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Qualitative Theory of Dynamical Systems

The Elusive Liapunov Periodic Solutions

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Abstract We illustrate the use of regular and singular reduction methods in conjunction with classical perturbation theory to find periodic solutions of Hamiltonian systems. In particular, we use these methods to find families of periodic solutions when the classical Liapunov center theorem fails due to a resonance.

Keywords Liapunov center theorem · Periodic solutions · Hamiltonian systems in resonance · Reduced space · Orbit space · Orbifold · Singular reduction

Mathematics Subject Classification 34C20 · 34C25 · 37J15 · 37J40 · 53D20 · 70F10 · 70K50 · 70K65

1 Introduction

The problem we discuss here has been a testing ground of many different methods for analyzing Hamiltonian systems—in particular finding periodic solutions and their stability. Here we illustrate the use of the method of singular reduction on this classic

Dedicated to Clark Robinson on his 70th birthday.

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problem. Normalization and singular reduction reduces the dimension of the problem studied and our test problem is a two-degree-of freedom Hamiltonian system in $\mathbb{R}^4$ which will be studied by a Hamiltonian system on a two dimensional real algebraic surface called an orbifold. The two dimensionality lends itself to a graphical representation with a better geometric insight.

Our starting point is the classic 1892 Liapunov center theorem, to wit, *consider the smooth Hamiltonian system*

$$\dot{z} = Az + \cdots = JSz + \cdots$$

defined in a neighborhood of the origin in $\mathbb{R}^4$, *let the eigenvalues of the Hamiltonian matrix A be the pure imaginaries* $\pm \mathrm{i}\omega_1,\ \pm \mathrm{i}\omega_2,\ \omega_1, \omega_2 \neq 0$, *then if* ω_1/ω_2 *is not an integer the system has a one parameter family of periodic solutions emanating from the origin of period near* $2\pi/|\omega_1|$.

Thus if ω_1/ω_2 and ω_2/ω_1 are not integers there are two families of periodic solutions emanating from the origin, but only one is guaranteed if one ratio is an integer different from ± 1 and none is guaranteed if the ratio is ± 1. The goal of this paper is to illustrate the method of singular reduction in deciding the existence of the other families of periodic solutions.

In the celebrated theorem of Weinstein two periodic solutions are found in each small energy level (H constant) provided the symmetric matrix S is definite, positive or negative. So we will need to consider the indefinite case in detail and we will find cases with none, one or two families of periodic solutions depending on a particular eigenvalue ratio and higher order terms.

The prototypical example is the Hamiltonian of the circular, planar three body problem at the Lagrange equilateral equilibrium, $\mathcal{L}_4$, for various values of the mass ratio parameter μ. The quadratic part of the Hamiltonian at $\mathcal{L}_4$ is indefinite and all ratios of the frequencies are found for various values of μ.

Notes An early method of finding periodic solutions, indeed the method used by Liapunov himself, was to construct a formal power series solution and then show that the series actually converges by obtaining estimates on the coefficients. The refined version of this method is known as the method of majorants. The classic 1892 paper of Liapunov was translated into French and reproduced in [11]. Power series proofs of the center theorem can also be found in [19,26]. Buchanan [3], Moulton [18], Roels [22] and many others carried on this series tradition.

Another early method still used today is the use of computers, human and electronic, to numerically calculate periodic solutions in specific equations such as the restricted three body problem. A famous example of the use of human computer was the series of papers published by Strömgren and colleagues in the Copenhagen Observatory Publication starting in 1913 where periodic solutions of the restricted three body problem with $\mu = 1/2$ where computed—see [27] and references therein. In the early 1960s Rabe and colleagues of the Cincinnati Observatory used the early electronic workhorse, the IBM 650, to compute Trojan periodic solutions [8,20,21]. Since then there has been an explosion of papers on computing periodic solutions to various problems.

The use of normal form techniques developed slowly from the works of Poincaré and Birkhoff to finally apply to the resonance cases and bifurcation problems as carried out by Schmidt [25] and Henrard [9,10] in the early 1970s. It is from this tradition that our procedure evolved, so additional notes will be found in later sections.

We will not touch on topological methods such as used by Weinstein [28,29] or the recent explosion of new families of periodic solutions to the N-body problem using variational methods and exploring various symmetries following the landmark paper by Chenciner and Montgomery [4]. These lines of research lead to a parallel and distinct universe which we shall not follow.

2 The Method

2.1 Invariants

We illustrate the use of singular reduction on a resonant system subject to small perturbations using normalization (i.e. averaging) and invariants. Consider the two degree-of-freedom system $\mathbb{H}_k = \frac{1}{2}\left[k(x_1^2 + y_1^2) - (x_2^2 + y_2^2)\right]$ where k is a positive integer and $z = (x_1, x_2, y_1, y_2)$ are rectangular coordinates. Change to action-angle coordinates $I_j = x_j^2 + y_j^2,\ \theta_j = \tan^{-1} y_j/x_j,\ \ j = 1, 2$ which is symplectic with multiplier 2. Then

$$\mathbb{H}_k = kI_1 - I_2$$

and the equations of motion are

$$\dot{I}_1 = 0, \quad \dot{\theta}_1 = -k, \quad \dot{I}_2 = 0, \quad \dot{\theta}_2 = 1.$$

This system has three independent *invariants* (integrals), namely I_1, I_2, $\theta_1 + k\theta_2$, which is enough since three independent invariants in a four dimensional system specify an orbit.

A fundamental set of polynomial invariants associated to the $k : -1$ resonance are

$$\begin{aligned}
a_1 &= I_1 \ = \ x_1^2 + y_1^2,\\
a_2 &= I_2 \ = \ x_2^2 + y_2^2,\\
a_3 &= I_1^{1/2}\, I_2^{k/2} \cos(\theta_1 + k\,\theta_2) = \Re\mathfrak{e}[(x_1 + \mathrm{i}\, y_1)(x_2 + \mathrm{i}\, y_2)^k],\\
a_4 &= I_1^{1/2}\, I_2^{k/2} \sin(\theta_1 + k\,\theta_2) = \Im\mathfrak{m}[(x_1 + \mathrm{i}\, y_1)(x_2 + \mathrm{i}\, y_2)^k],
\end{aligned}$$

subject to the *constraint*

$$a_3^2 + a_4^2 = a_1\, a_2^k, \quad a_1 \geq 0, \quad a_2 \geq 0,$$

which follows from the trig identity $\cos^2\phi + \sin^2\phi = 1$.

The Poisson brackets associated to the invariants are given in Table 1. Note that all Poisson brackets are polynomial, as k is a positive integer.

Table 1 Poisson brackets among the invariants $a_1,\ a_2,\ a_3,\ a_4$

{ , }	a_1	a_2	a_3	a_4
a_1	0	0	$-2\,a_4$	$2\,a_3$
a_2	0	0	$-2k\,a_4$	$2k\,a_3$
a_3	$2\,a_4$	$2k\,a_4$	0	$a_2^{k-1}(k^2\,a_1+a_2)$
a_4	$-2\,a_3$	$-2k\,a_3$	$-\,a_2^{k-1}(k^2\,a_1+a_2)$	0

The a_i's of the first column must be put in the left-hand side of the bracket, whereas the ones of the top row are placed on the right-hand side of the brackets

2.2 Orbit Space

Since $\mathbb{H}_k$ is an integral the set $\mathbb{N}=\{z\in\mathbb{R}^4:\mathbb{H}_k(z)=h\}$ is a smooth invariant submanifold of dimension 3 except possibly at $z=0$. The *orbit space* $\mathbb{O}$ is the quotient space obtained from $\mathbb{N}$ by identifying orbits to a point and let $\Pi:\mathbb{N}\to\mathbb{O}$ be the projection. Thus if $p\in\mathbb{O}$ then $\Pi^{-1}(p)\in\mathbb{N}$ is a circle (a periodic solution of the system defined by $\mathbb{H}_k$) or maybe just the origin. In general quotient spaces are not even Hausdorff, but since all solutions are periodic $\mathbb{N}$ is foliated by circles and the orbit space is a symplectic manifold or at least a symplectic *orbifold*. See Satake [24] where the concept of orbifold was introduced with the name *V-manifold*.

