$). In case VI, the unbounded component is again denoted by the subscript u, and the two bounded components are denoted by the subscripts 1 and 2 (e.g. $\mathfrak{M}_1(c)$ or $\mathfrak{r}_2(c)$).

TABLE 1. Homology of the planar integral manifolds.

$H_p(\mathfrak{m})$	0	1	2	$\chi(\mathfrak{m})$
I	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0
II	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0
III	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	0
IV	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	0
V	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}$	0
V.b	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0
V.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0
VI	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0
VI.1	$\mathbb{Z}$	$\mathbb{Z}$	0	0
VI.2	$\mathbb{Z}$	$\mathbb{Z}$	0	0
VI.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0

TABLE 2. Homology of the spatial integral manifolds.

$H_p(\mathfrak{M})$	0	1	2	3	4	$\chi(\mathfrak{M})$
I	$\mathbb{Z}$	0	$\mathbb{Z}^3$	0	$\mathbb{Z}^2$	6
II	$\mathbb{Z}$	0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	4
III	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^3$	4
IV	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	4
V	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	4
V.b	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	4
V.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
VI	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}^2$	0	0	4
VI.1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	2
VI.2	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	2
VI.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0

TABLE 3. Homology of the regularized planar integral manifolds.

$H_p(\mathfrak{r})$	0	1	2	3	$\chi(\mathfrak{r})$
I	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
II	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
III	$\mathbb{Z}$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}$	0	0
IV	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	0	0	0
V	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}$	0
V.b	$\mathbb{Z}$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}$	0
V.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
VI	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	0	$\mathbb{Z}^2$	0
VI.1	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
VI.2	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
VI.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0

TABLE 4. Homology of the regularized spatial integral manifolds.

$H_p(\mathfrak{R})$	0	1	2	3	4	5	$\chi(\mathfrak{R})$
I	$\mathbb{Z}$	0	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0	2
II	$\mathbb{Z}$	0	$\mathbb{Z}^2$	$\mathbb{Z}^3$	0	0	0
III	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	0
IV	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	0	0
V	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	0
V.b	$\mathbb{Z}$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	0
V.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
VI	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	0
VI.1	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
VI.2	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
VI.u	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0

In Tables 1 and 3 for the two planar problems there is no difference in the homology in cases I and II. It will be shown in §2.1 that there is, in fact, no difference in the topology of the planar manifolds in cases I and II. On the other hand, for the two spatial problems, Tables 2 and 4 show that there is a difference in the homology in cases I and II. This reflects the fact that $\mu(1-\mu)$ is not a critical value for the planar problems, but it is a critical value for the spatial problems and that $\mu(1-\mu)$ is a bifurcation value for the topology for the spatial problem. Thus, the Birkhoff conjecture is true in both of the planar problems and false in both of the spatial problems.

These tables also answer some questions on cross sections to the flow. Let M be a connected manifold of dimension m without boundary, $\Phi : \mathbb{R} \times M \to M$ a flow and C a submanifold of M of dimension $m-1$ with boundary ∂C of dimension $m-2$. Let $\operatorname{int} C = C \setminus \partial C$ be the interior of C. Then C is a *cross section* if:

(1) the boundary ∂C of C is invariant under the flow Φ;
(2) for each point $p \in M \setminus \partial C$ there is a $t(p) > 0$ such that $\Phi(t(p), p) \in \operatorname{int} C$;
(3) there is a continuous function, the *return time*, $\tau : \operatorname{int} C \to \mathbb{R}$ such that
  (a) $\Phi(t, p) \notin \operatorname{int} C$ for all $p \in \operatorname{int} C$ and $0 < t < \tau(p)$,
  (b) $\Phi(\tau(p), p) \in \operatorname{int} C$ for all $p \in \operatorname{int} C$;
(4) there is an open neighborhood U of $\operatorname{int} C \times \{0\}$ in $\operatorname{int} C \times \mathbb{R}$ such that $\Phi|_U$ is a homeomorphism from U to an open neighborhood of $\operatorname{int} C$ in $M \setminus \partial C$;
(5) the return time τ extends continuously to ∂C.

If $\partial C = \emptyset$ then C is called a *global cross section*.

Given a manifold M and a codimension two submanifold B, we developed in [**20**] necessary conditions in terms of the homology of M and B for the existence of a cross section C of finite type with $\partial C = B$. These results can also be applied when $B = \emptyset$, providing the necessary conditions on the homology of M for a global cross section of finite type exist in M. 'Of finite type' means that the homology of C is finitely generated.

With the homology tables of the restricted and regularized manifolds in hand, we can apply these results to investigate the existence of global cross sections of finite type for each of the four manifolds. Furthermore, since $\mathfrak{m}$ and $\mathfrak{r}$ are codimension two submanifolds of $\mathfrak{M}$ and $\mathfrak{R}$ respectively, we can also examine whether either of the planar manifolds can serve as the boundary of a cross section of finite type of the corresponding spatial manifold.

The details of this investigation are given in §5, but the results can be easily summarized. Table 5 displays the results. An entry of 'N' indicates that no cross section of finite type can exist and an entry of 'Y' means that all of the homological information is consistent with the existence of a cross section.

On the one hand, a 'N' still allows the possibility of a cross section whose homology is infinitely generated. While we cannot rule out such cross sections by homological arguments, the existence of such a cross section would seem to be of little practical value. Since cross sections are sought to reduce the complexity of the dynamics there would be no advantage in moving the investigation to a space with an infinitely complex topology. On the other hand, much more than homological consistency is required to demonstrate the existence of a cross section, so a 'Y' should be considered as an invitation to further investigation.

TABLE 5. Homological admissibility of cross sections.

	Global cross section				Cross section	
	$\mathfrak{m}$	$\mathfrak{M}$	$\mathfrak{r}$	$\mathfrak{R}$	$\mathfrak{m}$ in $\mathfrak{M}$	$\mathfrak{r}$ in $\mathfrak{R}$
I	Y	N	Y	N	N	N
II	Y	N	Y	N	N	N
III	Y	N	Y	Y	N	Y
IV	Y	N	Y	Y	N	Y
V	Y	N	N	N	N	Y
V.b	Y	N	N	N	N	Y
V.u	Y	Y	Y	Y	Y	Y
VI	Y	N	N	N	N	Y
VI.1	Y	N	N	N	N	Y
VI.2	Y	N	N	N	N	Y
VI.u	Y	Y	Y	Y	Y	Y

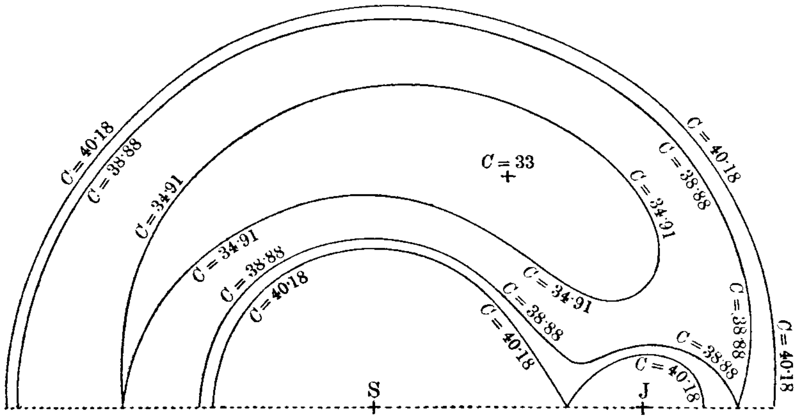
1.3. *History of the problem.* The restricted problem has a long history dating back to Euler's 1772 treatise [**10**] on the motion of the moon. Euler wrote the equations in sidereal coordinates; it was only later when Jacobi [**17**] wrote the equations in synodical coordinates that the integral was found.

Hill [**11**] discussed the zero velocity curves and Hill's regions for a limiting case of the restricted problem is known as Hill's lunar problem. In his simplified model he was able to assert that the orbit of the moon was bounded.

Darwin [**7**] obtained grants from the British government and the Royal Society to pay his computers (Messrs J. W. F. Allnutt, J. I. Craig and M. J. Berry) in order to calculate the zero velocity curves for the planar restricted problem. Figure 1 (reproduced by permission of the editors of Acta Mathematica) is the figure rendered by Mr Edwin Wilson. Darwin says of his computers ' . . . the trained computer, who is also a mathematician, is rare. I have thus found myself compelled to forgo the advantages of the rapidity and accuracy of the computer, for the higher qualities of mathematical knowledge and judgment' [**7**, p. 101].

The zero velocity surfaces for the spatial problem were first investigated by Picart [**27**]. Excellent figures of the zero velocity surfaces are found in Lundberg *et al* [**16**], who forwent the advantages of mathematical knowledge for the rapidity of an electronic computer. The zero velocity curves and surfaces are reproduced in many books on celestial mechanics [**6, 26, 31, 33**].

The removal of the singularity in Kepler's planar problem and the restricted problem seems to have been first noted by Thiele [**32**] in 1895. The most well-known method uses complex variables and it is due to Levi-Civita [**14, 15**]. Kustaanheimo and Stiefel [**18**] regularized the spatial problem using quaternions. Easton [**8**] gives a very general definition of regularization using Conley index ideas, but does not treat the spatial problems. (It is clear that Easton's general method would apply to the spatial problem.)



Curves of zero velocity, $10\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 + \frac{2}{\rho}\right) = C.$

FIGURE 1. Darwin's figure.

The only general method which treats the planar and spatial problems in a unified way is found in [**2, 3**], which will be summarized in §3.

The topology of the integral manifolds of the planar restricted and regularized problems is discussed by Birkhoff [**5**] using complex variable methods for regularization and by Lacomba [**12, 13**] using topological methods.

Hill's regions and integral manifolds have been investigated in the full (unregularized) three-body problem, see [**21, 28**] and the references therein.

2. *The restricted problem*

The planar and spatial integral manifolds are described by projecting them onto planar regions. In the end, the only homology groups which must be explicitly calculated will be those of subsets of the plane. For these, the homology groups can be derived from inspection of Figure 2. Thus, with the few theorems expressing the relationships between the homology groups of the various Hill's regions and integral manifolds and these simple planar calculations, all of the homology groups follow.

