NOTES ON DELAUNAY AND POINCARÉ ELEMENTS

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Abstract. This set of notes defines the Delaunay elements of celestial mechanics in two and three dimensions. It establishes the conditions for a doubly periodic orbit in these elements.

1. Introduction

The are many questions about polar type coordinates in celestial mechanics that have bothered me over the years. The treatment in many of the standard texts is rather poor. These notes contain a set of miscellaneous remarks and results about action-angle variables, Poincaré's Center Theorem, Delaunay and Poincaré variables in two and three dimensions. All the results contained here are well known to those who know them and nothing is new except may be the treatment.

2. The Action Variable for a Nonlinear Oscillator

In this section I discuss action-angle variables for a one degree of freedom Hamiltonian system near a center. Also I discuss the classical generating function, normalization at a center and give a symplectic form of Poincaré's Center Theorem.

2.1. **The action-angle variables.** Consider a Hamiltonian $H(x, y)$ defined in a neighborhood of the origin in $\mathbb{R}^2$, such that the origin is a center for the Hamiltonian flow. Thus the origin is encircled by periodic orbits. Assume that the origin is a local minimum of H, and $H(0, 0) = 0$. We seek symplectic action-angle variables (L, ℓ) where ℓ is an angle defined mod 2π and the Hamiltonian is to be of the form $H(L, \ell) = \Omega(L)$.

Let $R(h) =$ the component of $\{(x, y) \in \mathbb{R}^2 : H(x, y) \leq h\}$ which contains the origin. Since $dx \wedge dy = dL \wedge d\ell$ we must have

$$\int\int_{R(h)} dx \wedge dy = \int\int_{R(h)} dL \wedge d\ell = 2\pi L$$

or

$$L = \frac{1}{2\pi}\int\int_{R(h)} dx \wedge dy = \frac{1}{2\pi}\oint_{\partial R(h)} x dy.$$

Thus the variable L is just the area of the region $R(h)$. The last integral in the formula for L is classically called the 'action'. Since the equations of motion in the new coordinates are

$$\dot{L} = 0, \qquad \dot{\ell} = -\Omega'(L) \tag{1}$$

L is a constant and $\ell = \ell_0 - \Omega'(L)t$. Thus ℓ is a scaled time, scaled so that it is 2π-periodic.

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2.2. **The equation for the Hamiltonian.** Let the period of the orbit be $p(h)$. We wish to find symplectic variables (L, ℓ) defined in a neighborhood of the origin with ℓ an angle defined mod 2π and L is its conjugate momentum so that the Hamiltonian becomes a function of L alone. With $H = \Omega(L)$ the equations become (1). Thus the period of the orbit is $2\pi / \mid \Omega'(L) \mid$. Therefore, we must have

$$\frac{2\pi}{\Omega'(L)} = \pm p(\Omega(L))$$

or

$$\Omega' = \pm \frac{2\pi}{p(\Omega)}. \tag{2}$$

The differential equation above defines the function $\Omega(L)$ in terms of $p(h)$. The sign is chosen so that $\pm 2\pi/\Omega'(L)$ is positive (the period).

2.3. **Example–the Kepler Problem.** In the Kepler problem $H = -1/2s$, $P = 2\pi s^{3/2}$ where s is the semi-major axis. So $p(h) = \pi 2^{-1/2}(-h)^{-3/2}$. Thus the equation to solve is

$$\Omega' = 2^{3/2}(-\Omega)^{3/2}.$$

Separating variables gives

$$\begin{aligned}(-\Omega)^{-3/2} d\Omega &= 2^{3/2} dL\\ (-\Omega)^{-1/2} &= 2^{1/2} L.\end{aligned}$$

Thus the Hamiltonian must be

$$\Omega = -1/2L^2. \tag{3}$$

2.4. **The generating function.** Now consider the Hamiltonian system

$$H = \frac{1}{2}y^2 + F(x), \qquad F(x) = \int_0^x f(\tau)d\tau \tag{4}$$

where $xf(x) > 0$ for $0 < x < x_0$, so the origin is a center. So for small positive h the set $H = h$ is a closed orbit. Since $\dot{\ell}$ is constant, ℓ is a constant multiple of time, t, in particular $\ell = \ell_0 - \Omega'(L_0)t$. In order to fix initial conditions, let t and ℓ be measured from the positive x-axis. A one degree of freedom Hamiltonian system is integrable up to 'quadrature'. To this end, let a denote the point on the positive x-intercept of $H = h$, so $F(a) = \Omega(L)$. From the equation

$$\frac{1}{2}y^2 + F(x) = h$$

solve for y

$$y = \frac{dx}{dt} = \pm\{2h - 2F(x)\}^{1/2}$$

separate variables

$$dt = \pm\{2h - 2F(x)\}^{-1/2} dx$$

and so

$$t = \pm \int_a^x \frac{d\xi}{\{2h - 2F(\xi)\}^{1/2}}. \tag{5}$$

The angle ℓ is $-\Omega'(L_0)t$. The orbit is swept out in a clockwise direction, so if the angle ℓ is to be measured in the usual counterclockwise direction for small ℓ and t, the minus sign should be used in (5).