An orbit of the system is uniquely specified by the four invariants subject to the constraint and so the orbit space $\mathbb{O}$ is determined by the constraint and the integral

$$\mathbb{H}_k=ka_1-a_2=h.$$

Solve the integral for a_2 and substitute into the constraint to get the *orbit space equation*

$$a_3^2+a_4^2=a_1(k\,a_1-h)^k,$$

which defines a surface in the (a_1,a_3,a_4)-space and is a representation of the orbit space $\mathbb{O}$. We distinguish different situations according to the value of h. Note that the surface is a surface of revolution, so let $\rho,\ \psi$ be polar coordinates in the $(a_3,\ a_4)$-plane so that the equation becomes

$$\rho^2=a_1(k\,a_1-h)^k.$$

The surface of revolution is unbounded and it is smooth when the right-hand side is positive. A place where the orbit space is smooth we call a *plateau point*. As always $a_1\geq 0$ but $a_2\geq 0$ implies $a_1\geq h/k$. Refer to the subsequent sections for illustrative figures.

When $h<0$, the right-hand side of the orbit space equation is zero only at $a_1=0$ and nearby $\rho\sim c\,a_1^{1/2}$ with c a positive constant. Thus the surface $\mathbb{O}$ is smooth at $a_1=0$.

When $h=0$ the right hand side is zero at $a_1=0$ and nearby $\rho\sim c\,a_1^{(k+1)/2}$ with $c>0$ a constant. Thus the surface is cone-like when $k=1$ and is cusp-like when $k>1$.

When $h > 0$ the right hand side is zero at $a_1 = h/k$ and nearby $\rho \sim c\,(k\,a_1 - h)^{k/2}$ where $c > 0$ is a constant. Thus the surface is smooth at $a_1 = h/k$ when $k = 1$, is cone-like when $k = 2$ and is cusp-like when $k > 2$. These points with $k > 1$ we call *peaks*.

2.3 Preparation

Start with a real analytic Hamiltonian, $\mathcal{H}_k$, with an equilibrium point in $k : -1$ resonance, $k > 1$. The case $k = 1$ (with non diagonalizable A) is a little different, so it is treated in the last section. Assume that the equilibrium point is at the origin, time is scaled so that the frequencies are k and -1 (i.e. $\omega = 1$), symplectic coordinates have been chosen so that the quadratic terms in the Hamiltonian are in the form $\mathbb{H}_k$. Also assume that the Hamiltonian is in normal form up to degree $k+1$ and let the series truncated beyond the $k+1$ term be H_k. Now the Hamiltonian H_k can be written in terms of the invariants a_1, a_2, a_3, a_4.

The invariants a_1 and a_2 are the action variables I_1 and I_2 and are of degree 2 in z. The invariants a_3 and a_4 depend on the angle $\theta_1 + k\theta_2$ and are of degree $k+1$ in z. As we shall see the terms that contain angles are of prime importance in determining the existence and nature of some of the periodic solutions. We call these terms *angle terms*.

We consider the generic case where the angle term appears at the lowest degree, that is, at degree $k+1$. A linear combination of a_3 and a_4 can be combined into one by a shift of θ_1 i.e.

$$\begin{aligned}\alpha a_3 + \beta a_4 &= I_1^{1/2} I_2^{k/2}[\alpha\cos(\theta_1 + k\theta_2) + \beta\sin(\theta_1 + k\theta_2)]\\ &= G I_1^{1/2} I_2^{k/2}\cos(\theta_1 + k\theta_2 - \tilde{\theta}),\end{aligned}$$

where $G = \sqrt{\alpha^2 + \beta^2}$ and $\tan\tilde{\theta} = \beta/\alpha$. Shift θ_1 by $\theta_1 \to \theta_1 + \tilde{\theta}$.

Scale by $z \to \varepsilon z$ which is symplectic with multiplier ε^{-2}. This scaling indicates we are working near the equilibrium when ε is small. Thus we will look at systems of the form

$$\mathcal{H}_k = H_k + O(\varepsilon^k),$$

where

$$H_k = \mathbb{H}_k + \sum_{j=2}^{l} \varepsilon^{2j-2}\tilde{H}_k^j(I_1, I_2) + \varepsilon^{k-1} G I_1^{1/2} I_2^{k/2}\cos(\theta_1 + k\theta_2),$$

with $2l \le k+1$ and $\tilde{H}_k^j$ is a polynomial in I_1, I_2 of degree j. Here we have separated out the single angle term $a_3 = I_1^{1/2} I_2^{k/2}\cos(\theta_1 + k\theta_2)$. We refer to the system defined by $\mathcal{H}_k$ as the *full system* and the system defined by H_k as the *averaged system*.

2.4 Reduction

Since H_k is in normal form $\mathbb{H}_k$ is an integral, so hold it fixed by setting

$$h = \mathbb{H}_k = kI_1 - I_2 = ka_1 - a_2.$$

Solve for $a_2 = ka_1 - h,\ \ I_2 = kI_1 - h$ and so

$$H_k = h + \sum_{j=2}^{l} \varepsilon^{2j-2} \tilde{H}_k^j(a_1, h) + \varepsilon^{k-1} G a_3.$$

Pass to the averaged system on the orbit space by dropping the constant term and when $k > 2$ dividing by ε^2 (time scaling) so the *reduced averaged system* is

$$\bar{H} = \sum_{j=2}^{l} \varepsilon^{2j-4} \bar{H}_k^j(a_1, h) + \varepsilon^{k-3} G a_3.$$

When $k = 2$ the averaged system on the orbit space is obtained by dropping the constant term and dividing by ε (time scaling) so the *reduced averaged system* is

$$\bar{H} = G a_3.$$

Using the table of Poisson brackets given above we can obtain the reduced averaged flow on the orbit space by using $\dot{a}_i = \{a_i, \bar{H}\}, i = 1, 3, 4$. A critical point $p \in \mathbb{O}$ of this flow corresponds to a periodic solution $P = \Pi^{-1}(p) \in \mathbb{N}$ of the averaged system or to the origin. Likewise an orbit of the reduced averaged system which tends to the critical point p corresponds to a surface in $\mathbb{N}$ filled with orbits tending to the periodic solution P.

2.5 Perturbation Theory

We will encounter periodic solutions coming from peaks and plateau points on $\mathbb{O}$ and they give rise to short and long periodic families.