2.1. *Decomposition of the integral manifolds.* The projection of the integral manifolds onto position space are called *Hill's regions* and are denoted by

$$\mathfrak{h}(c) = \{x \in \mathbb{R}^2 \mid \exists y \in \mathbb{R}^2, (x, y) \in \mathfrak{m}(c)\},$$
$$\mathfrak{H}(c) = \{x \in \mathbb{R}^3 \mid \exists y \in \mathbb{R}^3, (x, y) \in \mathfrak{M}(c)\}.$$

Note that the planar manifold embeds in a natural way into the spatial manifold and similarly for the Hill's regions. We thus have the following diagram of embeddings and

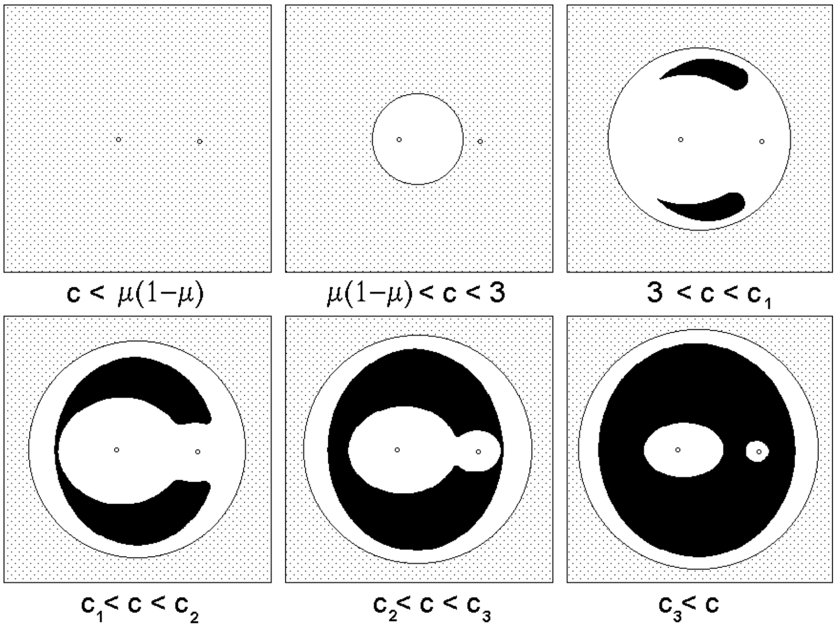


FIGURE 2. The planar Hill's regions.

projections:

$$\begin{array}{ccc} \mathfrak{m}(c) & \longrightarrow & \mathfrak{M}(c) \\ \downarrow \pi & & \downarrow \pi \\ \mathfrak{h}(c) & \longrightarrow & \mathfrak{H}(c) \end{array}$$

The dimensions of these spaces are

$$\dim(\mathfrak{m}(c)) = 3, \quad \dim(\mathfrak{M}(c)) = 5$$
$$\dim(\mathfrak{h}(c)) = 2, \quad \dim(\mathfrak{H}(c)) = 3.$$

The planar manifold $\mathfrak{m}(c)$ is defined by $C(x, y) = V(x) - |y|^2 = c$. Since y appears only in the term $|y|^2$, we can reformulate the condition $C(x, y) = c$ as

$$|y|^2 = V(x) - c.$$

That is, when $V(x) < c$, there can be no y such that $(x, y) \in \mathfrak{m}(c)$, and for $V(x) = c$ the only valid y is $y = 0$. When $V(x) > c$, then $(x, y) \in \mathfrak{m}(c)$ for all y in the circle of radius $V(x) - c$. Thus the Hill's region is

$$\mathfrak{h}(c) = \{x \in \mathbb{R}^2 \mid V(x) \geq c, x \neq (-\mu, 0), (1 - \mu, 0)\}.$$

The zero velocity curve is its boundary $\partial\mathfrak{h}(c)$, which is simply the level set $V(x) = c$. The planar Hill's regions for each of the six cases are shown in Figure 2. There, the solid black regions are excluded from the Hill's region; the white and shaded regions are included (the shading is relevant only to the spatial problem, and will be explained below).

There is a natural projection $\pi : \mathfrak{m}(c) \to \mathfrak{h}(c)$. It is worth noting that this projection admits a section. In fact, for any unit vector $\boldsymbol{v} \in S^1$, there is a section $s_{\boldsymbol{v}} : \mathfrak{h}(c) \to \mathfrak{m}(c)$ defined by $s_{\boldsymbol{v}}(x) = (x, (V(x) - c)\boldsymbol{v})$.

We can summarize this by describing $\mathfrak{m}(c)$ over $\mathfrak{h}(c)$, both locally and globally. Locally, we have the following.

PROPOSITION 2.1. *The projection $\pi : \mathfrak{m}(c) \to \mathfrak{h}(c)$ is an orientable singular S^1-bundle. That is, for $x \in \mathfrak{h}(c)$, the preimage $\pi^{-1}(x)$ is a circle when $V(x) > c$, and a single point when $V(x) = c$. The S^1-bundle over the interior of $\mathfrak{h}(c)$ is trivial.*

Globally, this can be used to describe the integral manifold as a quotient of a product space.

PROPOSITION 2.2. *The integral manifold $\mathfrak{m}(c)$ is a quotient space of the product $\mathfrak{h}(c) \times S^1$, obtained by collapsing the circle over each point in the boundary $\partial\mathfrak{h}(c)$ to a point.*

The structure of the spatial manifold and its projection $\pi : \mathfrak{M}(c) \to \mathfrak{H}(c)$ are very similar to those of the planar manifold. The Hill's region is the set

$$\mathfrak{H}(c) = \{x \in \mathbb{R}^3 \mid x \neq (-\mu, 0, 0), (1-\mu, 0, 0),\ V(x) \geq c\},$$

and its boundary $\partial\mathfrak{H}(c)$ is the level set $V^{-1}(c)$ in $\mathbb{R}^3$. For every $\boldsymbol{v} \in S^2$ there is a section $s_{\boldsymbol{v}} : \mathfrak{H}(c) \to \mathfrak{M}(c)$. In exactly the same manner as Propositions 2.1 and 2.2, we have the following.

PROPOSITION 2.3. *The projection $\pi : \mathfrak{M}(c) \to \mathfrak{H}(c)$ is an orientable singular S^2-bundle. That is, for $x \in \mathfrak{H}(c)$, the preimage $\pi^{-1}(x)$ is a sphere when $V(x) > c$, and a single point when $V(x) = c$. The S^2-bundle over the interior of $\mathfrak{h}(c)$ is trivial.*

PROPOSITION 2.4. *The integral manifold $\mathfrak{M}(c)$ is a quotient space of the product $\mathfrak{H}(c) \times S^2$, obtained by collapsing the sphere over each point in the boundary $\partial\mathfrak{h}(c)$ to a point.*

To further analyze the spatial problem, we follow Easton [**9**] and project the spatial Hill's region onto the planar Hill's region. Let $\mathfrak{h}^+(c) = \mathfrak{h}(c) \cup \{(-\mu, 0), (1-\mu, 0)\}$, and let $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the first two coordinates. To understand the structure of ρ, we must understand the x_3 dependence of V. Fortunately, this is quite simple. The quantity x_3 appears only in the potential terms

$$\frac{\mu}{\sqrt{(x_1 - 1 + \mu)^2 + x_2^2 + x_3^2}} + \frac{1-\mu}{\sqrt{(x_1 + \mu)^2 + x_2^2 + x_3^2}}.$$

This decreases monotonically to zero as $|x_3|$ goes to infinity, so for fixed x_1 and x_2, $V(x_1, x_2, x_3)$ limits to the quadratic

$$W(x_1, x_2) = x_1^2 + x_2^2 + \mu(1-\mu).$$

PROPOSITION 2.5. *$\rho : \mathfrak{H}(c) \to \mathbb{R}^2$ maps onto $\mathfrak{h}^+(c)$. The fiber of this projection is*

$$\rho^{-1}(x_1, x_2) = \begin{cases} *, & V(x_1, x_2) = c, \\ I, & W(x_1, x_2) < c < V(x_1, x_2) < \infty, \\ I \setminus \{0\}, & W(x_1, x_2) < c < V(x_1, x_2) = \infty, \\ \mathbb{R}, & c \le W(x_1, x_2),\ V(x_1, x_2) < \infty, \\ \mathbb{R} \setminus \{0\}, & c \le W(x_1, x_2),\ V(x_1, x_2) = \infty. \end{cases}$$

Note that the fiber I should be understood to be an interval that is symmetric about the origin, and the condition $V(x_1, x_2) = \infty$ should be understood as a short-hand for the two points $(x_1, x_2) = (1 - \mu, 0), (-\mu, 0)$. This local description can be globalized to describe $\mathfrak{H}(c)$ and $\partial\mathfrak{H}(c)$ in terms of $\mathfrak{h}(c)$ and $\partial\mathfrak{h}(c)$.

To do so, let $\mathfrak{s}(c)$ be a pair of circles in $\mathfrak{h}(c)$ about the two singularities $(-\mu, 0)$ and $(1 - \mu, 0)$, and let $\mathfrak{S}(c)$ be a pair of spheres in $\mathfrak{H}(c)$ about the singularities. Choose these so that the circles in $\mathfrak{s}(c)$ are the equators of the spheres in $\mathfrak{S}(c)$, and so that they do not intersect the boundary of the Hill's region. With these definitions we can construct strong deformation retracts of the spatial Hill's region and integral manifold.

PROPOSITION 2.6. *The spatial Hill's region $\mathfrak{H}(c)$ has the homotopy type of $\mathfrak{h}(c) \bigcup_{\mathfrak{s}(c)} \mathfrak{S}(c)$. The boundary of the spatial Hill's region is homeomorphic to $\partial\mathfrak{h}^+(c) \bigcup_{\partial\mathfrak{h}(c)} \mathfrak{h}^+(c)$.*

By combining Propositions 2.3 and 2.5, we have the following decomposition of $\mathfrak{M}(c)$.

PROPOSITION 2.7. *The projection $\rho \circ \pi : \mathfrak{M}(c) \to \mathfrak{h}^+(c)$ is surjective, with fiber*

$$\pi^{-1}\rho^{-1}(x_1, x_2) = \begin{cases} *, & V(x_1, x_2) = c, \\ S^3, & W(x_1, x_2) < c < V(x_1, x_2) < \infty, \\ S^3 \setminus S^2, & W(x_1, x_2) < c < V(x_1, x_2) = \infty, \\ \mathbb{R} \times S^2, & c \le W(x_1, x_2),\ V(x_1, x_2) < \infty, \\ (\mathbb{R} \setminus \{0\}) \times S^2, & c \le W(x_1, x_2),\ V(x_1, x_2) = \infty. \end{cases}$$

Given the very simple form of $W(x_1, x_2)$, it is a simple matter to precisely identify how the various regions change as c varies. First, for $c < \mu(1 - \mu)$, $c < W(x_1, x_2)$ for all (x_1, x_2). As c increases, the set $W(x_1, x_2) < c$ is the open disk $x_1^2 + x_2^2 < c - \mu(1 - \mu)$. At $c = \mu$ this passes through the point $(-\mu, 0)$ and at $c = 1 - \mu$ it passes through the point $(1 - \mu, 0)$. At $c = 3$ an excluded region appears inside this open disk, and continues to grow as c increases, just as in the planar problem. The evolution of the boundary $W(x_1, x_2) = c$ is displayed in Figure 2, where the shaded region represents the set $\{c < W(x_1, x_2)\}$.

Thus, the only parameter values at which the topology of $\mathfrak{H}(c)$ and $\mathfrak{M}(c)$ could possibly change are the values that already arose in the planar problem, 3, c_1, c_2 and c_3, and the new values $\mu(1-\mu)$, μ and $1-\mu$. At all of these values Proposition 2.4 implies that the topology of $\mathfrak{M}(c)$ can only change if the topology of $\mathfrak{H}(c)$ changes. We will see that the topology of $\mathfrak{H}(c)$ does change at $\mu(1 - \mu)$ and at 3, c_1, c_2 and c_3. However, at μ and $1 - \mu$ it is only the projection of $\mathfrak{H}(c)$ onto $\mathfrak{h}(c)$ that changes, not the topology of the space $\mathfrak{H}(c)$ itself.

PROPOSITION 2.8. *For all* $\mu(1-\mu) < c < 3$, $\mathfrak{H}(c) \cong (B^2 \times I) \setminus \{2 \text{ pts.}\}$.

Proof. For all c in this range, there is a positive function $h(x_1, x_2)$ defined on the open ball $x_1^2 + x_2^2 < c - \mu(1-\mu)$, with

$$\lim_{x_1^2+x_2^2 \to c-\mu(1-\mu)} h(x_1, x_2) = \infty,$$

such that the complement of $\mathfrak{H}(c)$ in $\mathbb{R}^3$ consists of the set $\{|x_3| > h(x_1, x_2)\}$ and the two points $\{(-\mu, 0, 0), (1-\mu, 0, 0)\}$. Applying $(2/\pi)\tan^{-1}$ to all coordinates, we can map this into $[-1, 1]^3$, and then scale x_3 by $\pi/2 \tan^{-1}(h(x_1, x_2))$ to map the set $\{x_3^2 > h(x_1, x_2)\}$ into the top and bottom faces. The resulting set, an open cube with two open disks appended to the top and bottom and two points deleted from the interior, is clearly homeomorphic to $(B^2 \times I) \setminus \{2 \text{ pts.}\}$. □

COROLLARY 2.1. *The homeomorphism type of* $\mathfrak{M}(c)$ *does not change on the parameter interval* $\mu(1-\mu) < c < 3$.