To construct the symplectic change of variables consider the generating function

$$W(x, L) = \int_a^x \{2\Omega(L) - 2F(\xi)\}^{1/2} d\xi \tag{6}$$

with

$$y = \frac{\partial W}{\partial x} = \{2\Omega(L) - 2F(x)\}^{1/2}$$

so

$$\frac{1}{2}y^2 + F(x) = \Omega(L)$$

and

$$\begin{aligned} \ell &= \frac{-\partial W}{\partial L = -\{2\Omega(L) - 2F(a\}^{1/2}\dfrac{da}{dL} - \displaystyle\int_a^x \frac{\Omega'(L)d\xi}{\{2\Omega(L_0) - 2F(\xi)\}^{1/2}}} \\ &= -t\Omega'(L). \end{aligned}$$

(The first term in the formula for ℓ is zero by the definition of a.) Thus the change of variables is symplectic. We shall call these variables *action-angle variables (for the Hamiltonian H)*. If $H = (x^2 + y^2)/2$, the usual harmonic oscillator, then $\ell = \tan^{-1}(y/x)$ and $L = (x^2 + y^2)/2$.

2.5. **Normalized rectangular variables.** Go back to 'rectangular' coordinates u, v by

$$u = (2L)^{1/2}\cos\ell, \qquad v = (2L)^{1/2}\sin\ell.$$

One computes that $du \wedge dv = dL \wedge d\ell$ so this change of variables is symplectic also. The Hamiltonian becomes

$$H(u, v) = \Omega((u^2 + v^2)/2). \tag{7}$$

Question: Is the symplectic change of variable $(x, y) \to (L, \ell) \to (u, v)$ smooth/analytic at the origin?

To answer this question let us investigate the smoothness of the period function. If F is an even function (the force function f is odd) then the quarter period is

$$T = \int_0^a \frac{d\xi}{\{2F(a) - 2F(\xi)\}^{1/2}}.$$

(If F is not even then the half period is made of two such integrals and so the present discussion can easily be carried over to that case.) Assume that F has a power series expansion of the form

$$2F(x) = \alpha x^2 + \cdots, \qquad \alpha > 0$$

x small. Note that the lowest order term is αx^2 which insures that the origin is a center.

Write

$$2F(a) - 2F(x) = (a - x)G(a, x)$$

where $G(a,a) = 2\alpha a + \cdots$. We have factored out the zero of $2F(a) - 2F(x)$ and so $G(a,x) > 0$ for $0 \le x \le a$, a small. In the integral for T make the change of variables $u^2 = (a - \xi)/a$ so

$$T = 2a \int_0^1 \frac{du}{\{aG(a, a(1-u^2)\}^{1/2}} = 2 \int_0^1 \frac{du}{H(a,u)^{1/2}}$$

where

$$aG(a, a(1-u^2) = a^2 H(a,u),$$

and $H(a,u)$ is analytic in both variables, $H(0,u) = 2\alpha + \cdots$ and $H(0,u) > 0$ for $0 \le u \le 1$, so

$$T = \beta + \cdots, \qquad \beta = 2\int_0^1 \frac{du}{H(0,u)^{1/2}} > 0.$$

Thus I have shown:

Lemma 2.1. *If the system (4) is analytics and* $F(x) = \alpha x^2 + \cdots, \alpha > 0$ *then the period is analytic in and* $T(0) = \beta > 0$.

However, I would like to prove:

Theorem 2.1. *Let the system (4) be analytic and* $F(x) = \alpha x^2 + \cdots, \alpha > 0$. *Then the system can be transformed to the analytic normal form (7).*

2.6. **A Symplectic Poincaré's Center Theorem.** Consider an real analytic system with an equilibrium at the origin of the form

$$\dot{x} = y + \cdots, \qquad \dot{y} = -x + \cdots. \tag{8}$$

Poincaré's Center Theorem [3, 6] states that if (8) can be formally transformed to a system of the form

$$\dot{u} = u\{1 + f(u^2+v^2)\}, \qquad \dot{v} = -v\{1 + f(u^2+v^2)\}, \tag{9}$$

by a formal change of variables of the form $u = x + \cdots$, $v = y + \cdots$ then there is an analytic change of variable of the same form which takes (8) to (9). That is, if the system is formally a center then it is analytically a center. Equation (9) is the standard normal form for a center.

There are two caveats. First, since the formal transformation carrying (8) to (9) is not unique the formal transformation may not be the same one that Poincaré proved to be analytic. Second, even though the original system (8) is Hamiltonian the analytic transformation may not be symplectic. In this section, we shall over come the second problem.