For peaks: *Let $k > 1$, $\Pi(p) = \bar{p} \in \mathbb{O}$ be a peak. Then the solution of the full system through p for $\varepsilon = 0$ is periodic with period $2\pi/k$ and characteristic multipliers are $1, 1, e^{2\pi \mathrm{i}/k}, e^{-2\pi \mathrm{i}/k}$. For $\varepsilon > 0$ and small, the full system has a periodic solution near p of period and multipliers near the above.*

For plateau points: *Let $\bar{H}$ have a nondegenerate equilibrium point at $\Pi(p) = \bar{p} \in \mathbb{O}$ with characteristic exponents $\mu, -\mu$. Then for ε small the full system has a periodic solution near p with period near 2π and multipliers near $1, 1, 1 + \varepsilon^2\mu, 1 - \varepsilon^2\mu$.*

Thus, for $k > 1$ there is always a short period family and so the quest is to find long period families.

Notes Bifurcation theory based on regular reduction can be found in our paper [30], whereas the theory based on singular reduction can be found in [15] and the references therein.

3 Examples

3.1 Resonance $2 : -1$

After preparation the averaged system in $2 : -1$ resonance is

$$H = 2a_1 - a_2 + \varepsilon G a_3 + \cdots = 2I_1 - I_2 + \varepsilon G I_1^{1/2} I_2 \cos(\theta_1 + 2\theta_2) + \cdots .$$

Note that in this discussion the subscript 2 of H is omitted. The reduced averaged Hamiltonian is $\bar{H} = Ga_3$ and since we assume $G \neq 0$ by scaling time again $G = 1$, so $\bar{H} = a_3$.

The geometry of the problem is obtained by considering the intersection of two surfaces in $\mathbb{R}^3$. The first surface, the orbit space $\mathbb{O}$, is given by the orbit space equation

$$a_3^2 + a_4^2 = a_1(2a_1 - h)^2, \quad a_1 \geq 0, \;\; 2a_1 \geq h,$$

for various values of h and the second surface is the reduced averaged Hamiltonian

$$\bar{H} = a_3 = \bar{h}$$

for various values of $\bar{h}$. Near the origin in a-space the orbit space is a paraboloid of revolution when $h < 0$ and hence smooth, it is a rotated cusp when $h = 0$, and it is a cone when $h > 0$—see Fig. 1.

Of course $\bar{H} = a_3 =$ constant is just a plane parallel to the a_1, a_4 coordinate plane. The flow on the orbit spaces is defined by the equations of motion

$$\dot{a}_1 = \{a_1, \bar{H}\} = -2a_4, \quad \dot{a}_3 = \{a_3, \bar{H}\} = 0, \quad \dot{a}_4 = \{a_4, \bar{H}\} = -a_2(4a_1 + a_2).$$

Recall that on the orbit space $a_2 = 2a_1 - h$. For an equilibrium point one must have $a_4 = 0$ and $a_2(4a_1 + a_2) = 0$ and since not both a_1 and a_2 can be zero or negative the conditions for an equilibrium are $a_4 = a_2 = 0$. But on the orbit space $a_2 = 0$ only when $a_1 = h/2$ and that only occurs when $h \geq 0$.

Look at the flow lines in Fig. 1. The flow lines lie in $\bar{H} = a_3 =$ constant and a_4 is decreasing.

When $h = 0$ the origin in a-space corresponds to the origin in $\mathbb{R}^4$. There is an orbit on $\mathbb{O}$ tending to the origin as $t \to +\infty$ and there is an orbit on $\mathbb{O}$ tending to the origin as $t \to -\infty$. These represent a surface of solutions that spiral to the origin as $t \to \pm\infty$. Thus the equilibrium point is unstable.

When $h > 0$ there is an equilibrium (a peak) on $\mathbb{O}$ at $a_1 = h/2, \; a_3 = a_4 = 0$. This gives rise to a periodic solution of period $T \sim \pi$ for each $h \geq 0$. These solutions are the short periodic family given by Liapunov center theorem. Note that here too there

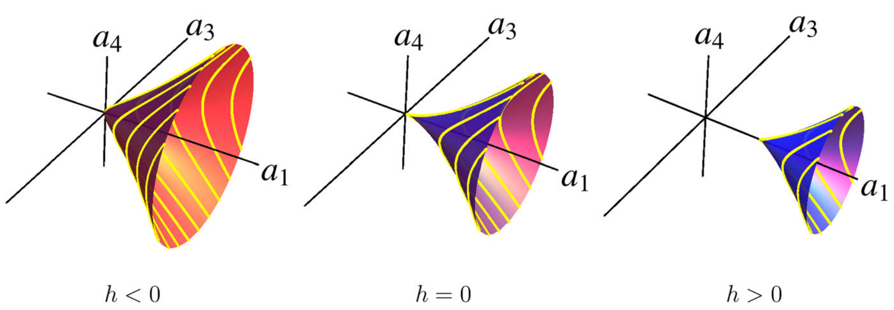

Fig. 1 $\mathbb{O}$ when $k = 2$

is an orbit on $\mathbb{O}$ tending to the equilibrium as $t \to \pm\infty$. Thus the solutions in the short period family are unstable.

When $h < 0$ there are no equilibria and so all solutions recede far away as $t \to \pm\infty$. Thus, there is no long period family.

Only a little more care is needed to show that a_4 is a Chetaev function for the full system. Thus, *in the case of* $2 : -1$ *resonance there is only one family of periodic solutions, the short period family of Liapunov, and it is unstable.*

3.2 Resonance $3 : -1$

Now consider the $3 : -1$ system

$$H = 3I_1 - I_2 + \frac{\varepsilon^2}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \varepsilon^2 G I_1^{1/2} I_2^{3/2} \cos(\theta_1 + 3\theta_2),$$

where A, B, C, G are constants. Introduce the constants $D = \frac{1}{2}(A + 6B + 9C)$ and $R = B + 3C$.

In KAM theory D is called the twist coefficient. Note that in the present $3 : -1$ example the twist coefficient D and the angle coefficient G are both defined at the same order of ε, i.e., at order ε^2. In the previous $2 : -1$ example the angle coefficient is of lower order than the twist coefficient, whereas in the next example $k : -1$, $k \geq 4$ the order is reversed. So we are now looking at the case when the twist and the angle are competing.

Passing to the averaged system we get

$$\bar{H} = Da_1^2 - Rha_1 + Ga_3$$

which is defined on the orbit space, $\mathbb{O}$,

$$a_3^2 + a_4^2 = a_1(3a_1 - h)^3, \quad a_1 \geq 0, \quad 3a_1 \geq h.$$

Again the geometry of the problem is depicted by the intersection in $\mathbb{R}^3$ of the surface of the orbit space for various values of h and the surfaces of the averaged Hamiltonian $\bar{H} = \bar{h}$ for various values of $\bar{h}$. Near the origin in a-space the orbit space is smooth when $h < 0$, it is a rotated parabola when $h = 0$ and a rotated cusp when $h > 0$. The surface $\bar{H} = \bar{h} =$ constant is just a translation of a parabola—translated in the a_4 direction.