2.2. *Homology of the restricted manifolds.* In the previous section we showed that the topology of the integral manifolds could be understood in terms of the topology of the Hill's regions and that the topology of the spatial Hill's region could be understood in terms of the planar Hill's region. In this section we pursue the homological implications of these reductions. We will show that the homology groups of $\mathfrak{m}(c)$ and $\mathfrak{M}(c)$ can be calculated from the homology of $\mathfrak{h}(c)$, $\mathfrak{h}^+(c)$ and $\partial\mathfrak{h}(c)$ and the inclusion map $\iota : \mathfrak{s}(c) \to \mathfrak{h}(c)$. Specifically, we will require knowledge of the homology groups $H_*(\mathfrak{h}(c))$ and $H_*(\mathfrak{h}(c), \partial\mathfrak{h}(c))$ and the homomorphisms $\iota_* : H_*(\mathfrak{s}(c)) \to H_*(\mathfrak{h}(c))$ and $\iota_* : H_*(\mathfrak{s}(c)) \to H_*(\mathfrak{h}(c), \partial\mathfrak{h}(c))$. As all of these spaces are 1-complexes (or retract onto 1-complexes), these groups and homomorphisms are easily computed from Figure 2.

PROPOSITION 2.9. *In all cases, the homology groups of* $\mathfrak{h}$ *and* $(\mathfrak{h}(c), \partial\mathfrak{h}(c))$ *are torsion-free. The Betti numbers and the ranks of the maps* $\iota_* : H_p(\mathfrak{s}(c)) \to H_*(\mathfrak{h}(c))$ *and* $\hat{\iota}_* : H_p(\mathfrak{s}(c)) \to H_*(\mathfrak{h}(c), \partial\mathfrak{h}(c))$ *are given in Table 6.*

TABLE 6. Homology of the planar Hill's regions.

	$H_*(\mathfrak{h})$		rank(ι_{p*})		$H_*(\mathfrak{s})$		rank$(\hat{\iota}_{p*})$		$H_*(\mathfrak{h}, \partial\mathfrak{h})$	
	0	1	0	1	0	1	0	1	0	1
I	1	2	1	2	2	2	1	2	1	2
II	1	2	1	2	2	2	1	2	1	2
III	1	4	1	2	2	2	0	2	0	3
IV	1	3	1	2	2	2	0	2	0	2
V	2	3	1	2	2	2	0	1	0	1
VI	3	3	2	2	2	2	0	0	0	0

With these values in hand, we can now compute the homology groups of the Hill's regions and integral manifolds.

PROPOSITION 2.10. *The homology of the spatial Hill's region is*

$$H_p(\mathfrak{H}(c)) \cong \operatorname{coker}(\iota_{p*}) \oplus \tilde{H}_p(\mathfrak{S}(c)).$$

For $c > \mu(1-\mu)$, the homology of the pair $(\mathfrak{H}(c), \partial\mathfrak{H}(c))$ is

$$H_p(\mathfrak{H}(c), \partial\mathfrak{H}(c)) \cong H_{p-1}(\mathfrak{h}(c), \partial\mathfrak{h}(c)).$$

For $c < \mu(1-\mu)$, $\partial\mathfrak{H}(c) = \emptyset$ and

$$H_p(\mathfrak{H}(c), \partial\mathfrak{H}(c)) \cong H_p(\mathfrak{H}(c)).$$

Proof. Both formulae are established via Mayer–Vietoris arguments. For $\mathfrak{H}(c)$, Proposition 2.6 supplies the decomposition $\mathfrak{h}(c)$ and $\mathfrak{S}(c)$, which have overlap $\mathfrak{s}(c)$. The homology sequence is

$$\to H_p(\mathfrak{s}) \xrightarrow{\begin{bmatrix} \iota_* \\ i_* \end{bmatrix}} H_p(\mathfrak{h}) \oplus H_p(\mathfrak{S}) \to H_p(\mathfrak{H}) \to .$$

Except for $p = 0$, $i_{p*} : H_p(\mathfrak{s}) \to H_p(\mathfrak{S})$ is trivial, so $I_{p*} : H_p(\mathfrak{S}) \to H_p(\mathfrak{H})$ is injective. Thus, the sequence breaks up as

$$0 \to \operatorname{coker}(\iota_{p*}) \oplus H_p(\mathfrak{H}(c)) \to \ker(\iota_{p-1*}) \to 0.$$

Since $\ker(\iota_{p-1*})$ is always torsion-free, the sequence splits.

In dimension 0, I_{0*} is not injective, but taking reduced homology supplies the appropriate correction.

For the homology of pairs, we employ a different Mayer–Vietoris argument. Let

$$\mathfrak{H}_+ = \{(x_1, x_2, x_3) \in \mathfrak{H}(c) | x_3 \geq 0\},$$
$$\mathfrak{H}_- = \{(x_1, x_2, x_3) \in \mathfrak{H}(c) | x_3 \leq 0\}.$$

and let $\partial\mathfrak{H}_\pm = \mathfrak{H}_\pm \cap \partial\mathfrak{H}$. Then the pairs $(\mathfrak{H}_+, \partial\mathfrak{H}_+)$ and $(\mathfrak{H}_-, \partial\mathfrak{H}_-)$ give a decomposition of $(\mathfrak{H}_+, \partial\mathfrak{H}_+)$ and $(\mathfrak{H}_-, \partial\mathfrak{H}_-)$ with intersection $(\mathfrak{H}_+, \partial\mathfrak{H}_+)$ and $(\mathfrak{H}_-, \partial\mathfrak{H}_-)$.

The crucial observation is that, when $c > \mu(1-\mu)$, $\mathfrak{H}_\pm$ retracts onto $\partial\mathfrak{H}_\pm$, so $H_*(\mathfrak{H}_\pm, \partial\mathfrak{H}_\pm) = 0$. Thus, the Mayer–Vietoris sequence of the pairs becomes

$$0 \to H_p(\mathfrak{H}(c), \partial\mathfrak{H}(c)) \to H_{p-1}(\mathfrak{h}(c), \partial\mathfrak{h}(c)) \to 0. \qquad \square$$

COROLLARY 2.2. *There is a commutative diagram*

$$\begin{array}{ccc} H_p(\mathfrak{S}) & \xrightarrow{\hat{I}_{p*}} & H_p(\mathfrak{H}, \partial\mathfrak{H}) \\ \partial \downarrow & & \downarrow \partial \\ H_{p-1}(\mathfrak{s}) & \xrightarrow{\hat{\iota}_{p-1*}} & H_{p-1}(\mathfrak{h}, \partial\mathfrak{h}) \end{array}$$

For convenience, we will denote the preimage $\pi^{-1}(\partial\mathfrak{h}(c))$ as $\partial\mathfrak{m}(c)$. It follows from Proposition 2.1 that $\pi : \partial\mathfrak{m}(c) \to \partial\mathfrak{h}(c)$ is a homeomorphism.

THEOREM 2.1. *The homology of the planar integral manifold is given by*

$$H_p(\mathfrak{m}(c)) \cong H_p(\mathfrak{h}(c)) \oplus H_{p-1}(\mathfrak{h}(c), \partial\mathfrak{h}(c)).$$

The homology of the spatial integral manifold is given by

$$H_p(\mathfrak{M}(c)) \cong H_p(\mathfrak{H}(c)) \oplus H_{p-2}(\mathfrak{H}(c), \partial\mathfrak{H}(c)).$$

Proof. The two arguments are identical, so it suffices to describe the planar case. We will suppress all of the dependence on (c), and simply write $\mathfrak{m}$, etc.

Consider the exact sequences of the pairs $(\mathfrak{m}, \partial\mathfrak{m})$ and $(\mathfrak{h}, \partial\mathfrak{h})$.

$$\begin{array}{ccccccccc} \longrightarrow & H_p(\partial\mathfrak{m}) & \longrightarrow & H_p(\mathfrak{m}) & \longrightarrow & H_p(\mathfrak{m}, \partial\mathfrak{m}) & \longrightarrow \\ & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* & \\ \longrightarrow & H_p(\partial\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{h}, \partial\mathfrak{h}) & \longrightarrow . \end{array}$$

Since π admits the section s, π_* is surjective. Since $\pi : \partial\mathfrak{m} \to \partial\mathfrak{h}$ is one-to-one, the resulting map on homology is an isomorphism.

The pair $\pi : (\mathfrak{m}, \partial\mathfrak{m}) \to (\mathfrak{h}, \partial\mathfrak{h})$ is an orientable relative S^1-bundle, so there is a Gysin sequence

$$\to H_{p-1}(\mathfrak{h}, \partial\mathfrak{h}) \to H_p(\mathfrak{m}, \partial\mathfrak{m}) \to H_p(\mathfrak{h}, \partial\mathfrak{h}) \to H_{p-2}(\mathfrak{h}, \partial\mathfrak{h}) \to .$$

The existence of the section s implies that this sequence splits, with

$$H_p(\mathfrak{m}, \partial\mathfrak{m}) \cong H_p(\mathfrak{h}, \partial\mathfrak{h}) \oplus H_{p-1}(\mathfrak{h}, \partial\mathfrak{h}),$$

with $\pi_* : H_p(\mathfrak{m}, \partial\mathfrak{m}) \to H_p(\mathfrak{h}, \partial\mathfrak{h})$ surjective.

Thus, the sequences of the pairs $(\mathfrak{m}, \partial\mathfrak{m})$ and $(\mathfrak{h}, \partial\mathfrak{h})$ can be written as

$$\begin{array}{ccccccccc} \longrightarrow & H_p(\partial\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{m}) & \longrightarrow & H_p(\mathfrak{h}, \partial\mathfrak{h}) \oplus H_{p-1}(\mathfrak{h}, \partial\mathfrak{h}) & \longrightarrow \\ & \downarrow \mathrm{id} & & \downarrow \pi_* & & \downarrow pr_1 & \\ \longrightarrow & H_p(\partial\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{h}, \partial\mathfrak{h}) & \longrightarrow . \end{array}$$

From this, it is a simple diagram chase to see that $\pi_* : H_*(\mathfrak{m}) \to H_*(\mathfrak{h})$ is surjective, with the kernel isomorphic to $H_*(\mathfrak{h}, \partial\mathfrak{h})$. □

Note that both the statement and proof of Theorem 2.1 parallel those of [**19**, Theorem 1.3]. With the values for the homology of the Hill's regions computed, the homology of the integral manifolds can be read off directly.

COROLLARY 2.3. *For all values of c, the homology groups of the planar and spatial integral manifolds are torsion-free. The Betti numbers for the planar manifold are given by*

$$\beta_p(\mathfrak{m}(c)) = \beta_p(\mathfrak{h}(c)) + \beta_{p-1}(\mathfrak{h}(c), \partial\mathfrak{h}(c)).$$

For $c > \mu(1-\mu)$, the Betti numbers of the spatial manifold are given by

$$\beta_p(\mathfrak{M}(c)) = \beta_p(\mathfrak{h}(c)) + \beta_{p-3}(\mathfrak{h}(c), \partial\mathfrak{h}(c)) - \mathrm{rank}(\iota_{p*}) + \delta_{p,0} + 2\delta_{p,2}.$$

For $c > \mu(1-\mu)$, the Betti numbers of the spatial manifold are given by

$$\beta_p(\mathfrak{M}(c)) = \beta_p(\mathfrak{h}(c)) + \beta_{p-2}(\mathfrak{h}(c)) - \mathrm{rank}(\iota_{p*}) - \mathrm{rank}(\iota_{p-2*}) + \delta_{p,0} + 3\delta_{p,2} + 2\delta_{p,4}.$$

The homology groups so computed are given in Tables 1 and 2.