Theorem 2.2. *If equation (8) is analytic and Hamiltonian then it can be normalized by an analytic, symplectic, near identity transformation. That is, if (8) is an analytic system with Hamiltonian of the form*

$$H(x,y) = \frac{1}{2}(x^2 + y^2) + \cdots, \tag{10}$$

there is an analytic, near identity, symplectic change of coordinates $(x,y) \longrightarrow (\xi,\eta)$ *such that the Hamiltonian becomes*

$$H(\xi,\eta) = K((\xi^2+\eta^2)/2), \qquad K(s) = 1 + \cdots. \tag{11}$$

Proof. Let g be the analytic transformation of the form $u = x + \cdots$, $v = y + \cdots$ which takes (8) to (9), see [3, 6]. Then g takes the symplectic form $\Omega = dx \wedge dy$ to $\Omega = h(u,v)du \wedge dv$ where h is analytic and $h(0,0) = 1$. The u, v flow preserves $\Omega = h(u,v)du \wedge dv$ and the circles $u^2 + v^2 = constant$, so $\Omega = s((u^2+v^2)/2)du \wedge dv$, $s(0) = 1$.

Change to action-angle coordinated (I, θ), $I = (x^2+y^2)/2$, $\theta = \tan^{-1}(y/x)$ so the form becomes $\Omega = s(I)dI \wedge d\theta$. Define

$$S(I) = \int s(I)dI$$

and change coordinates by $J = S(I)$. This last change of coordinates is analytic in the rectangular coordinates, preserves the form of the equations and take the form Ω to $dJ \wedge d\theta$. Thus, Hamiltonian has been normalized by a symplectic change of variables. □

3. The Planar Kepler Problem

In this section the ideas of the previous section are used to create action-angle variables for the Kepler problem (the central force problem with inverse square law attraction.) These variables are called *Delaunay elements*, named after the French astronomer of the last century who used these coordinates to develop his theory of the moon. Delaunay elements are valid only in the domain in phase space where there are elliptic orbit for the Kepler problem. I follow Poincaré to give a set of variables which are valid in a neighborhood of the circular orbits. These are the *Poincaré elements.*

3.1. Delaunay elements for the planar problem.

The Hamiltonian of the Kepler problem in symplectic polar coordinate (r, θ, R, Θ) is

$$H = \frac{1}{2}\left\{R^2 + \frac{\Theta^2}{r^2}\right\} - \frac{1}{r},$$

see [4]. Angular momentum, Θ, is an integral so for fixed $\Theta \neq 0$ this is a one-degree of freedom system of the form discussed above, except the origin of the center is at $r = \Theta$ (the circular orbit). Set $H = -/2L^2$ and solve for the r value of perigee (when $R = 0$) to get that $a = L[L - (L^2 - \Theta^2)^{1/2}]$. Note that $L^2 \geq \Theta^2$ and $L = \pm\Theta$ corresponds to the circular orbits of the Kepler problem. By the discussion given above we should use as our generating function

$$W_1(r, L) = \int_a^r \left\{-\frac{\Theta^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{L^2}\right\}^{1/2} d\xi$$

so $H = -1/L^2$, $\dot{L} = 0$, $\dot{\ell} = 1/L^3$. ℓ is known as the mean anomaly and is measured from perigee.

To change to Delaunay variables (ℓ, g, L, G) we use the generating function

$$W(r, \theta, L, G) = \theta G + \int_a^r \left\{-\frac{G^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{L^2}\right\}^{1/2} d\xi$$

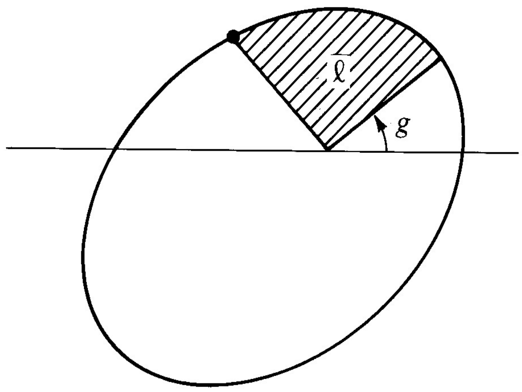

FIGURE 1. Delaunay angles in the planar problem.

where $a = a(L, G) = L[L - (L^2 - G^2)^{1/2}]$. Thus

$$
\begin{aligned}
R = \quad & \frac{\partial W}{\partial r} = \quad \left\{ -\frac{G^2}{r^2} + \frac{2}{r} - \frac{1}{L^2} \right\}^{1/2}, \\
\Theta = \quad & \frac{\partial W}{\partial \theta} = \quad G, \\
\ell = \quad & \frac{\partial W}{\partial L} = \quad - \left\{ -\frac{G^2}{a^2} + \frac{2}{a} - \frac{1}{L^2} \right\}^{1/2} \frac{\partial a}{\partial L} + \int_a^r \left\{ -\frac{G^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{L^2} \right\}^{-1/2} d\xi L^{-3} = \\
& \qquad t/L^3, \\
g = \quad & \frac{\partial W}{\partial G} = \quad \theta + \int_a^r \left\{ -\frac{G^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{L^2} \right\}^{-1/2} \left(-\frac{G}{\xi^2} \right) d\xi = \theta - f
\end{aligned} \tag{12}
$$