The associated vector field of $\bar{H}$, i.e. the reduced system

$$\begin{aligned}
\dot{a}_1 &= \{a_1, \bar{H}\} = -2Ga_4,\\
\dot{a}_3 &= \{a_3, \bar{H}\} = 2a_4(2Da_1 - Rh),\\
\dot{a}_4 &= \{a_4, \bar{H}\} = -2a_3(2Da_1 - Rh) - Ga_2^2(9a_1 + a_2)\\
&\qquad\qquad\quad = -2a_3(2Da_1 - Rh) - G(3a_1 - h)^2(12a_1 - h),
\end{aligned}$$

gives the precise flow on the orbit space. First find a critical point of these equations on the orbit space. Clearly $a_4 = 0$ (we consider $G \neq 0$) and so we must find a solution of the equations

$$\begin{aligned} a_3^2 &= a_1(3a_1 - h)^3, \\ 2a_3(2Da_1 - Rh) &+ G(3a_1 - h)^2(12a_1 - h) = 0. \end{aligned}$$

Solve the second equation for a_3, square it, then substitute in a_3^2 from the first equation, cancel some terms and expand to get

$$16(D^2 - 27G^2)a_1^3 + 8h(27G^2 - 2DR)a_1^2 + h^2(4R^2 - 27G^2)a_1 + G^2h^3 = 0.$$

This is a cubic polynomial in a_1 with parameters D, G, R and h, but we can reduce by one the number of parameters by defining $\alpha = D/G$, $\beta = R/G$ and dividing the cubic polynomial by G^2. We seek roots $a_1 \geq \max\{0, h/3\}$. The number of roots changes when the resultant of the polynomial with its derivative with respect to a_1 vanishes, i.e. when

$$1024(\alpha^2 - 27)(\alpha - 6\beta)^2(729 + 108\alpha^2 + 648\beta^2 - 48\beta^4 + 8\alpha\beta(-81 + 4\beta^2))h^6 = 0.$$

Also the number of critical points of the equations on the orbit space changes when one root of the cubic polynomial meets the peak at $a_1 = h/3$ when $h \geq 0$, i.e. when after replacing a_1 by $h/3$ in the cubic we get

$$\frac{4}{27}(2\alpha - 3\beta)^2h^3 = 0.$$

We will distinguish the cases $h < 0$, $h = 0$ and $h > 0$ and analyze the equilibria and bifurcations as functions of α and β.

The case $h = 0$ is simple because the above vanishes and the resultant becomes

$$16a_1^3(\alpha^2 - 27) = 0.$$

The only equilibrium is the origin in a-space, which corresponds to the origin in $\mathbb{R}^4$. It is a peak that changes its stability when crossing the line $\alpha^2 - 27 = 0$. See Fig. 2 for the evolution of the flow.

When $h \neq 0$ we have the two bifurcation diagrams appearing in Figs. 3 and 4. The equations for the bifurcation lines become

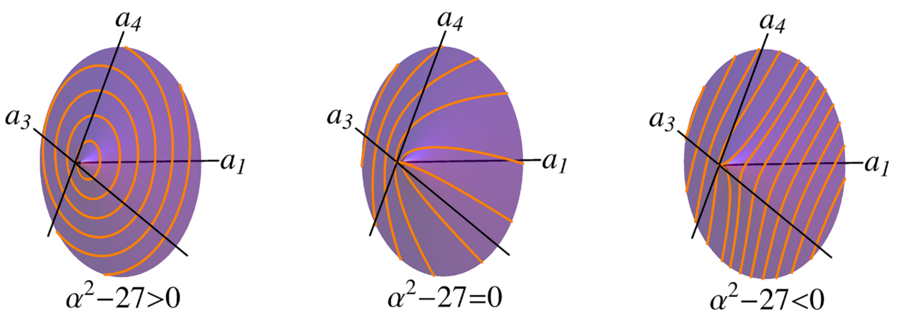

Fig. 2 Evolution of the flow for the $3 : -1$ resonance when $h = 0$

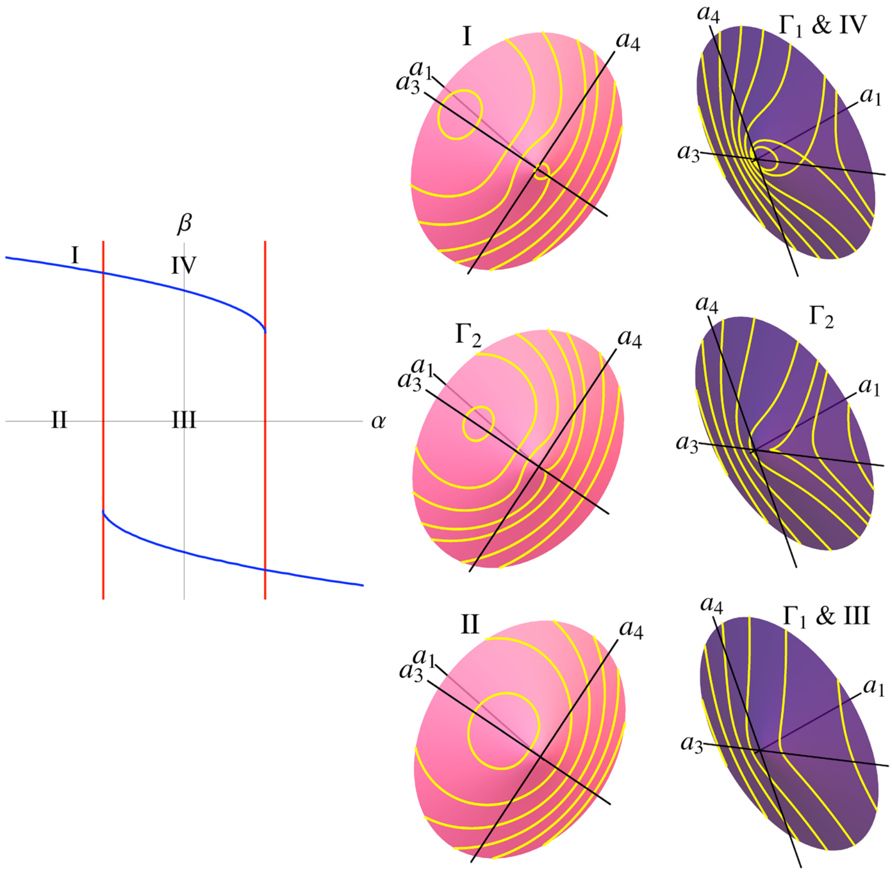

Fig. 3 Bifurcation diagram and flows for the $3 : -1$ resonance when $h < 0$

Γ_1: $\alpha^2 - 27 = 0$ (red lines);
Γ_2: $729 + 108\alpha^2 + 648\beta^2 - 48\beta^4 + 8\alpha\beta(-81 + 4\beta^2) = 0$ (blue curves);
Γ_3: $2\alpha - 3\beta = 0$ (green curve).

The diagrams are symmetric with respect to the origin. The blue curves correspond to a saddle-center or extremal bifurcation of critical points thus, an extremal bifurcation of periodic solutions, see [15]. On the red lines, the leading term of the cubic vanishes, so only two zeros are possible. The green curve is a bifurcation of the peak.

Case $h < 0$, *Fig.* 3 The surface $\mathbb{O}$ is smooth, so all the critical points are in the plateau and correspond to periodic orbits with $T \sim 2\pi$. In Region I the cubic polynomial has three different positive roots, so there are three critical points on the surface, namely two centers and one saddle. On the blue curve one of the centers and the saddle collide giving rise to an extremal critical point that disappears in Region II, where only the other center survives. In Region IV, only one center and the saddle are present. On the blue curve, this center and the saddle collide in another extremal critical point thus there are no critical points in Region III.