3. *The regularized problem*

Belbruno [**2, 3**] showed that the Kepler problem in $\mathbb{R}^n$ can be regularized and the regularized flow on an energy level E is equivalent to the geodesic flow on a manifold of constant curvature $-E$. (Also see the survey [**24**].) His theorem extends the work of Conley and Moser [**25**] who showed that the Kepler problem with negative energy can be regularized and the regularized flow is equivalent to the geodesic flow on the unit tangent bundle of the n-sphere. The regularization is accomplished by the construction of a symplectomorphism which can also be used to remove the singularities of the restricted problem. We have chosen this approach to regularization over the many others since in this approach the planar and spatial problems are treated in a unified way.

Here we shall summarize the salient points of this method. In our summary we have changed the order to simplify the presentation. We took the square root of the Hamiltonian and reversed the roles of x and y at the start of the discussion instead of at the end, as in [**2, 25**].

3.1. *Negative energy.*

Let $\xi = (\xi_0, \xi_1, \dots, \xi_n)$ and $\eta = (\eta_0, \eta_1, \dots, \eta_n)$ be coordinates on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, TS^n its tangent bundle and $T_\delta S^n$ the δ-sphere bundle of the unit sphere. So

$$S^n = \{|\xi| = 1\}, \quad TS^n = \{|\xi| = 1, \xi \cdot \eta = 0\}, \quad T_\delta S^n = \{|\xi| = 1, \xi \cdot \eta = 0, |\eta| = \delta\}.$$

The geodesic flow on an embedded manifold is such that the acceleration is normal to the manifold, so the geodesic flow on S^n is defined by the equation

$$\ddot{\xi} = \lambda \xi.$$

Since this flow must satisfy $|\xi|^2 = \xi \cdot \xi = 1$ we have, by differentiating twice, that $\lambda = -|\dot{\xi}|^2$. This can be written as a Hamiltonian system with Hamiltonian $|\xi|^2|\eta|^2/2$ [**25**], but we shall take the square root and consider the system with Hamiltonian

$$G = |\xi||\eta|, \tag{3}$$

with equations of motion on the δ-sphere bundle

$$\dot{\xi} = \frac{\partial G}{\partial \eta} = \delta^{-1}\eta, \quad \dot{\eta} = -\frac{\partial G}{\partial \xi} = -\delta\xi.$$

The flows of G and $G^2/2$ on the unit sphere bundle are precisely the same—on other level sets they are reparameterizations of one another.

Let $\hat{S}^n$ denote the sphere punctured at the north pole, i.e. $\hat{S}^n = S^n \setminus \{(1, 0, \dots, 0)\}$. The stereographic projection of $\hat{S}^n$ onto $\mathbb{R}^n$ is given by

$$y_k = \frac{\xi_k}{1 - \xi_0}, \quad k = 1, \dots, n \tag{4}$$

where $\xi \in \hat{S}^n$ and $y \in \mathbb{R}^n$. The inverse is given by

$$\xi_0 = \frac{|y|^2 - 1}{|y|^2 + 1}, \quad \xi_k = \frac{2y_k}{|y|^2 + 1}, \quad k = 1, \dots, n. \tag{5}$$

(In the Kepler problem at a collision, as position $x \to 0$ the velocity $y \to \infty$. This is why it is the y space that is projected onto the sphere.)

Moser extends the stereographic projection (4) to a mapping $\psi : \mathbb{R}^n \times \mathbb{R}^n \to T\hat{S}^n$ by

$$\eta_0 = x \cdot y, \quad \eta_k = \frac{|y|^2 + 1}{2} x_k - (x \cdot y) y_k, \quad k = 1, \ldots, n \tag{6}$$

with inverse

$$x_k = \eta_k (1 - \xi_0) + \xi_k \eta_0, \quad k = 1, \ldots, n. \tag{7}$$

PROPOSITION 3.1. *The extended stereographic mapping $\psi : \mathbb{R}^n \times \mathbb{R}^n \to T\hat{S}^n$ given by formulas (4), (5), (7) and (6) is a symplectomorphism.*

The Hamiltonian (3) of the flow on $T\hat{S}^n$ in the symplectic coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ becomes

$$G = \tfrac{1}{2}(|y|^2 + 1)|x|.$$

In order to treat all negative energy levels $(-h)$ we need to scale the variables by $x \to x$, $y \to (2h)^{-1/2} y$ which is symplectic with multiplier $(2h)^{1/2}$. Now

$$G = \tfrac{1}{2}(2h)^{-1/2}(|y|^2 + 2h)|x|.$$

Let $\delta = \mu(2h)^{-1/2}$ and observe

$$G - \delta = (2h)^{-1/2}|x|\{L + h\}, \quad \text{where } L = \frac{1}{2}|y|^2 - \frac{\mu}{|x|},$$

and L is the Hamiltonian of the Kepler problem with central mass μ.

For the moment let $z = (x, y)$, $\phi(z) = (2h)^{-1/2}|x|$ and J be the usual skew symmetric matrix of Hamiltonian theory. Now define a new time τ by $d\tau = (2h)^{-1/2}|x|\,dt = \phi(z)\,dt$ and $' = d/d\tau$. Then

$$\dot{z} = J\nabla(G - \delta) = \{L + h\}J\nabla\phi + \phi J\nabla L.$$

On the set $G = \delta$ or $L = -h$ this says

$$z' = J\nabla L,$$

i.e. the flow on $G = \delta$ is a reparameterization of the Kepler flow on $L = -h$. The flow on the δ-sphere bundle of the unit sphere is by definition the flow of the regularized Kepler problem.

This symplectomorphism can be used to define local coordinates about each primary of the restricted problem and thus regularize the singularities one at a time. The flow will no longer be the same as the flow defined by G but a perturbation thereof.

Consider the restricted problem where one primary is at the origin, i.e. replace x_1 by $x_1 + 1 - \mu$ and y_2 by $y_2 + 1 - \mu$. The Hamiltonian (1) becomes

$$H = \frac{1}{2}|y|^2 - \frac{\mu}{|x|} - x^T K y - \frac{\mu - 1}{d_2} + -(1 - \mu)x_1 - \frac{1}{2}(1 - \mu)^2, \tag{8}$$

where $d_2^2 = (x_1+1)^2 + x_2^2 + x_3^3$. Change to (ξ, η) coordinates so that

$$\begin{aligned}\phi(H+h) &= |\xi||\eta| - \delta + \phi(\xi,\eta)\left\{-\xi_2\eta_1 + \xi_1\eta_2 - \frac{1-\mu}{d_2} - \frac{1}{2}(1-\mu)^2\right\}\\ &= |\xi||\eta| - \delta + O(\varepsilon),\end{aligned}$$

where $\varepsilon = |x|$. As in the Kepler problem the change of time $d\tau = \phi\, dt$ shows that the flow of the restricted problem on $H = -h$ is a reparameterization of the flow on $|\xi||\eta| = \delta + O(\varepsilon)$ which is a small perturbation of the geodesic flow on the δ-sphere bundle. Thus, the singularity of the restricted problem has been regularized.

Performing the sequence of transformation used for the Kepler problem on the neighborhood $\{|x| \le \varepsilon^2, |y| \ge \varepsilon^{-1}\}$ of the singularity yields a perturbation of the geodesic flow on a neighborhood of the north pole. Thus, the regularization of this singularity can be viewed as excising this neighborhood of the singularity and attaching the northern hemisphere of the unit tangent bundle of the sphere.

Let us look closely at the operation of excising and attaching. Since the geometry is the same as that of the Kepler problem, we may rescale the variables so that $\mu = 1$, $h = 1/2$. We then rescale the dimensions by $x \to \varepsilon^2 x$, $y \to \varepsilon^{-1} y$ so that the neighborhood is of the form $\{|x| \le 1, |y| \ge 1\}$. Let $N = \{(x,y) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n : L(x,y) = -1/2\}$, so N is a negative energy level of the Kepler problem. Let

$$\begin{gathered}A = \{1 \le |x| \le 2, 0 \le |y| \le 1\} \cap N, \quad B = \{0 \le |x| \le 1, 1 \le |y| < \infty\} \cap N,\\ T = A \cap B = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| = |y| = 1\},\end{gathered}$$

and

$$\begin{gathered}A^\dagger = \{(\xi,\eta) \in T_1 S^n : -1 \le \xi_0 \le 0\}, \quad B^\dagger = \{(\xi,\eta) \in T_1 S^n : 0 \le \xi_0 \le 1\},\\ T^\dagger = A^\dagger \cap B^\dagger = \{(\xi,\eta) \in TS^n : \xi_0 = 0\}.\end{gathered}$$

The symplectomorphism σ takes A, B, T to $A^\dagger, B^\dagger, T^\dagger$ respectively, $B \cong D^n \setminus \{0\} \times S^{n-1}$ is a neighborhood of the singularity in $H = 0$ and $B^\dagger \cong D^n \times S^{n-1}$ is a neighborhood of the north pole in $T_1 S^n$. Thus, our definition of regularization is excising two copies of B, one about each singularity, and attaching two copies of $B^\dagger$ using the symplectomorphism. The details of the attaching map are important for our computations.

The symplectomorphism takes $T \cong S^{n-1} \times S^{n-1}$ diffeomorphically to $T^\dagger$. To understand the structure of the corresponding homology map ν_*, we first look at the symplectomorphism on the boundaries T and $T^\dagger$ in the planar case when $n = 2$. We place angular coordinates θ, ϕ on T by

$$\begin{aligned}x_1 &= \cos\theta, \quad x_2 = \sin\theta,\\ y_1 &= \cos\phi, \quad y_2 = \sin\phi.\end{aligned}$$

The symplectomorphism on T in these coordinates is

$$\begin{gathered}\xi_0 = 0, \quad \xi_1 = \cos\phi, \quad \xi_2 = \sin\phi,\\ \eta_0 = \cos(\theta-\phi), \quad \eta_1 = \cos\theta - \cos(\theta-\phi)\cos\phi, \quad \eta_2 = \sin\theta - \cos(\theta-\phi)\sin\phi.\end{gathered}$$

We place angular coordinates $\theta^\dagger, \phi^\dagger$ on $T^\dagger$ by

$$\xi_1 = \cos\phi^\dagger, \quad \xi_2 = \sin\phi^\dagger$$

$$\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi^\dagger & -\sin\phi^\dagger\\ 0 & \sin\phi^\dagger & \cos\phi^\dagger\end{bmatrix}\begin{bmatrix}\cos\theta^\dagger & 0 & \sin\theta^\dagger\\ 0 & 1 & 0\\ -\sin\theta^\dagger & 0 & \cos\theta^\dagger\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Let $\alpha = \{\theta \text{ arbitrary}, \phi = 0\}$ and $\beta = \{\theta = 0, \phi \text{ arbitrary}\}$ be the generators of $H_{n-1}(T)$ and similarly let $\alpha^\dagger = \{\theta^\dagger \text{ arbitrary}, \phi^\dagger = 0\}$ and $\beta^\dagger = \{\theta^\dagger = 0, \phi^\dagger \text{ arbitrary}\}$ be the generators of $H_{n-1}(T^\dagger)$. These generators have the usual orientation in mathematics, i.e. the generators are traversed by increasing the angles.

The symplectomorphism on α is

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = 0, \quad \eta_0 = \cos\theta, \quad \eta_1 = 0, \quad \eta_2 = \sin\theta$$

$$\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\end{bmatrix} = \begin{bmatrix}\cos\theta\\ 0\\ \sin\theta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\pi/2-\theta) & 0 & \sin(\pi/2-\theta)\\ 0 & 1 & 0\\ -\sin(\pi/2-\theta) & 0 & \cos(\pi/2-\theta)\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

So $\phi^\dagger = 0, \theta^\dagger = \pi/2 - \theta$ or $\alpha \to -\alpha^\dagger$.