$G = \Theta$, so G is angular momentum. The expression for R implies that $H = -1/2L^2$ as is expected. The first quantity in the definition of ℓ is zero by the definition of a and the integral in the second quantity is just time, t. So $\ell = -t/L^3$ where t is measured from perigee, so ℓ is measured from perigee also. Recall that to change independent variable from t (time) to f (true anomaly) in the solution of the Kepler problem we set $df = (\Theta/r^2)dt = (G/r^2)dt$. Thus since $dt = (-G^2/\xi^2 + 2/\xi - 1/L^2)^{-1/2} d\xi$ the integrand in the definition of g is just df, and the integral gives the true anomaly, f, measured from perigee. Thus $g = \theta - f$ is the argument of the perigee. See Figure 1

3.2. **Poincaré elements for the planar problem.** The argument of the perigee is clearly undefined for the circular orbits; so, Delaunay elements are not valid coordinates in a neighborhood of the circular orbits. To overcome this problem Poincaré introduced what he call Kepler variables, but which have become known as *Poincaré elements*.

Recall that $L^2 > G^2$ is the elliptic domain and $L = \pm G$ corresponds to the circular orbits. Using the ideas in Subsection 2.5 to go from action-angle variables to rectangular variables one should make the symplectic change of variables

$$\begin{aligned} Q_1 &= \ell + g, \qquad & Q_2 &= \{2(L-G)\}^{1/2} \cos \ell, \\ P_1 &= L, & P_2 &= \{2(L-G)\}^{1/2} \sin \ell. \end{aligned} \tag{13}$$

Q_1 is an angular variable defined $\mod 2\pi$ and P_1 is the conjugate radial coordinate; Q_2, P_2 are rectangular coordinates with $Q_2 = 0, P_2 = 0$ corresponding to the circular orbits of the Kepler problem.

The Hamiltonian of the Kepler problem in Poincaré elements becomes

$$H = -\frac{1}{P_1^2}.$$

4. The Spatial Kepler Problem

Usually the three dimensional Kepler problem is reduced to the planar problem because conservation of angular momentum implies the motion takes place in a plane perpendicular to the angular momentum vector. However, now I want to give action-angle type variables for the three dimensional problem. So it is necessary to look at the Kepler problem in three dimensions. Once the three dimensional Kepler problem is understood, the three dimensional Delaunay variables can be given.

4.1. **The Kepler problem in spherical coordinates.**

Before we discuss Delaunay coordinates in three dimensions we need to solve the Kepler problem in spherical coordinates (ρ, θ, ϕ), the radius, longitude, and azimuth. The standard definition of spherical coordinates is

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi. \tag{14}$$

To extend this point transformation use Mathieu generating function

$$S = X\rho \sin\phi \cos\theta + Y\rho \sin\phi \sin\theta + Z\rho \cos\phi,$$

so

$$\begin{aligned} P = \frac{\partial S}{\partial \rho} &= X \sin\phi \cos\theta + Y \sin\phi \sin\theta + Z \cos\phi = (xX + yY + zZ)/\rho = \dot\rho, \\ \Theta = \frac{\partial S}{\partial \theta} &= -X\rho \sin\phi \sin\theta + Y\rho \sin\phi \cos\theta = -Xy + Yx = \rho^2\dot\theta \\ \Phi = \frac{\partial S}{\partial \phi} &= X\rho \cos\phi \cos\theta + Y\rho \cos\phi \sin\theta - Z\rho \sin\phi = \rho^2 \cos^2\phi\dot\phi. \end{aligned} \tag{15}$$

Thus R is radial momentum, and Θ is the z-component of angular momentum. From these expressions compute

$$\begin{gathered} Z = P\cos\phi - (\Phi/\rho)\sin\phi, \\ P\sin\phi + (\Phi/\rho)\cos\phi = X\cos\theta + Y\sin\theta, \\ \Theta/(\rho\sin\phi) = -X\sin\theta + Y\cos\theta. \end{gathered}$$