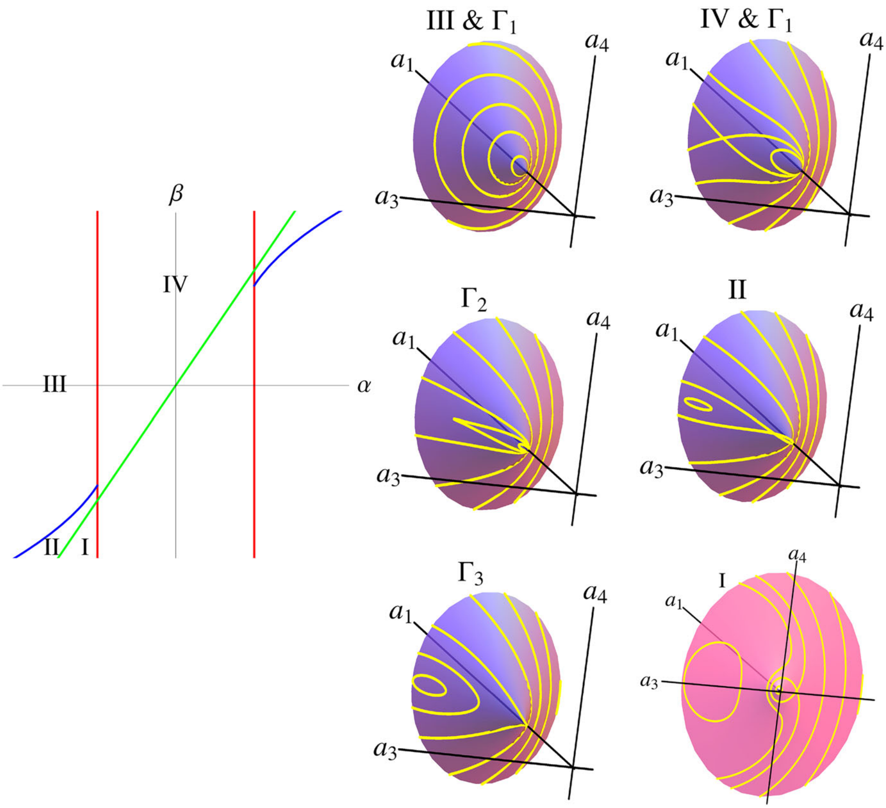

Fig. 4 Bifurcation diagram and flows for the $3 : -1$ resonance when $h > 0$

Case $h > 0$, *Fig.* 4 Here the peak $a_1 = h/3$ is always an equilibrium. It corresponds to a periodic orbit in the full system with period close to $2\pi/3$ (the short period family).

We are interested in roots of the cubic which are bigger than $h/3$. In Region I the cubic has two roots that are different from $h/3$. So, there are three equilibria in total: the peak and two of plateau-type. The peak is a center, there is one saddle close to the peak and another center is relatively far from the other two points. On the green curve the saddle collides with the peak and the situation of Region I is recovered in Region II. On the blue curve the saddle and the center (that is not the peak) collide and then they disappear in Region III, where only the peak stays as a center up to the red line. After crossing the red line, in Region IV, a saddle appears and the peak continues to be a center.

Notes Markeev [12] and Alfriend [1,2] established that the Lagrange point $\mathcal{L}_4$ of the restricted three body problem is unstable in the $2 : -1$ and $3 : -1$ cases. Their proofs cover the general instability cases. A proof based on Chetaev theorem can be found in [13]. Roels [23] established that there are 1 or 3 long period families in the $3 : -1$ case depending on the number of zeros of a certain cubic polynomial. Schmidt [25] and Henrard [9,10] describe the unfolding of these periodic solutions.

3.3 Resonances $k : -1,\ k \geq 4$

Now consider the $k : -1$ system when $k \geq 4$, i.e.,

$$H = kI_1 - I_2 + \frac{\varepsilon^2}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \cdots,$$

where A, B, C are constants. Again introduce the constants $D = \frac{1}{2}(A + 2kB + k^2C)$ and $R = B + kC$. At first we ignore the higher order terms, but they will appear later.

As before D is the twist coefficient. Note that the angle term is not yet included since it is of higher order. Passing to the averaged system we get

$$\bar{H} = Da_1^2 - hRa_1,$$

which is defined on the orbit space, $\mathbb{O}$,

$$a_3^2 + a_4^2 = a_1(ka_1 - h)^k, \quad a_1 \geq 0,\ ka_1 \geq h.$$

Again the geometry of the problem is depicted by the intersection in $\mathbb{R}^3$ of the surface of the orbit space for various values of h and the surfaces of the averaged Hamiltonian $\bar{H} = \bar{h}$ for various values of $\bar{h}$. Near the origin in a-space the orbit space is smooth when $h < 0$ and it is a rotated cusp when $h \geq 0$.

The associated vector field of $\bar{H}$, the reduced system, on the orbit space is

$$\begin{aligned}
\dot{a}_1 &= \{a_1, \bar{H}\} = 0,\\
\dot{a}_3 &= \{a_3, \bar{H}\} = 2a_4(2Da_1 - Rh),\\
\dot{a}_4 &= \{a_4, \bar{H}\} = -2a_3(2Da_1 - Rh),
\end{aligned}$$

which gives the precise flow on the orbit space. Note that a_1 is an integral, so the flow curves lie in $a_1 =$ constant planes.

When $h \geq 0$ the orbit space is a trumpet with a peak at $a_1 = h/k$ and this corresponds to an elliptic orbit with period $T \sim 2\pi/k$ and multipliers approximately $1, 1, e^{2\pi \mathrm{i}/k}, e^{-2\pi \mathrm{i}/k}$. This is of course the short period family of Liapunov.

When $h < 0$ the orbit space consists of only plateau points. The orbit space, $\mathbb{O}$, and the planes $a_1 =$ constant defined by the averaged equation are tangent at the origin in (a_1, a_3, a_4)-space which corresponds to $a_2 = -h$. (For $D = 0$ and $R \neq 0$ it is the only tangency point). This is a critical point of the averaged equations on the orbit space and thus corresponds to a periodic solution with period near 2π. The intersection of the planes and the orbit space are circles on the orbit space suggesting the orbit is elliptic. At that point a_3, a_4 are (non-symplectic) coordinates and the matrix of the linearized equations is

$$\begin{bmatrix} 0 & -2hR \\ 2hR & 0 \end{bmatrix}$$

with eigenvalues $\pm 2hR\mathrm{i}$.

Thus, *if $h < 0$ and $R = B + kC \neq 0$ there is an elliptic family of periodic solutions (Roels' long period family) of the full system of period near 2π with characteristic multipliers* $1, 1, 1 + \varepsilon^2 4\pi h R\mathrm{i} + \cdots, 1 - \varepsilon^2 4\pi h R\mathrm{i} + \cdots$.

There are other critical points on the orbit space when $D \neq 0$. In particular there are circles with coordinates (a_1, a_3, a_4) that are parameterized by h with

$$\mathcal{C}: \quad a_1 = \frac{hR}{2D}, \quad a_3^2 + a_4^2 = R\left(\frac{h}{2D}\right)^{k+1}(-A - kB)^k.$$

These circles make sense only when the right hand sides are positive.

These circles represent tori filled with periodic solutions. The persistence of these tori filled with periodic solutions for the full problem cannot be deduced straightforwardly since they are not isolated nondegenerate critical points. Using fixed point methods one can show that at least two periodic solutions persist but the precise number and nature will depend on higher order terms in the normal form especially the angle terms.