The symplectomorphism on β is

$$\xi_0 = 0, \quad \xi_1 = \cos\phi, \quad \xi_2 = \sin\phi$$

$$\begin{aligned}\begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\end{bmatrix} &= \begin{bmatrix}\cos\phi\\ \sin^2\phi\\ -\cos\phi\sin\phi\end{bmatrix}\\ &= \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos(\phi+\pi/2) & 0 & \sin(\phi+\pi/2)\\ 0 & 1 & 0\\ -\sin(\phi+\pi/2) & 0 & \cos(\phi+\pi/2)\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}.\end{aligned}$$

Thus $\phi^\dagger = \phi, \theta^\dagger = \phi + \pi/2$ or $\beta \to \alpha^\dagger + \beta^\dagger$. Thus the homology map $\nu_* : H_1(T) \to H_1(T^\dagger)$ is

$$\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}.$$

In the spatial case we take generators of T and $T^\dagger$ consistent with the definitions given in the planar case. Let

$$\alpha = \{|x| = 1, y_1 = 1, y_2 = y_3 = 0\}, \quad \beta = \{x_1 = 1, x_2 = x_3 = 0, |y| = 1\},$$
$$\alpha^\dagger = \{\xi_0 = 1, \xi_0 = \xi_2 = \xi_3 = 0, \eta_1 = 0, \eta_0^2 + \eta_2^2 + \eta_3^2 = 1\},$$
$$\beta^\dagger = \{\xi_0 = 0, \xi_1^2 + \xi_2^2 + \xi_3^2 = 1, \eta_0 = 1, \eta_1 = \eta_2 = \eta_3 = 0\}.$$

Each of these two-spheres generators have a great circle with the same name as a generator for the planar problem. These generators are oriented in a manner consistent with the planar convention and usual mathematical practice. That is, as we traverse the great circle of the planar problem in the positive sense, using the right-hand rule the thumb will point in the direction of the positive third coordinate. By (6) the mapping on α is $\eta_0 = x_1, \eta_1 = 0, \eta_2 = x_2, \eta_3 = x_3$, but the coordinates on α are η_0, η_3, η_2 in that order, and so $\alpha \to -\alpha^\dagger$.

To understand the mapping on β observe that the choice of how the great circles of the planar problem sit in the spatial problem is arbitrary and that any consistent choice of the orientation of the circle will yield the same results for the planar problem. Any great circle in β is mapped onto a great circle of $\alpha^\dagger$ and of $\beta^\dagger$, thus $\beta \to \pm\alpha^\dagger \pm \beta^\dagger$. The point where $x_1 = 1, x_2 = x_3 = 0, y_1 = 0, y_2 = y_3 = 0$ is mapped to the point where $\xi_0 = 0, \xi_1 = 1, \xi_2 = 0, \xi_3 = 0, \eta_0 = 1, \eta_1 = 0, \eta_2 = 0, \eta_3 = 0$. At these points we choose oriented coordinates in the generators as follows: x_2, x_3 for α; y_2, y_3 for β; η_3, η_2 for $\alpha^\dagger$ and ξ_2, ξ_3 for $\beta^\dagger$ in that order. The Jacobian determinants at these points are

$$\frac{\partial(\eta_3, \eta_2)}{\partial(y_2, y_3)} = -1, \qquad \frac{\partial(\xi_2, \xi_3)}{\partial(y_2, y_3)} = 1.$$

Therefore, $\beta \to -\alpha^\dagger + \beta^\dagger$ and the homology map $\nu_* : H_2(T) \to H_2(T^\dagger)$ is

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

We will also need to understand the inclusion $T^\dagger \to B^\dagger$. That is, we need to determine the homology map $H_{n-1}(T^\dagger) \to H_{n-1}(B^\dagger)$. Let $\alpha_1^\dagger, \beta_1^\dagger$ be the projections of $\alpha^\dagger$ and $\beta^\dagger$ onto S^n, and let $\alpha_2^\dagger, \beta_2^\dagger$ be the tangential components. Let

$$R'(t) = \begin{bmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{bmatrix}, \qquad R(t) = \begin{bmatrix} R'(t) & 0 \\ 0 & I_{n-1} \end{bmatrix},$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Define $F, G : S^{n-1} \to T_1 S^n$ by

$$F(q,t) = (R(t)\alpha_1^\dagger(q), R(t)\alpha_2^\dagger(q))$$
$$G(q,t) = \left(\cos\left(\frac{\pi t}{2}\right)\beta_1^\dagger(q) + \sin\left(\frac{\pi t}{2}\right)\beta_2^\dagger(q), -\sin\left(\frac{\pi t}{2}\right)\beta_1^\dagger(q) + \cos\left(\frac{\pi t}{2}\right)\beta_2^\dagger(q)\right).$$

Then $F_0 = \alpha^\dagger$, $G_0 = \beta^\dagger$, while

$$F(q,1) = (1, 0, \dots, 0, \alpha_{21}^\dagger(q), -\alpha_{20}^\dagger(q), \alpha_{22}^\dagger(q), \dots, \alpha_{2n}^\dagger(q))$$
$$G(q,1) = (1, 0, \dots, 0, -\beta_{10}^\dagger(q), -\beta_{11}^\dagger(q), \dots, -\beta_{1n}^\dagger(q)).$$

When $n = 2$, we can evaluate $\alpha_2^\dagger$ and $\beta_1^\dagger$ directly to see that $F_1 = G_1$. That is, $\alpha^\dagger$ and $\beta^\dagger$ have the same image in $H_1(B^\dagger)$, and the inclusion map $H_1(T^\dagger) \to H_1(B^\dagger)$ can be taken as $[1 \quad 1]$. In the spatial case, we can use spherical coordinates to write

$$\alpha_2(\phi^\dagger, \gamma^\dagger) = (\cos(\phi^\dagger)\sin(\gamma^\dagger), 0, -\sin(\phi^\dagger)\sin(\gamma^\dagger), -\cos(\gamma))$$
$$\beta_1(\phi^\dagger, \gamma^\dagger) = (0, \cos(\phi^\dagger)\sin(\gamma^\dagger), \sin(\phi^\dagger)\sin(\gamma^\dagger), \cos(\gamma)),$$

with the signs chosen to embed the planar case as $\{\gamma^\dagger = \pi/2\}$ and preserve the right-hand rule. With these choices, we again see that $F_1 = G_1$, so that $H_1(T^\dagger) \to H_1(B^\dagger)$ is given by $[1 \quad 1]$.

3.2. *Positive energy.* Refer to [**2, 3**] for the details that are summarized here. As before let ξ, η be coordinates in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, but define an Lorentz inner product $\langle \xi, \eta \rangle = -\xi_0\eta_0 + \xi_1\eta_1 + \cdots + \xi_n\eta_n = \xi^T \Lambda \eta$ where Λ is the $(n+1) \times (n+1)$ square matrix $\Lambda = \mathrm{diag}(-1, 1, \ldots, 1)$. This inner product is positive definite on the hyperboloid $\langle \xi, \xi \rangle = -1$ and defines a Riemannian metric with constant negative curvature. Let $\mathcal{S}^n$ be one sheet of this hyperboloid, $T\mathcal{S}^n$ its tangent bundle and $T_\delta\mathcal{S}^n$ its δ-sphere bundle, so

$$\begin{aligned} \mathcal{S}^n &= \{\xi \in \mathbb{R}^{n+1} \mid \langle \xi, \xi \rangle = -1, \xi_0 > 0\}, \\ T\mathcal{S}^n &= \{(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \langle \xi, \xi \rangle = -1, \xi_0 > 0, \langle \xi, \eta \rangle = 0\}, \\ T_\delta\mathcal{S}^n &= \{(\xi, \eta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \langle \xi, \xi \rangle = -1, \xi_0 > 0, \langle \xi, \eta \rangle = 0, \langle \eta, \eta \rangle = \delta^2\}. \end{aligned}$$

The geodesic flow on $\mathcal{S}^n$ is defined by the equation $\ddot{\xi} = \lambda\xi$ where $\lambda = \langle \dot{\xi}, \dot{\xi} \rangle$. This can be written as a Hamiltonian system with Hamiltonian $-\frac{1}{2}\langle \xi, \xi \rangle \langle \eta, \eta \rangle$, but we shall take the square root and consider the system with Hamiltonian

$$G = \{-\langle \xi, \xi \rangle \langle \eta, \eta \rangle\}^{1/2}, \tag{9}$$

with equations of motion on the δ-sphere bundle

$$\dot{\xi} = \Lambda \frac{\partial G}{\partial \eta} = \delta^{-1}\eta, \quad \dot{\eta} = -\Lambda \frac{\partial G}{\partial \xi} = \delta\xi.$$

The flows of G and $G^2/2$ on the unit sphere bundle are precisely the same—on other level sets they are reparameterizations of one another.

Let $\hat{\mathcal{S}}^n$ denote the hyperboloid punctured at the point $\xi^\dagger = (1, 0, \ldots, 0)$, i.e. $\hat{\mathcal{S}}^n = \mathcal{S}^n \setminus \{(1, 0, \ldots, 0)\}$. The desired projection of $\hat{\mathcal{S}}^n$ onto $\mathcal{D} = \{y \in \mathbb{R}^n \mid |y| > 1\}$ is given by

$$y_k = \frac{\xi_k}{\xi_0 - 1}, \quad k = 1, \ldots, n. \tag{10}$$

The inverse is given by

$$\xi_0 = \frac{|y|^2 + 1}{|y|^2 - 1}, \quad \xi_k = \frac{2y_k}{|y|^2 - 1}, \quad k = 1, \ldots, n. \tag{11}$$

Belbruno extends the stereographic projection (10) to a mapping $\omega : \mathbb{R}^n \times \mathcal{D} \to T\hat{\mathcal{S}}^n$ by

$$\eta_0 = -x \cdot y, \quad \eta_k = \frac{|y|^2 - 1}{2} x_k - (x \cdot y) y_k, \quad k = 1, \ldots, n \tag{12}$$

with inverse

$$x_k = \eta_k(\xi_0 - 1) - \xi_k\eta_0, \quad k = 1, \ldots, n. \tag{13}$$

PROPOSITION 3.2. *The extended stereographic mapping $\omega : \mathbb{R}^n \times \mathcal{D} \to T\hat{\mathcal{S}}^n$ given by formulae (10), (11), (13) and (12) is a symplectomorphism of $\mathbb{R}^n \times \mathcal{D}$ and $T\hat{\mathcal{S}}^n$.*

The Hamiltonian (9) of the flow on $T\hat{\mathcal{S}}^n$ in the symplectic coordinates $(x, y) \in \mathbb{R}^n \times \mathcal{D}$ becomes

$$G = \tfrac{1}{2}(|y|^2 - 1)|x|.$$

In order to treat all positive energy levels we need to scale the variables by $x \to x$, $y \to (2h)^{-1/2}y$ which is symplectic with multiplier $(2h)^{1/2}$. Now

$$G = \tfrac{1}{2}(2h)^{-1/2}(|y|^2 - 2h)|x|.$$

Let $\delta = \mu(2h)^{-1/2}$ and observe,

$$G - \delta = (2h)^{-1/2}|x|\{L - h\}, \quad \text{where } L = \frac{1}{2}|y|^2 - \frac{\mu}{|x|},$$

and L is the Hamiltonian of the Kepler problem with central mass μ.

As in the previous case define a new time τ by $d\tau = \phi(x)\,dt$ where $\phi = (2h)^{-1/2}|x|$. Then the flow on $G = \delta$ is a reparameterization of the Kepler flow on $L = h$. The flow on the δ-sphere bundle of the $\mathcal{S}^n$ is by definition the flow of the regularized Kepler problem.