From the last two formulas compute X^2+Y^2 without computing $X\&Y$. You will find that the Hamiltonian of the Kepler problem in spherical coordinates is

$$H = \frac{1}{2}\left\{P^2 + \frac{\Phi^2}{\rho^2} + \frac{\Theta^2}{\rho^2\sin^2\phi}\right\} - \frac{1}{\rho} \tag{16}$$

and the equations of motion are

$$\begin{aligned}
&\dot{\rho} = H_P = P, && \dot{P} = -H_\rho = \frac{\Phi^2}{\rho^3} + \frac{\Theta^2}{\rho^3\sin^2\phi} - \frac{1}{\rho^2},\\
&\dot{\theta} = H_\Theta = \frac{\Theta}{\rho^2\sin^2\phi}, && \dot{\Theta} = -H_\theta = 0\\
&\dot{\phi} = H_\Phi = \frac{\Phi}{\rho^2}, && \dot{\Phi} = -H_\phi = \left(\frac{\Theta^2}{\rho^2}\right)\frac{\cos\phi}{\sin^3\phi}.
\end{aligned} \tag{17}$$

Clearly, Θ, the z-component of angular momentum, is an integral, but so is G defined by

$$G^2 = \left(\frac{\Theta^2}{\sin^2\phi} + \Phi^2\right). \tag{18}$$

We shall show that G is the magnitude of total angular momentum.

4.2. **A plane in spherical coordinates.** The equation of a plane through the origin is of the form $\alpha x + \beta y + \gamma z = 0$ or in spherical coordinates

$$\alpha\sin\phi\cos\theta + \beta\sin\phi\sin\theta + \gamma\cos\phi = 0$$

or

$$a\sin(\theta - \theta_0) = b\cot\phi. \tag{19}$$

Let the plane meet the x, y-plane in a line through the origin with polar angle $\theta = \Omega$ (the longitude of the node) and be inclined to the x, y-plane by an angle i (the inclination).

When $\theta = \Omega, \phi = \pi/2$ so we may take $\theta_0 = \Omega$. Let ϕ_m be the minimum ϕ takes on the plane, so $\phi_m + i = \pi/2$. ϕ_m gives the maximum value of $\cot\phi$ and sin has its maximum value of $+1$. Thus from (19) $a = b\cot\phi_m$ or $a\sin\phi_m = b\cos\phi_m$. Take $a = \cos\phi_m = \sin i$ and $b = \sin\phi_m = \cos i$. Therefore, the equation of a plane in spherical coordinates with the longitude of the node Ω and inclination i is

$$\sin i\sin(\theta - \Omega) = \cos i\cot\phi. \tag{20}$$

4.3. **The equation of the invariant plane.** Use (18) to solve for Φ and substitute it into the equation for $\dot{\phi}$, then eliminate ρ^2 from the equations for $\dot{\phi}$ and $\dot{\theta}$, to obtain

$$\dot{\phi} = \frac{\Phi}{\rho^2} = \left\{G^2 - \frac{\Theta^2}{\sin^2\phi}\right\}^{1/2}\frac{1}{\rho^2} = \left\{G^2 - \frac{\Theta^2}{\sin^2\phi}\right\}^{1/2}\left\{\frac{\sin^2\phi\dot{\theta}}{\Theta}\right\}.$$

Separate variables and let $\theta = \Omega$ when $\phi = \pi/2$, so that Ω is the longitude of the node. Thus

$$
\begin{aligned}
\int_0^\phi \left\{G^2 - \frac{\Theta^2}{\sin^2\phi}\right\}^{-1/2} \sin^{-2}\phi d\phi = &\int_\Omega^\theta \Theta^{-1} d\theta = (\theta-\Omega)/\Theta \\
-\int_0^u \{G^2 - \Theta^2(1+u^2)\}^{-1/2} du = & \\
-\Theta^{-1}\int_0^u \{\beta^2 - u^2\}^{-1/2} du = & \\
\Theta^{-1}\sin^{-1}(u/\beta) = &
\end{aligned} \tag{21}
$$

The first substitution is $u = \cot\phi$ and β is defined by $\beta^2 = (G^2 - \Theta^2)/\Theta^2$. Therefore,

$$
-\cot\phi = \pm\beta\sin(\theta-\Omega).
$$

Finally

$$
\cos i \cot\phi = \sin i \sin(\theta - \Omega) \tag{22}
$$

where

$$
\beta^2 = \frac{G^2-\Theta^2}{\Theta^2} = \tan^2 i = \frac{\sin^2 i}{\cos^2 i}. \tag{23}
$$

Equation (22) is the equation of the invariant plane. The above gives $\Theta = \pm G\cos i$. Since i is the inclination and Θ is the z-component of angular momentum this means that G is the magnitude of total angular momentum. In the above take θ_0 to be Ω the longitude of the node.