Natural Centers in the General Case To complete our understanding of the higher order resonance we will consider the case when $k = 4$ in detail and refer to the literature for the similar discussion of the case when $k > 4$. To find the periodic solutions on these tori we must include the angle term, so consider

$$H = 4I_1 - I_2 + \frac{\varepsilon^2}{2}(AI_1^2 + 2BI_1I_2 + CI_2^2) + \varepsilon^3 G a_3 + \cdots,$$

with $G \neq 0$. The averaged system is given by

$$\bar{H} = Da_1^2 - hRa_1 + \varepsilon G a_3,$$

and the associated vector field of $\bar{H}$

$$\begin{aligned} \dot{a}_1 &= \{a_1, \bar{H}\} = -2\varepsilon G a_4, \\ \dot{a}_3 &= \{a_3, \bar{H}\} = 2a_4(2Da_1 - Rh), \\ \dot{a}_4 &= \{a_4, \bar{H}\} = -2a_3(2Da_1 - Rh) - \varepsilon G a_2^3(16a_1 + a_2) \\ &= -2a_3(2Da_1 - Rh) - \varepsilon G(4a_1 - h)^3(20a_1 - h). \end{aligned}$$

Let us find the critical points of these equations on the orbit space. Since $G \neq 0$ we must have $a_4 = 0$ and so the orbit space equation is $a_3^2 = a_1(4a_1 - h)^4$. Therefore, we must solve

$$\begin{aligned} &a_3^2 - a_1(4a_1 - h)^4 = 0, \\ &-2a_3(2Da_1 - Rh) - \varepsilon G(4a_1 - h)^3(20a_1 - h) = 0. \end{aligned}$$

To eliminate a_3 we can solve the second equation for a_3, square it, then substitute into the first equation and simplify; or we can just eliminate a_3 by computing the resultant of the two polynomials given by the left-hand side of the equations. The resultant is

$$\mathrm{Res}(a_1) = (4a_1 - h)^4 \left(4a_1(2Da_1 - hR)^2 - \varepsilon^2 G^2 (4a_1 - h)^2 (20a_1 - h)^2\right).$$

Thus, the values of a_1 corresponding to the possible critical points have to satisfy $\mathrm{Res}(a_1) = 0$ and $a_1 \geq \max\{0, h/4\}$.

First we find the solution $a_1 = h/4$ (for $h \geq 0$) which gives the short period family discussed above. Now we look for roots of the long factor of the resultant. First set $\varepsilon = 0$ in this factor to get $a_1 = 0$, and $a_1 = hR/(2D)$. The root $a_1 = 0$ gives Roels' long period family already discussed, so we look at the case $a_1 = hR/(2D)$ in more detail.

To this end set $a_1 = hR/(2D) + \varepsilon\eta + \cdots$ and insert it in the resultant. We obtain two specific values for η namely

$$\eta_{\mp} = \mp hGZ \frac{(D - 10R)}{4D^2} \sqrt{\frac{h}{2DR}},$$

where $Z = A + 4B$. Henceforth, assume $hDR > 0$ to assure that $\eta_{\pm}$ are real. When $Z \neq 0$ (i.e., $D \neq 2R$), $D \neq 10R$ and $G \neq 0$ there are two different values of η while when $Z = 0$ or $D = 10R$ there is only one value of η and this would correspond to bifurcation lines as we will see. Finally when $hDR < 0$ there are no real values for η.

For a_3 we get two values after replacing in the second equation of the approximation obtained for a_1. Thus, after collecting the intermediate results we end up with two approximate critical points

$$\begin{aligned}
a_1 &= \frac{hR}{2D} \mp \varepsilon hGZ \frac{(D - 10R)}{4D^2} \sqrt{\frac{h}{2DR}}, \\
a_3 &= \pm \frac{h^2 Z^2}{4D^2} \sqrt{\frac{hR}{2D}} + \varepsilon h^3 G Z^2 \frac{(D - 10R)}{2D^4}, \\
a_4 &= 0.
\end{aligned}$$

The upper signs in the expressions of a_1 and a_3 correspond to one critical point, say P_3, whereas the lower ones correspond to the other critical point, say P_4.

Let us describe the bifurcations as functions of the parameters h, D, R, G and ε. In fact, as in the $3 : -1$ analysis we can reduce the number of parameters by defining $\alpha = D/G$ and $\beta = R/G$. The bifurcations take place when the last factor of $\mathrm{Res}(a_1)$ has a multiple valid root or when one of its roots is 0 or $h/4$. The expression of the last factor of $\mathrm{Res}(a_1)$ as a function of α and β after dividing by G^2 is:

$$4a_1(2\alpha a_1 - h\beta)^2 - \varepsilon^2 (4a_1 - h)^2 (20a_1 - h)^2.$$

The case $h = 0$ is simple because the polynomial gets:

$$16a_1^3(\alpha^2 - 400 a_1 \varepsilon^2).$$

When $h \neq 0$ the polynomial has a multiple root when the discriminant of the polynomial is 0, i.e. when:

$$(\alpha - 10\beta)^2(\alpha - 2\beta)^2[-8\alpha^3\beta^3 + 3(9\alpha^4 - 72\alpha^3\beta - 8\alpha^2\beta^2 - 1440\alpha\beta^3 + 3600\beta^4)h_\varepsilon + 1536(9\alpha^2 - 100\alpha\beta + 180\beta^2){h_\varepsilon}^2 - 327{,}680{h_\varepsilon}^3] = 0,$$

where we have defined $h_\varepsilon = h\varepsilon^2$. When $h > 0$ a root of $\mathrm{Res}(a_1)$ is $h/4$ when $\alpha = 2\beta$. When $h < 0$ the roots of $\mathrm{Res}(a_1)$ are never 0. Thus, the bifurcation lines are defined by the previous discriminant to be 0. In Figs. 5 and 6 we represent the bifurcation planes in α, β for $h > 0$ and $h < 0$, respectively. Both planes are symmetric with respect to the origin. Let us describe the evolution of the flow and the bifurcations in both cases.

Case $h > 0$, Fig. 5 When starting in Region I there are only two equilibria: the peak, $a_1 = h/4$, which is linearly and nonlinearly stable, and a plateau that is a saddle. The