Consider the restricted problem where one primary is at the origin, i.e. equation (8). Change to (ξ, η) coordinates so that

$$\begin{aligned}\phi(H - h) = \{-\langle\xi, \xi\rangle\langle\eta, \eta\rangle\}^{1/2} - \delta \\ + \phi(\xi, \eta)\left\{-\xi_2\eta_1 + \xi_1\eta_2 - \frac{1-\mu}{d_2} - \frac{1}{2}(1-\mu)^2\right\} \\ = \{-\langle\xi, \xi\rangle\langle\eta, \eta\rangle\}^{1/2} - \delta + O(\varepsilon),\end{aligned}$$

where $\varepsilon = |x|$. As in the Kepler problem the change of time $d\tau = \phi\,dt$ shows that the flow of the restricted problem on $H = h$ is a reparameterization of the flow on $\{-\langle\xi, \xi\rangle\langle\eta, \eta\rangle\}^{1/2} = \delta + O(\varepsilon)$ which is a small perturbation of the geodesic flow on the δ-sphere bundle. Thus, the singularity of the restricted problem has been regularized.

Performing the sequence of transformations used for the Kepler problem on a neighborhood of the singularity yields a perturbation of the geodesic flow on a neighborhood of the $\xi^\dagger$. Thus, the regularization of this singularity can be viewed as excising this neighborhood of the singularity and attaching a neighborhood of $\xi^\dagger$ in the δ-sphere bundle of the hyperboloid.

Let us look closely at the operation of excising and attaching. For purposes of discussing the geometry of the Kepler problem we may assume that $\mu = 1$, $h = 1/2$ and rescale the dimensions so that the neighborhood is of the form $\{|x| \le 2, |y|^2 \ge 2\}$. Let $P = \{(x, y) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n : L(x, y) = 1/2\}$, so P is a positive energy level of the Kepler problem. Let

$$A = \{2 \le |x| < \infty, 1 < |y|^2 \le 2\} \cap P, \quad B = \{0 \le |x| \le 2, 2 \le |y|^2 < \infty\} \cap P,$$
$$T = A \cap B = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| = 2, |y|^2 = 2\},$$

and

$$A^\dagger = \{(\xi, \eta) \in T_1\mathcal{S}^n : 3 \le \xi_0 < \infty\}, \quad B^\dagger = \{(\xi, \eta) \in T_1\mathcal{S}^n : 1 \le \xi_0 \le 3\},$$
$$T^\dagger = A^\dagger \cap B^\dagger = \{(\xi, \eta) \in T\mathcal{S}^n : \xi_0 = 3\}.$$

The symplectomorphism takes A, B, T to $A^\dagger, B^\dagger, T^\dagger$ respectively, $B \cong D^n \setminus \{0\} \times S^{n-1}$ is a neighborhood of the singularity in $L = 1/2$ and $B^\dagger$ is a neighborhood of the $\xi^\dagger \cong D^n \times S^{n-1}$ in $T_1\mathcal{S}^n$.

The symplectomorphism takes $T \cong S^{n-1} \times S^{n-1}$ diffeomorphically to $T^\dagger$. As in the negative energy case, we first look at the symplectomorphism on the boundaries T and $T^\dagger$ in the planar case $n = 2$. We take coordinates θ, ϕ on T by

$$x_1 = 2\cos\theta, \quad x_2 = 2\sin\theta$$
$$y_1 = \sqrt{2}\cos\phi, \quad y_2 = \sqrt{2}\sin\phi$$

and angular coordinates $\theta^\dagger, \phi^\dagger$ on $T^\dagger$ by

$$\xi_0 = 3, \quad \xi_1 = 2\sqrt{2}\cos\phi^\dagger, \quad \xi_2 = 2\sqrt{2}\sin\phi^\dagger$$
$$\eta = \psi R_3(\phi^\dagger) R_1^{-1} R_2(\theta^\dagger) R_1 v,$$

where

$$\eta = \begin{bmatrix}\eta_0\\ \eta_1\\ \eta_2\end{bmatrix}, \quad v = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad R_1 = \begin{bmatrix}-\frac{3}{\sqrt{17}} & \frac{2\sqrt{2}}{\sqrt{17}} & 0\\ -\frac{2\sqrt{2}}{\sqrt{17}} & -\frac{3}{\sqrt{17}} & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$R_2(\theta^\dagger) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta^\dagger & \sin\theta^\dagger\\ 0 & -\sin\theta^\dagger & \cos\theta^\dagger\end{bmatrix}, \quad R_3(\phi^\dagger) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi^\dagger & -\sin\phi^\dagger\\ 0 & \sin\phi^\dagger & \cos\phi^\dagger\end{bmatrix}.$$

The vector v is a tangent vector to $\mathcal{S}^n$ at the point $(3, 2\sqrt{2}, 0)$, $R_1^{-1}R_2(\theta^\dagger)R_1$ rotates v about the normal $(-3, 2\sqrt{2}, 0)$ by an angle $\theta^\dagger$, so it is still a tangent vector. The matrix $R_3(\phi^\dagger)$ rotates this tangent vector about the ξ_0 axis by an angle $\phi^\dagger$.

In this case we are dealing with both Euclidian and Lorentzian geometry. η is a unit vector in the Lorentz metric ($\langle\eta, \eta\rangle = 1$) whereas the vector v is a unit vector in Euclidian geometry and the matrices R_1, R_2, R_3 are orthogonal matrices. The positive scalar ψ is the Euclidean norm of η. Both η and the rotations of v lie in the tangent plane of the hyperboloid and so given η and $\phi^\dagger$ one can solve for $\theta^\dagger$.

The symplectomorphism on T in these coordinates is

$$\xi_0 = 3, \quad \eta_0 = -2\sqrt{2}\cos(\theta - \phi),$$
$$\xi_1 = 2\sqrt{2}\cos\phi, \quad \eta_1 = \cos\theta - 4\cos(\theta - \phi)\cos\phi,$$
$$\xi_2 = 2\sqrt{2}\sin\phi, \quad \eta_2 = \sin\theta - 4\cos(\theta - \phi)\sin\phi.$$

Let $\alpha = \{\theta \text{ arbitrary}, \phi = 0\}$ and $\beta = \{\theta = 0, \phi \text{ arbitrary}\}$ be the generators of $H_{n-1}(T)$ and, similarly, let $\alpha^\dagger = \{\theta^\dagger \text{ arbitrary}, \phi^\dagger = 0\}$ and $\beta^\dagger = \{\theta^\dagger = 0, \phi^\dagger \text{ arbitrary}\}$ be the generators of $H_{n-1}(T^\dagger)$. These generators have the usual orientation in mathematics, i.e. the generators are traversed by increasing the angles.

The symplectomorphism on α is

$$\xi_0 = 3, \quad \xi_1 = 2\sqrt{2}, \quad \xi_2 = 0, \quad \eta_0 = -2\sqrt{2}\cos\theta, \quad \eta_1 = -3\cos\theta, \quad \eta_2 = \sin\theta.$$

Unlike the case of negative energy we cannot identify θ and $\theta^\dagger$, since we are dealing with two geometries. Therefore we must consider the equation

$$\begin{bmatrix} -2\sqrt{2}\cos\theta \\ -3\cos\theta \\ \sin\theta \end{bmatrix} = \psi R_1^{-1} R_2(\theta^\dagger) R_1 v = \psi \begin{bmatrix} -\frac{2\sqrt{2}}{\sqrt{17}}\sin\theta^\dagger \\ -\frac{3}{\sqrt{17}}\sin\theta^\dagger \\ \cos\theta^\dagger \end{bmatrix} = \psi \begin{bmatrix} -\frac{2\sqrt{2}}{\sqrt{17}}\cos(\pi/2-\theta^\dagger) \\ -\frac{3}{\sqrt{17}}\cos(\pi/2-\theta^\dagger) \\ \sin(\pi/2-\theta^\dagger) \end{bmatrix}.$$

Although they are not equal as θ completes one revolution in the positive sense $\theta^\dagger$ completes one revolution in the negative sense, so $\alpha \to -\alpha^\dagger$.

The symplectomorphism on β is

$$\xi_0 = 3, \quad \xi_1 = 2\sqrt{2}\cos\phi, \quad \xi_2 = 2\sqrt{2}\sin\phi,$$
$$\tilde{\eta}(\phi) = \begin{bmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2}\cos\phi \\ 1-4\cos^2\phi \\ -4\cos\phi\sin\phi \end{bmatrix}.$$

We must consider

$$\psi R_1 R_3^{-1}(\phi)\tilde{\eta}(\phi) = R_2(\theta^\dagger) R_1 v$$

or

$$\psi \begin{bmatrix} 0 \\ \sqrt{17}\cos\phi \\ -\sin\phi \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta^\dagger \\ \cos\theta^\dagger \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(\theta^\dagger-\pi/2) \\ -\sin(\theta^\dagger-\pi/2) \end{bmatrix}.$$

Thus, as ϕ makes one complete revolution so does $\theta^\dagger$ or $\beta \to \alpha^\dagger + \beta^\dagger$.

In the spatial case we take generators of T and $T^\dagger$ consistent with the definitions given in the planar case. Let

$$\alpha = \{|x| = 2, y_1 = \sqrt{2}, y_2 = y_3 = 0\}, \quad \beta = \{x_1 = 2, x_2 = x_3 = 0, |y|^2 = 2\},$$
$$\alpha^\dagger = \{\xi_0 = 3, \xi_1 = 2\sqrt{2}, \xi_2 = \xi_3 = 0, \langle\xi,\eta\rangle = 0, \langle\eta,\eta\rangle = 1\},$$
$$\beta^\dagger = \{\xi_0 = 3, \langle\xi,\xi\rangle = -1, \eta_0 = \eta_3 = 0, \langle\xi,\eta\rangle = 0, \langle\eta,\eta\rangle = 1\}.$$

Each of these two-sphere generators have a great circle with the same name as a generator for the planar problem. These generators are oriented in a manner consistent with the planar convention and usual mathematical practice. That is, as we traverse the great circle of the planar problem in the positive sense, using the right-hand rule the thumb will point in the direction of the positive third coordinate. By (12) the mapping on α is $\xi_0 = 3$, $\xi_1 = 2\sqrt{2}$, $\xi_2 = \xi_3 = 0$, $\eta_0 = -\sqrt{2}x_1$, $\eta_1 = -\frac{3}{2}x_1$, $\eta_2 = -\frac{1}{2}x_2$, $\eta_3 = \frac{1}{2}x_3$, but the coordinates on α are η_0, η_3, η_2 in that order, and so $\alpha \to -\alpha^\dagger$.

The point where $x_1 = 2$, $x_2 = x_3 = 0$, $y_1 = \sqrt{2}$, $y_2 = y_3 = 0$ is mapped to the point where $\xi_0 = 3$, $\xi_1 = \sqrt{2}$, $\xi_2 = \xi_3 = 0$, $\eta_0 = -2\sqrt{2}$, $\eta_1 = -3$, $\eta_2 = \eta_3 = 0$. At these points we choose oriented coordinates in the generators as follows: x_2, x_3 for α; y_2, y_3 for β; η_3, η_2 for $\alpha^\dagger$ and ξ_2, ξ_3 for $\beta^\dagger$ in that order. The Jacobian determinants at these points are

$$\frac{\partial(\eta_3,\eta_2)}{\partial(y_2,y_3)} = -8, \quad \frac{\partial(\xi_2,\xi_3)}{\partial(y_2,y_3)} = \frac{4}{9}.$$

Therefore, $\beta \to -\alpha^\dagger + \beta^\dagger$ and the homology map $\nu_* : H_2(T) \to H_2(T^\dagger)$ is

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

That is, the homological results for positive energy are the same as those for negative energy: $H_{n-1}(T) \to H_{n-1}(T^\dagger)$ and $H_{n-1}(T^\dagger) \to H_{n-1}(B^\dagger)$ are given by $\left[\begin{smallmatrix} -1 & (-1)^n \\ 0 & 1 \end{smallmatrix}\right]$ and $[1 \quad 1]$ respectively.