4.4. **Delaunay elements in three dimensions.** We shall change from spherical coordinates $(\rho, \theta, \phi,\ P, \Theta, \Phi)$ to Delaunay elements $(\ell, g, k,\ L, G, K)$ where the first three variables are angles defined $\mod 2\pi$. Consider the generating function

$$
\begin{aligned}
W(\rho,\theta,\phi,L,G,K) = \ &\theta K + \int_{\pi/2}^\phi \left\{G^2 - \frac{K^2}{\sin^2\zeta}\right\}^{1/2} d\zeta + \\
&\int_a^\rho \left\{-\frac{G^2}{\xi^2} + \frac{2}{\xi} - \frac{1}{L^2}\right\}^{1/2} d\xi,
\end{aligned} \tag{24}
$$

where $a = a(L,G) = L[l - (L^2 - G^2)]$. The change of coordinates is

$$
\begin{aligned}
P &= \frac{\partial W}{\partial \rho} &&= \left\{-\frac{G^2}{\rho^2}+\frac{2}{\rho}-\frac{1}{L^2}\right\}^{1/2}\\
\Theta &= \frac{\partial W}{\partial \theta} &&= K\\
\Phi &= \frac{\partial W}{\partial \phi} &&= \left\{G^2-\frac{K^2}{\sin^2\phi}\right\}^{1/2}\\
\ell &= \frac{\partial W}{\partial L} &&= \int_a^r \left\{-\frac{G^2}{\xi^2}+\frac{2}{\xi}-\frac{1}{L^2}\right\}^{-1/2} d\xi L^{-3} = -t/L^3\\
g &= \frac{\partial W}{\partial G} &&= -\int_{\pi/2}^{\phi} \left\{G^2-\frac{K^2}{\sin^2\zeta}\right\}^{-1/2} G d\zeta - \int_a^r \left\{-\frac{G^2}{\xi^2}+\frac{2}{\xi}-\frac{1}{L^2}\right\}^{-1/2}\left(\frac{G}{\xi^2}\right) d\xi\\
& &&= \sigma - f\\
k &= \frac{\partial W}{\partial K} &&= \theta - \int_{\pi/2}^{\phi} \left\{G^2-\frac{K^2}{\sin^2\zeta}\right\}^{-1/2}\left(\frac{K}{\sin^2\zeta}\right) d\zeta = \Omega.
\end{aligned}
\tag{25}
$$

Since $\Theta = K$, K is the z-component of angular momentum, the expression for Φ gives that G is the magnitude of total angular momentum, and the expression for P insures that $H = -1/2L^2$. $\ell = -t/L^3$ where t is measured from perigee, and so ℓ is the mean anomaly. The integral in the definition of k is the first integral in (21), so $k = \theta - (\theta - \Omega) = \Omega$ the longitude of the node.

The first integral in the formula for g is integrated as follows.

$$
\begin{aligned}
\int_{\pi/2}^{\phi} \left\{G^2-\frac{K^2}{\sin^2\zeta}\right\}^{-1/2} G d\zeta &= \int_{\pi/2}^{\phi} \left\{1-\frac{\cos^2 i}{\sin^2\zeta}\right\}^{-1/2} d\zeta\\
&= \int_{\pi/2}^{\phi} \frac{\sin\zeta d\zeta}{\sqrt{\sin^2\zeta-\cos^2 i}}\\
&= -\int_0^{\cos\phi} \frac{du}{\sqrt{1-u^2-\cos^2 i}}, \qquad (u=\cos\zeta)\\
&= -\int_0^{\cos\phi} \frac{du}{\sqrt{\sin^2 i-u^2}}\\
&= -\sin^{-1}\left(\frac{\cos\phi}{\sin i}\right)\\
&= -\sigma.
\end{aligned}
$$

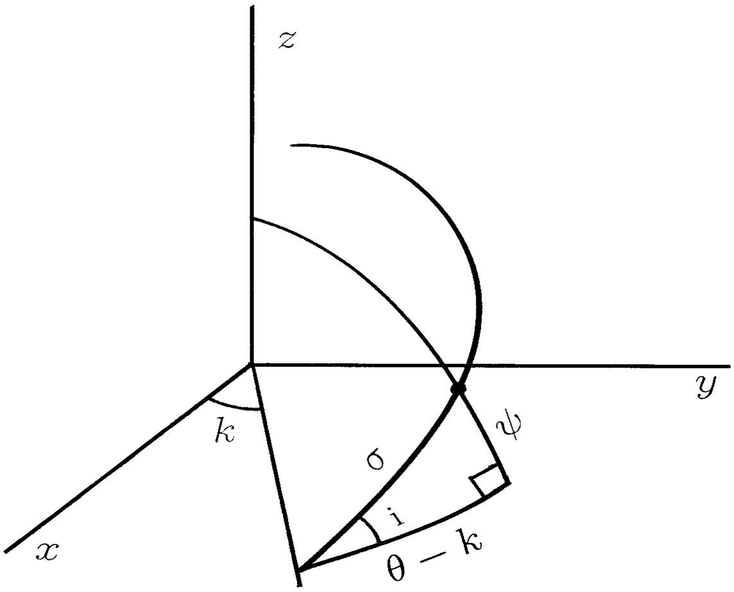

Figure 2. The definition of σ.