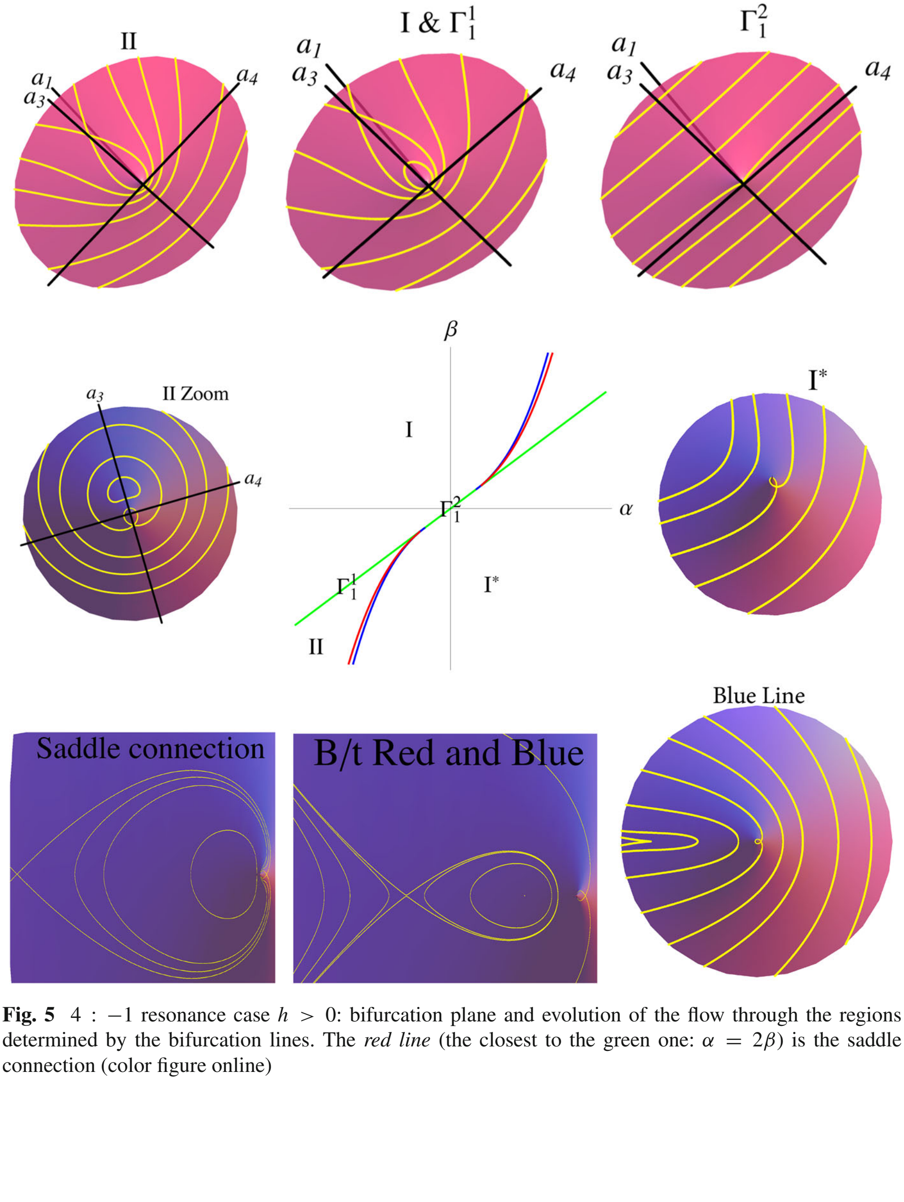

Fig. 5 $4 : -1$ resonance case $h > 0$: bifurcation plane and evolution of the flow through the regions determined by the bifurcation lines. The *red line* (the closest to the green one: $\alpha = 2\beta$) is the saddle connection (color figure online)

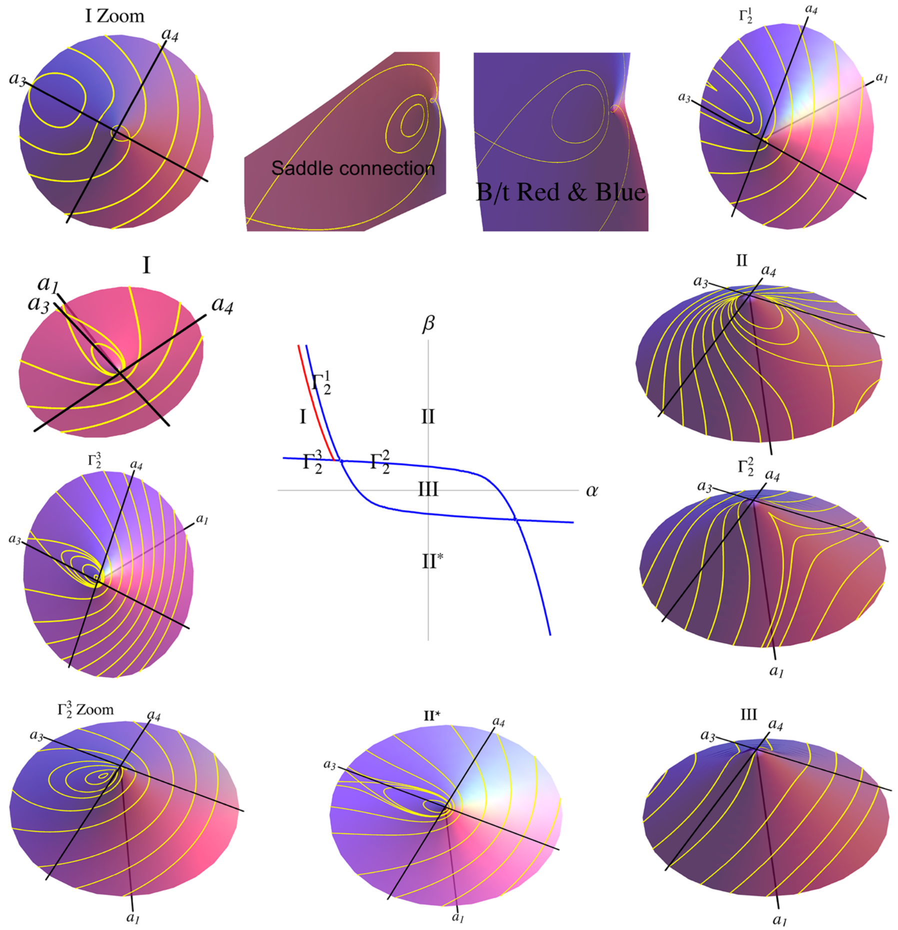

Fig. 6 $4 : -1$ resonance case $h < 0$: bifurcation plane and evolution of the flow through the regions determined by the bifurcation lines. The *red line* (the one to the left of Γ_2^1) is the saddle connection (color figure online)

green line corresponds to the bifurcation line $\alpha - 2\beta = 0$. On the first part of this line, Γ_1^1, we still have the two equilibria, but the peak is linearly degenerate and nonlinearly stable. It bifurcates after crossing this line onto Region II. Once in Region II, the saddle already present in Region I continues and the peak, which is again linearly and nonlinearly stable has bifurcated to give a new saddle and a new center (see “II Zoom” in Fig. 5). So, we have four equilibria. Still in this region, when approaching the red line the energies of the two saddles get closer and on the red line a saddle connection takes place. In this way, after crossing the red line towards the blue one, the center that was initially attached to the saddle that is close to the peak, gets attached to the saddle which comes from Region I and is relatively far from the peak. On the blue line (when the last factor in the discriminant is zero), a saddle-center bifurcation takes place. It involves the center that is not the peak and the saddle coming from Region I. Once in

Region I* we recover the situation of Region I: we only have two equilibria: the peak, which is a center, and a saddle. On the middle part of the green bifurcation line, Γ_1^2 we only have one equilibrium, the peak, that is linearly degenerate and nonlinearly unstable.

Case $h < 0$, *Fig.* 6 The only local bifurcations are of saddle-center type and come from the last factor in the discriminant to be zero. The surface is smooth and the origin in a-space, $(a_1, a_2, a_3) = (0, -h, 0)$, is never an equilibrium. Starting in Region I we have two saddles and two centers. One saddle is relatively far from the origin, whereas the other saddle and the two centers are very close to it (see "I Zoom" in Fig. 6). On the red line the two saddles have the same energy, so a saddle connection takes place (see "Saddle connection" in Fig. 6). Previous to the saddle connection, the center that is not very close to the origin is attached to the saddle near the origin and after the saddle connection, this center gets related to the saddle far from the origin (see "B/t Red and Blue" in Fig. 6). This center and this saddle get closer and closer until a saddle-center bifurcation occurs on the branch Γ_2^1 of the blue line. Once in Region II only one center very close to the origin and one saddle are present. On the second branch of the blue line, Γ_2^2, the center and the saddle collide in an extremal critical point that disappears once in Region III. There are no equilibria in this region. If we cross from Region I to Region II* through Γ_2^3 it is the center and the saddle that are very close to the origin that collide in an extremal critical point (see "Γ_2^3 Zoom") that disappears once in Region II*. In this region only the other saddle and the other center remain and we recover the situation of Region II.