The case of zero energy can be handled in the same way using the regularization in [**3**], or one can consider it as a small perturbation of the previous cases.

3.3. *Construction of the regularized manifolds.* We are now ready to construct the regularized manifolds $\mathfrak{r}(c)$ and $\mathfrak{R}(c)$. We describe in detail the construction for the spatial problem—the planar problem will simply be the restriction to the appropriate planar subspaces. For the spatial problem, choose small balls about the singularities $(-\mu, 0)$, $(1-\mu, 0)$ in the Hill's region $\mathfrak{H}(c)$. The boundaries of these disks are the spheres $\mathfrak{S}(c)$ introduced in §2.1. Let $\mathfrak{N}$ be the preimage of these balls in $\mathfrak{M}(c)$. Then (up to a rescaling), $\mathfrak{N}$ is conjugate to the sets B constructed in §3.1 and §3.2. Then take $\mathfrak{D} = B^\dagger$. For the attaching map $\sigma : \mathfrak{N} \to \mathfrak{D}$, take ν (as in §3.1) for $c > \mu(1-\mu)$, and take ω (as in §3.2) for $c < \mu(1-\mu)$. In either case, the regularized manifold is

$$\mathfrak{R}(c) = \mathfrak{M}(c) \cup_\sigma \mathfrak{D}.$$

The construction for the planar problem can be viewed in two ways. On the one hand, the same construction can be employed, choosing disks in $\mathfrak{h}$ with boundary circles $\mathfrak{s}(c)$, taking the preimage $\mathfrak{n}$ in $\mathfrak{m}(c)$ and attaching $\mathfrak{d} \cong D^2 \times S^1$ via either ν or ω to produce

$$\mathfrak{r}(c) = \mathfrak{m}(c) \cup_\sigma \mathfrak{d}.$$

On the other hand, $\mathfrak{r}(c)$ can also be viewed as a submanifold of $\mathfrak{R}(c)$, obtained by restricting both $\mathfrak{M}(c)$ and $\mathfrak{D}$ to the appropriate invariant submanifolds.

4. *Homology of the regularized manifolds*

We are now ready to compute the homology groups for the integral manifolds in the regularized problem. Having observed that $\mathfrak{r}(c) = \mathfrak{m}(c) \cup_\sigma \mathfrak{d}$ and $\mathfrak{R}(c) = \mathfrak{M}(c) \cup_\Sigma \mathfrak{D}$, it is natural to use these as Mayer–Vietoris decompositions. The sequences have the form

$$\to H_p(\mathfrak{n}) \overset{j_*}{\to} H_p(\mathfrak{m}) \oplus H_p(\mathfrak{d}) \to H_p(\mathfrak{r}) \to$$

and

$$\to H_p(\mathfrak{N}) \overset{J_*}{\to} H_p(\mathfrak{M}) \oplus H_p(\mathfrak{D}) \to H_p(\mathfrak{R}) \to .$$

Since the homology groups of $\mathfrak{n}$, $\mathfrak{N}$, $\mathfrak{m}$, $\mathfrak{M}$, $\mathfrak{d}$ and $\mathfrak{D}$ are all known, we only need to determine the maps j_* and J_* to compute the homology of $\mathfrak{r}$ and $\mathfrak{R}$. That is, there are short exact sequences

$$0 \longrightarrow \operatorname{coker}(j_{p*}) \longrightarrow H_p(\mathfrak{r}) \longrightarrow \ker(j_{p-1*}) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{coker}(J_{p*}) \longrightarrow H_p(\mathfrak{R}) \longrightarrow \ker(J_{p-1*}) \longrightarrow 0.$$

Moreover, since $\ker(j_*)$ and $\ker(J_*)$ are torsion-free, these sequences split and

$$H_p(\mathfrak{r}) \cong \operatorname{coker}(j_{p*}) \oplus \ker(j_{p-1*})$$
$$H_p(\mathfrak{R}) \cong \operatorname{coker}(J_{p*}) \oplus \ker(J_{p-1*}).$$

The maps j_* and J_* are simply the induced homomorphisms of the inclusion maps $\mathfrak{m} \leftarrow \mathfrak{n} \rightarrow \mathfrak{d}$ and $\mathfrak{M} \leftarrow \mathfrak{N} \rightarrow \mathfrak{D}$. To assemble the information on these homomorphisms, we need to use the decompositions

$$H_p(\mathfrak{n}) \cong H_p(\mathfrak{s}) \oplus H_{p-1}(\mathfrak{s}), \quad H_p(\mathfrak{m}) \cong H_p(\mathfrak{h}) \oplus H_{p-1}(\mathfrak{h}, \partial\mathfrak{h})$$
$$H_p(\mathfrak{N}) \cong H_p(\mathfrak{S}) \oplus H_{p-2}(\mathfrak{S}), \quad H_p(\mathfrak{M}) \cong H_p(\mathfrak{H}) \oplus H_{p-1}(\mathfrak{H}, \partial\mathfrak{H}).$$

The attaching maps $\sigma_* : H_*(\mathfrak{n}) \rightarrow H_*(\mathfrak{d})$ and $\Sigma : H_*(\mathfrak{N}) \rightarrow H_*(\mathfrak{D})$ are simply the compositions $T \rightarrow T^\dagger \rightarrow B^\dagger$ of §§3.1 and 3.2. That is,

$$\sigma_* = [1 \quad 1]\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = [-1 \quad 2]$$
$$\Sigma_* = [1 \quad 1]\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = [-1 \quad 0].$$

Using the decompositions of $H_*(\mathfrak{n})$ and $H_p(\mathfrak{N})$ (with the maps $H_1(\mathfrak{n}) \rightarrow H_1(\mathfrak{s})$ and $H_2(\mathfrak{N}) \rightarrow H_2(\mathfrak{S})$ both mapping α to the generator), the maps

$$\sigma_*^1 : H_p(\mathfrak{s}) \rightarrow H_p(\mathfrak{d}), \quad \sigma_*^2 : H_{p-1}(\mathfrak{s}) \rightarrow H_p(\mathfrak{d})$$
$$\Sigma_*^1 : H_p(\mathfrak{S}) \rightarrow H_p(\mathfrak{D}), \quad \Sigma_*^2 : H_{p-2}(\mathfrak{S}) \rightarrow H_p(\mathfrak{D})$$

can be written explicitly as:

$$\sigma_{0*}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_{0*}^2 = 0, \quad \sigma_{1*}^1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_{1*}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\Sigma_{0*}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_{0*}^2 = 0, \quad \Sigma_{2*}^1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Sigma_{2*}^2 = 0.$$

Similarly, the homology maps induced by the inclusions $\mathfrak{n} \rightarrow \mathfrak{m}$ and $\mathfrak{N} \rightarrow \mathfrak{M}$ can be derived from Theorem 2.1. Consider $\mathfrak{n} \rightarrow \mathfrak{m}(c)$ first. The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{p-1}(\mathfrak{s}) & \longrightarrow & H_p(\mathfrak{n}) & \underset{\longleftarrow\, s_{e_1}}{\longrightarrow} & H_p(\mathfrak{s}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{\iota}_{p-1*}} & & \downarrow & & \downarrow{\scriptstyle \iota_{p*}} & & \\
0 & \longrightarrow & H_{p-1}(\mathfrak{h}, \partial\mathfrak{h}) & \longrightarrow & H_p(\mathfrak{m}) & \underset{\longleftarrow\, s_{e_1}}{\longrightarrow} & H_p(\mathfrak{s}) & \longrightarrow & 0
\end{array}$$

commutes and splits naturally, so there is a commutative diagram

$$\begin{array}{ccc}
H_p(\mathfrak{n}) & \longrightarrow & H_p(\mathfrak{m}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H_p(\mathfrak{s}) \oplus H_{p-1}(\mathfrak{s}) & \xrightarrow{\begin{bmatrix} \iota_{p*} & 0 \\ 0 & \hat{\iota}_{p-1*} \end{bmatrix}} & H_p(\mathfrak{h}) \oplus H_{p-1}(\mathfrak{h}, \partial\mathfrak{h})
\end{array}$$

In exactly the same fashion, there is a commutative diagram

$$\begin{array}{ccc} H_p(\mathfrak{N}) & \longrightarrow & H_p(\mathfrak{M}) \\ \downarrow\cong & & \downarrow\cong \\ H_p(\mathfrak{S})\oplus H_{p-2}(\mathfrak{S}) & \xrightarrow{\begin{bmatrix} I_{p*} & 0 \\ 0 & \hat{I}_{p-2*}\end{bmatrix}} & H_p(\mathfrak{H})\oplus H_{p-2}(\mathfrak{H},\partial\mathfrak{H}) \end{array}$$

Combining these, the homomorphisms $j_* : H_p(\mathfrak{n}) \to H_p(\mathfrak{m}) \oplus H_p(\mathfrak{d})$ and $J_* : H_p(\mathfrak{N}) \to H_p(\mathfrak{M}) \oplus H_p(\mathfrak{D})$ can be expressed as

$$H_p(\mathfrak{s})\oplus H_{p-1}(\mathfrak{s}) \xrightarrow{\begin{bmatrix} \iota_{p*} & 0 \\ 0 & \hat{\iota}_{p-1*} \\ \sigma^1_{p*} & \sigma^2_{p*}\end{bmatrix}} H_p(\mathfrak{h})\oplus H_{p-1}(\mathfrak{h},\partial\mathfrak{h})\oplus H_p(\mathfrak{d})$$

and

$$H_p(\mathfrak{S})\oplus H_{p-2}(\mathfrak{S}) \xrightarrow{\begin{bmatrix} I_{p*} & 0 \\ 0 & \hat{I}_{p-2*} \\ \Sigma^1_{p*} & \Sigma^2_{p*}\end{bmatrix}} H_p(\mathfrak{H})\oplus H_{p-2}(\mathfrak{H},\partial\mathfrak{H})\oplus H_p(\mathfrak{D})$$

For the planar problem, j_* can only be non-trivial in dimensions 0, 1 and 2. In these dimensions, $H_*(\mathfrak{n})$ has rank 2, 4 and 2 respectively, and

$$j_{0*} = \begin{bmatrix} \iota_{0*} \\ \mathrm{id} \end{bmatrix}, \quad j_{1*} = \begin{bmatrix} \iota_{1*} & 0 \\ 0 & \hat{\iota}_{0*} \\ -\mathrm{id} & 2\mathrm{id} \end{bmatrix}, \quad j_{2*} = \hat{\iota}_{1*}.$$

Clearly, j_0 is injective for all c. Consulting Table 6 shows that ι_{1*} is injective for all c, $\hat{\iota}_{0*}$ is surjective for all c and j_2 has full rank for all c.

For $c > 3$, $H_0(\mathfrak{h}, \partial\mathfrak{h}) = 0$, so

$$j_{1*} = \begin{bmatrix} \iota_{1*} & 0 \\ -\mathrm{id} & 2\mathrm{id} \end{bmatrix},$$

with ι_{1*} injective. Thus j_{1*} is injective, but the two generators of $H_0(\mathfrak{s})$ each map to twice a generator. For $c < 3$, $\hat{\iota}_{0*}$ is an isomorphism, so j_{1*} can be row-reduced (over $\mathbb{Z}$) to $\left[\begin{smallmatrix} I_4 \\ 0 \end{smallmatrix}\right]$.