Therefore,

$$\sin\sigma = \frac{\sin(\pi/2-\phi)}{\sin i} = \frac{\sin\psi}{\sin i}.$$

The angle σ is defined by the spherical triangle with sides (arcs measured in radians) θ, $\psi = \pi/2-\phi$, σ and spherical angle i. Recall the law of sines for spherical triangles and see Figure 2.

Thus σ measures the position of the particle in the invariant plane. Since f is the true anomaly measured from perigee in the invariant plane $g = \sigma - f$ is the argument of the perigee measured in the invariant plane.

5. Delaunay elements using Arnold's Theorem

The material in this section is adapted from the Moser-Zhender notes [5]. First a few general results which are easy to verify.

Lemma 5.1. *Let $F, G, H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^1$ be smooth, then*

$$\{FG, H\} = F\{G, H\} + G\{F, H\}.$$

Lemma 5.2. *Let $s : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ and $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^1$ be smooth. Then the flow $dK^{\#}$ where $K = s(H)$ is a reparameterization of the flow $dH^{\#}$ where*

$$dK^{\#}: \quad du/dt = JgradK, \qquad dH^{\#}: du/d\tau = JgradH$$

and $d\tau = s'(H)dt$. In particular, if an periodic orbit has a t-period of p then it has a τ-period of $s'(H)p(H)$.

Return to three-space and let angular momentum by denoted by

$$A = r \times R = A_x i + A_y j + A_z k.$$

An easy calculation shows:

Lemma 5.3. $\{A_x, A_y\} = A_z, \quad \{A_z, A_x\} = A_y, \quad \{A_y, A_z\} = A_x.$

Define three functions L,G,K by

$$L = 1/(-2H)^{1/2}, \qquad G = \| A \|, \qquad K = A_z \tag{26}$$

where in (26) H is the Hamiltonian of the Kepler problem. G and K are defined on all of $\mathbb{R}^6$ but L is defined on the open subset of $\mathbb{R}^6$ where H is defined and negative.

Proposition 5.1. *L, G, K are in involution.*

Proof. $\{H, G\} = \{H, K\} = 0$ since G and K are integrals of the flow $dH^{\#}$. Reversing the roles this means that H and hence L is constant on the trajectories of $dG^{\#}$ and $dK^{\#}$, thus $\{L, G\} = \{L, K\} = 0$. Now $\{G^2, K\} = \{A_x^2 + A_y^2 + A_z^2, A_z\} = 2A_x\{A_x, A_z\} + 2A_y\{A_y, A_z\} + 2A_z\{A_z, A_z\} = 2A_x(-A_y) + 2A_y(A_x) = 0$. □

Let $\mathfrak{D}$ denote the open domain in $\mathbb{R}^6$ where , $G^2 > K^2 > 0$ and all orbits of $dH^{\#}$ are ellipses. We will call this the *Delaunay domain*.

Proposition 5.2. *On $\mathfrak{D}$ all orbits of $dL^{\#}$, $dG^{\#}$, and $dK^{\#}$ are periodic with period 2π. The orbits of $dL^{\#}$ are elliptic orbits of the Kepler problem and the orbits of $dG^{\#}$, $dK^{\#}$ are circles described below.*

Proof. In coordinates $(r, R) = (x, y, z, X, Y, Z)$ the function $K = A_z = xY - yX$, and the equations of $dK^{\#}$ are

$$\begin{aligned} \dot{x} &= K_X = -y & \dot{X} &= -K_x = -Y \\ \dot{y} &= K_Y = -x & \dot{Y} &= -K_y = X \\ \dot{z} &= K_Z = 0 & \dot{Z} &= -K_z = 0 \end{aligned}$$

This is just two harmonic oscillators of period 2π. Note there are circles about the z-axis and Z-axis.

Let E be the two dimensional plane spanned by r, R in $\mathbb{R}^6$. Recall $A = r \times R \neq 0$ so r, R are linearly independent. Let Q be the rotation matrix which rotates E to the x, y-plane. In that plane $G = \pm K$ and so by the above in that plane all orbits are of period 2π and circles. These orbits would be circles in the plane spanned by r, R.

In the Kepler problem the period of the orbits are $p(h) = (\pi/\sqrt{2})(1/(-h)^{3/2}$. So $L = (-2H)^{-1/2}$ gives $s(h) = (-2h)^{-3/2}$ which makes the orbits all 2π periodic. □

Lemma 5.4. *On $\mathfrak{D}$, $L^2 > G^2 > K^2 > 0$ and dK, dG, and dL are independent.*

Proof. Clearly, $G^2 \geq K^2 \geq 0$ and by definition of $\mathfrak{D}$ we require strict inequality. If at some point $K^2 > 0$ then the solution of $dK^{\#}$ through that point lies on a circle and so $dK \neq 0$. If at some point $G^2 > K^2 > 0$ then the solution of $G^{\#}$ is a circle different from the solution of $dK^{\#}$. Thus dK and dG are independent.