Notes Palmore found numerically that in certain situations in the restricted problem there were two more long period families which he called natural centers and one was stable and one was unstable. Meyer and Palmore [16] proved their existence by a topological argument and Schmidt [25] established their existence and stability analytically.

3.4 Resonance $1 : -1$

We place the $1 : -1$ at the end because the preliminary work is slightly different. We start with a Hamiltonian matrix A that has eigenvalues $\pm i$ with multiplicity two and A is not diagonalizable. The standard normal form for the Hamiltonian is

$$\mathbb{H} = \mathbb{H}_1 = x_2 y_1 - x_1 y_2 + \frac{\delta}{2}(x_1^2 + x_2^2)$$

where $\delta = \pm 1$. Again for this discussion we drop the subscript 1. The linear system of equations is $\dot{z} = Az$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & -1 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}.$$

The characteristic polynomial of A is $p(\lambda) = (\lambda^2 + 1)^2$ with repeated eigenvalues $\pm$i. But not all solutions are 2π periodic, since there are secular terms like $t \sin t,\ t \cos t$.

The four invariants usually associated with this Hamiltonian are just

$$b_1 = x_2 y_1 - x_1 y_2, \quad b_2 = \frac{1}{2}(x_1^2 + x_2^2), \quad b_3 = \frac{1}{2}(y_1^2 + y_2^2), \quad b_4 = x_1 y_1 + x_2 y_2,$$

with the constraint

$$b_1^2 + b_4^2 = 4\, b_2 b_3.$$

The nonzero Poisson brackets are

$$\{b_2, b_3\} = -\{b_3, b_2\} = b_4, \quad \{b_2, b_4\} = -\{b_4, b_2\} = 2b_2, \quad \{b_4, b_3\} = -\{b_3, b_4\} = 2b_3.$$

Consider the nonlinear Hamiltonian system $\mathcal{H}$ which has $\mathbb{H} = b_1 + \delta b_2$ as its quadratic part and has been normalized to Sokol'skii normal form through the fourth order terms in the rectangular coordinates, i.e. let

$$\mathcal{H} = b_1 + \delta b_2 + (\alpha b_1^2 + 2\beta b_1 b_3 + \gamma b_3^2) + \cdots,$$

where the ellipsis stands for terms that are at least fifth order.

Use the Meyer–Schmidt scaling

$$\begin{aligned} x_1 &\to \varepsilon^2 x_1, \quad x_2 \to \varepsilon^2 x_2, \\ y_1 &\to \varepsilon y_1, \quad y_2 \to \varepsilon y_2, \end{aligned}$$

which is symplectic with multiplier ε^{-3}, so the Hamiltonian becomes

$$\mathcal{H} = b_1 + \varepsilon(\delta b_2 + \gamma b_3^2) + O(\varepsilon^2).$$

Now with $\varepsilon = 0$ all solutions are periodic with least period 2π, so the orbit space is a manifold and we use regular reduction. Let

$$H = b_1 + \varepsilon(\delta b_2 + \gamma b_3^2)$$

play the role of the averaged system.

The orbit space, $\mathbb{O}$, is specified by the invariants subject to the constraint and

$$H_{\varepsilon=0} = b_1 = h.$$

Thus the equation of the orbit space is

$$h^2 + b_4^2 = 4\, b_2\, b_3,$$

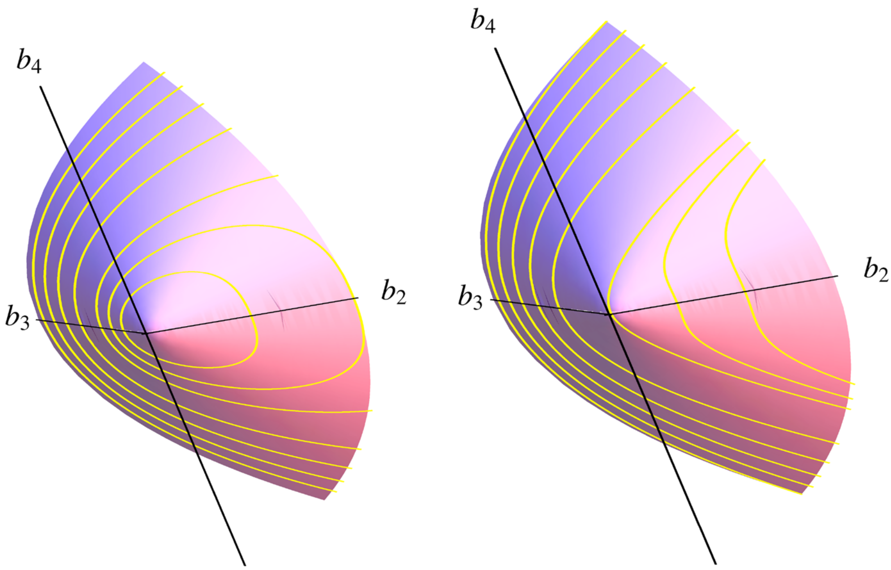

Fig. 7 Flows in the $1 : -1$ resonance. On the *left* $\delta\gamma > 0$. On the *right* $\delta\gamma < 0$

which is a two sheeted hyperboloid in (b_2, b_3, b_4)-space when $h \neq 0$, but we only look at the sheet where $b_2 > 0$, $b_3 > 0$. This one sheet represents both $h > 0$ and $h < 0$.

The reduced averaged Hamiltonian is

$$\bar{H} = \delta b_2 + \gamma b_3^2 = \bar{h}.$$

The Hamiltonian $\bar{H}$ has a critical point on the orbit space if $b_4 = 0$ and $\gamma b_3 = \delta b_2$ and so a critical point exists if $\gamma h^2 = \delta b_2^2$. Hence there is a critical point of $\bar{H}$ on $\mathbb{O}$ if and only if δ and γ have the same sign, see Fig. 7.

The equations of motion are

$$\dot{b}_2 = \{b_2, \bar{H}\} = 2\gamma b_3 b_4, \quad \dot{b}_3 = \{b_3, \bar{H}\} = 0, \quad \dot{b}_4 = \{b_4, \bar{H}\} = -2\delta b_2 + 4\gamma b_3^2.$$

Thus b_3 is a positive constant, say $b_3 = p$, and then the equations in b_2, b_4 stand for a harmonic oscillator when $\delta\gamma > 0$.

Thus, *in the case of* $1 : -1$ *resonance there are two families of nearly* 2π *elliptic periodic solutions emanating from the origin when* $\delta\gamma > 0$. *One family exists for* $\mathcal{H} > 0$ *and one for* $\mathcal{H} < 0$. *There are no nearby* 2π *periodic solutions when* $\delta\gamma < 0$.

The designation short and long period family is meaningless in this case, it is better to distinguish the two families by the sign of $\mathcal{H}$.

Notes In 1941 Buchanan [3] provided a proof by power series of the existence of families of periodic solutions in the restricted three body problem at the Lagrange point $\mathcal{L}_4$ with mass ratio parameter μ_1 provided the sign of a certain coefficient in the series expansion of the Hamiltonian is of the right sign. This is a $1 : -1$ resonance problem. Deprit and Henrard computed that term in [7] thus effectively computing γ in the restricted problem. These early papers did not use normal form methods.

The proof of the existence of such periodic solutions is a byproduct of the Hamilton-Hopf bifurcation analysis as found in [13, 17]. For a straightforward proof using normal form methods see Appendix C of [14].

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