The result of all this is that there is 2-torsion in $H_1(\mathfrak{r})$ for $c > 3$, and the Betti numbers for $\mathfrak{r}$ are

$$\begin{aligned} \beta_0(\mathfrak{r}) &= \beta_0(\mathfrak{h}) \\ \beta_1(\mathfrak{r}) &= \beta_1(\mathfrak{h}) + \beta_0(\mathfrak{h}, \partial\mathfrak{h}) - 2 \\ \beta_2(\mathfrak{r}) &= \beta_1(\mathfrak{h}, \partial\mathfrak{h}) - \mathrm{rank}(\hat{\iota}_{1*}) \\ \beta_3(\mathfrak{r}) &= 2 - \mathrm{rank}(\hat{\iota}_{1*}) \end{aligned}$$

The values are shown in Table 3.

The calculations for $\mathfrak{R}$ are simpler, because J_* only occurs in even dimensions. For $p = 0, 2, 4$, $H_p(\mathfrak{R}) \cong \mathrm{coker}(J_{p*})$, while for $p = 1, 3, 5$, $H_p(\mathfrak{R}) \cong H_p(\mathfrak{M}) \oplus \ker(J_{p-1*})$.

The values for J_* are:

$$J_{0*} = \begin{bmatrix} I_{0*} \\ \mathrm{id} \end{bmatrix}, \quad J_{2*} = \begin{bmatrix} I_{2*} & 0 \\ 0 & \hat{I}_{0*} \\ -\mathrm{id} & 0 \end{bmatrix}, \quad J_{4*} = \hat{I}_{2*}.$$

We need to consider the cases $c < \mu(1-\mu)$ and $c > \mu(1-\mu)$ separately. For $c > \mu(1-\mu)$, $H_p(\mathfrak{H}, \partial\mathfrak{H}) \cong H_{p-1}(\mathfrak{h}, \partial\mathfrak{h})$ and $\hat{I}_{p*}$ is conjugate to $\hat{\imath}_{p-1*}$. That is,

$$J_{0*} = \begin{bmatrix} I_{0*} \\ \mathrm{id} \end{bmatrix}, \quad J_{2*} = \begin{bmatrix} I_{2*} & 0 \\ -\mathrm{id} & 0 \end{bmatrix}, \quad J_{4*} = \hat{\imath}_{1*},$$

so no J_* creates torsion and $\mathrm{rank}(J_{0*}) = \mathrm{rank}(J_{2*}) = 2$, $\mathrm{rank}(J_{4*}) = \mathrm{rank}(\hat{\imath}_{1*})$.

There is no torsion in $H_*(\mathfrak{R}(c))$ and the Betti numbers are:

$$\begin{aligned}
\beta_0(\mathfrak{R}) &= \beta_0(\mathfrak{H}) \\
\beta_1(\mathfrak{R}) &= \beta_1(\mathfrak{M}) \\
\beta_2(\mathfrak{R}) &= \beta_2(\mathfrak{H}) \\
\beta_3(\mathfrak{R}) &= \beta_3(\mathfrak{M}) + 2 \\
\beta_4(\mathfrak{R}) &= \beta_1(\mathfrak{h}, \partial\mathfrak{h}) - \mathrm{rank}(\hat{\imath}_{1*}) \\
\beta_5(\mathfrak{R}) &= \beta_5(\mathfrak{M}) + 2 - \mathrm{rank}(\hat{\imath}_{1*}).
\end{aligned}$$

These are the formulae obtained by direct evaluation. It is worth noting that these can be reformulated, by comparing them to the Betti number formulae for $\mathfrak{M}(c)$. That is, $\beta_0(\mathfrak{H}) = \beta_0(\mathfrak{M})$ and $\beta_2(\mathfrak{H}) = \beta_2(\mathfrak{M})$, while $\beta_1(\mathfrak{h}, \partial\mathfrak{h}) = \beta_4((M))$. That is,

$$\begin{aligned}
\beta_0(\mathfrak{R}) &= \beta_0(\mathfrak{M}) \\
\beta_1(\mathfrak{R}) &= \beta_1(\mathfrak{M}) \\
\beta_2(\mathfrak{R}) &= \beta_2(\mathfrak{M}) \\
\beta_3(\mathfrak{R}) &= \beta_3(\mathfrak{M}) + 2 \\
\beta_4(\mathfrak{R}) &= \beta_4(\mathfrak{M}) - \mathrm{rank}(\hat{\imath}_{1*}) \\
\beta_5(\mathfrak{R}) &= \beta_5(\mathfrak{M}) + 2 - \mathrm{rank}(\hat{\imath}_{1*}).
\end{aligned}$$

These values are listed in Table 4.

For $c < \mu(1-\mu)$, $H_p(\mathfrak{H}, \partial\mathfrak{H}) \cong H_p(\mathfrak{H})$ and $\hat{I}_{p*}$ is conjugate to I_{p*}. Clearly, $I_{0*} = [1 \quad 1]$, while I_{2*} is an isomorphism. The matrices are then (up to a choice of bases in $H_2(\mathfrak{H})$)

$$J_{0*} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_{2*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad J_{4*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus none of the J_* homomorphisms create torsion, and $\mathrm{rank}(J_{0*}) = \mathrm{rank}(J_{4*}) = 2$, $\mathrm{rank}(J_{0*}) = 3$. Since $\mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{D}$ all have non-trivial homology only in dimensions 0, 2 and 4, it is a simple matter to now read off the Betti numbers of $\mathfrak{R}$. These values are listed in Table 4.

5. *Cross sections*

Typically, the homology of a manifold can only be directly related to the dynamics on the manifold when the manifold is compact. One exception is the existence of a cross section. For any space (compact or not), the existence of a cross section imposes restrictions on the homology of the space. These were formulated in [**20**] as the following.

THEOREM 5.1. *If the flow $\Phi : \mathbb{R} \times M \to M$ on the manifold M admits a cross section C, then:*

- *$M \setminus \partial C$ is a fiber bundle over S^1 with fiber $C \setminus \partial C$;*
- *there is a long exact homology sequence*

$$\to H_{k+1}(M, \partial C) \to H_k(C, \partial C) \xrightarrow{\mathrm{id}-P_*} H_k(C, \partial C) \to H_k(M, \partial C) \to;$$

- *if M, C and ∂C are of finite type, then there exists a polynomial $Q(t)$ with*

$$-\min\{P_{\partial C}(t), tP_{(C,\partial C)}(t)\} \leq Q(t) \leq P_{(C,\partial C)}(t)$$

  such that $P_M(t) - P_{\partial C}(t) = (1+t)Q(t)$;
- *if M and ∂C are of finite type, then $\chi(M) = \chi(\partial C)$.*

In the case of a global cross section when $\partial C = \emptyset$, this theorem implies $\chi(M) = 0$, $H_1(M)$ must have a factor $\mathbb{Z}$ and the polynomial $Q(t)$ must have non-negative coefficients.

Since $\mathfrak{m}$, $\mathfrak{M}$, $\mathfrak{r}$ and $\mathfrak{R}$ all have finitely generated homology, the only hypothesis required to apply the theorem is that the cross section C is of finite type. As described in §1.2, these can be interpreted as necessary conditions for any of the four manifolds $\mathfrak{m}$, $\mathfrak{M}$, $\mathfrak{r}$, $\mathfrak{R}$ to admit a global cross section of finite type; and also as necessary conditions for $\mathfrak{m}$ and $\mathfrak{r}$ to be the boundaries of cross sections of finite type to $\mathfrak{M}$ and $\mathfrak{R}$.

If we look first for global cross sections, the Euler characteristic requirement shows that the spatial restricted manifold $\mathfrak{M}$ can never admit a global cross section, nor can the spatial regularized manifold $\mathfrak{R}$ in case I. The requirement that H_1 has a factor of $\mathbb{Z}$ further rules out a global cross section for $\mathfrak{R}$ in case II. When the manifolds are disconnected, these requirements must be satisfied on all components; but in cases V and VI, the bounded components of both $\mathfrak{r}$ and $\mathfrak{R}$ fail to have a factor of $\mathbb{Z}$ in H_1. Thus global cross sections are ruled out for both $\mathfrak{r}$ and $\mathfrak{R}$ in cases V and VI.

A priori, the requirement that the Poincaré polynomial factors as $(1+t)Q(t)$, with $Q(t)$ positive, is a stronger requirement that might exclude further cases. In this instance, it does not. In all remaining cases, the Poincaré polynomial is divisible by $(1+t)$, and the quotient has positive coefficients. So, for $\mathfrak{m}$ in all cases, for $\mathfrak{r}$ in cases I–IV and for $\mathfrak{R}$ in cases III and IV, there is no homological obstruction to the existence of a global cross section.

We next consider if the planar manifolds can serve as boundaries for cross sections to the spatial manifolds. Since the Euler characteristics of the planar and spatial restricted manifolds are never equal, $\mathfrak{m}$ is never the boundary of a cross section to the flow of $\mathfrak{M}$. For the regularized problem, the results are more equivocal. In case I the Euler characteristics of the planar and spatial manifolds are different, ruling out a cross section for $\mathfrak{R}$ with boundary $\mathfrak{r}$. In all other cases the Euler characteristics of the planar and spatial manifolds are both zero. In case II some of the finer structure of Theorem 5.1 can be used to rule out a cross section with boundary $\mathfrak{r}$.

Namely, if there were a section C with boundary $\mathfrak{r}$, then $H_0(C, \mathfrak{r}) = 0$. For, if not, then there must be a component C_0 to C which does not intersect $\mathfrak{r}$. As the image of C_0 under the flow is clearly connected and disjoint from $\mathfrak{r}$, there must be a component to $\mathfrak{R}$ which is disjoint from $\mathfrak{r}$; but there is no such component, so $H_0(C, \mathfrak{r}) = 0$. Thus the Poincaré polynomial of any such pair $(C, \mathfrak{r})$ must have constant term zero. On the other hand,

$$P_{\mathfrak{R}}(t) - P_{\mathfrak{r}}(t) = -t + 2t^2 + 3t^3 = (1+t)(-t+3t^2).$$

The inequality $-tP_{C,\mathfrak{r}}(t) \leq Q(t)$ would then require the constant term of $P_{\mathfrak{r}}(t)$ to be at least 1.

In the remaining cases, III–VI, there is no homological obstruction. Indeed, in case VI, there is a very natural candidate for a cross section C. Both $\mathfrak{r}$ and $\mathfrak{R}$ have three components: one around each of the regularized singularities and one unbounded component. Each of the bounded components of $\mathfrak{R}$ is conjugate to the geodesic flow on T_1S^3, with the corresponding component of $\mathfrak{r}$ conjugate to the geodesic flow on T_1S^2. If $T_1S^2 = \{(\xi, \eta) \in T_1S^3 \mid \xi_3 = 0, \eta_3 = 0\}$, then let $C = \{(\xi, \eta) \in T_1S^3 \mid \xi_3 = 0, \eta_3 \geq 0\}$. Then C is a cross section for the geodesic flow on T_1S^3, and has boundary T_1S^2.

On the unbounded component, let $C_u(c) = \{(x, y) \in \mathfrak{M}_u(c) | x_3 = 0, y_3 \geq 0\}$. Then $\partial C_u(c) = \mathfrak{m}_u(c)$, and $\dot{x}_3 = y_3 \geq 0$, so the flow is transverse to C_u on $C_u \setminus \mathfrak{m}_u$. What remains open is whether or not all orbits in $\mathfrak{M}_u(c)$ intersect $C_u(c)$. If they do, then $C_u(c)$ is a cross section to $\mathfrak{M}_u(c)$. It would suffice to show that the forward orbit of every point in $C_u(c)$ intersects the set $\{(x, y) \in \mathfrak{M}_u(c) \mid x_3 \geq 0, y_3 = 0\}$. This is an open question at the moment.

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