For the Kepler problem $H = -1/2s$ where s is the semi-major axis so $L^2 = s$. But $G^2 = (1 - \epsilon^2)s$, so $L^2 \geq G^2$ and $L^2 = G^2$ only on circular orbits, so on $\mathfrak{D}$ we have $L^2 > G^2 > K^2 > 0$.

If dL depends on dG and dK then $dL^{\#} = d(\alpha G + \beta K)^{\#}$ where α and β are constants. But, the solutions of $d(\alpha G + \beta K)^{\#}$ are circles again ($\alpha G + \beta K$ is just a new momentum vector), but the solution of $dL^{\#}$ is an ellipse. Thus, $dL^{\#} \neq d(\alpha G + \beta K)^{\#}$ and the vectors are independent. $\square$

Theorem 5.1. { *Arnold's Theorem*} *If an n-degree of freedom system has n-independent integrals $F_1, \dots, F_n$ in involution and the level set $\mathbb{T}_0$: $F_1 = const, \dots, F_n = const$ is compact then $\mathbb{T}_0$ is an n-torus and one can introduce action-angle variables around $\mathbb{T}_0$.*

Proof. {Outline:} Holding n independent functions fixed in a $2n$-dimensional system defined an n-manifold – so $\mathbb{T}_0$ is an n manifold. The integrals in involution implies that $dF_1^{\#}, \dots, dF_n^{\#}$ are commuting vector fields on $\mathbb{T}_0$. It is an old theorem in topology that a compact manifold with n-commuting vector fields is an n-torus.

To construct the action-angle coordinates one must find n integrals $I_1 = I_1(F_1, \dots, F_n)$, $\dots$, $I_n = I_n(F_1, \dots, F_n)$ such that the integrals are defined in a neighborhood of $\mathbb{T}_0$ and are still in involution, but such that all solutions of $dI_1^{\#}, \dots, dI_n^{\#}$ are periodic with the same period. (Arnold's proof has a gap at this point, but the gap was filled in [2].) For the Kepler problem in space we have constructed the three such independent integrals in involution, namely L, G, K.)

Since the integrals are defined in a neighborhood of $\mathbb{T}_0$ it lies in an n-parameter family of n-tori which will be denoted by $\mathbb{T}(I_1, \dots, I_n)$. Pick a local Lagrangian manifold S transversal to $\mathbb{T}(I_1, \dots, I_n)$ so $S \cap \mathbb{T}(I_1, \dots, I_n)$ is a single point, $p(I_1, \dots, I_n)$, and let $\Phi_i(t, \zeta)$ be the solution of $dI_i^{\#}$ with $\Phi_i(0, \zeta) = \zeta$.

Action-angle coordinates are defined by the map

$$\mathbb{A} : (I_1, \dots, I_n, \phi_1, \dots, \phi_n) \longrightarrow \Phi_n(\phi_n, \Phi_{n-1}(\phi_{n-1}, \dots, \Phi_1(\phi_1, p(I_1, \dots, I_n)) \cdots)).$$

That is, to find the point with coordinates $(I_1, \dots, I_n, \phi_1, \dots, \phi_n)$ first find the point p on the intersection of the Lagrangian manifolds S and $\mathbb{T}(I_1, \dots, I_n)$, then follow the flow $dI_1^{\#}$ by time ϕ_1, then the flow $dI_2^{\#}$ by a time ϕ_2 etc. $\square$

Return to the Kepler problem in space. We will describe the angles in reverse order. Given a point in $\mathfrak{D}$ flow back by a time $-\ell$ by the flow $dL^{\#}$ until you reach the perigee of the Kepler orbit, so ℓ is the mean anomaly measured from the perigee. Then flow by a time $-g$ until the point is in the x, y-plane, so g is the argument of the perigee. You are still in the invariant plane, so now you are at the intersection of the invariant plane and the x, y-plane or at the line of the node. Now flow by $-k$ until you reach the x-axis, so k is the longitude of the node.

This gives the Lagrangian manifold $S = \{x, y, z, X, Y, Z) : y = z = 0, X = 0\}$, i.e. the initial conditions for the Kepler problem that have the perigee on the x-axis.

6. Notes on Poincaré Elements and Initial Conditions

The three dimensional Kepler problem in spherical coordinates is

$$H = \frac{1}{2}\{R^2 + \frac{\Phi^2}{\rho^2} + \frac{\Theta^2}{\rho^2 \sin^2 \phi}\} - \frac{1}{\rho}$$

Letting $G^2 = \Phi^2 + \frac{\Theta^2}{\sin^2 \phi}$ and $R = 0$, we can set $H = -1/2L^2$ to solve for the ρ value of the perigee. This value is $L[L - (L^2 - G^2)^{1/2}]$.

On circular solutions, where there is no perigee,

$$0 = \dot{\rho} = \dot{R} = \frac{1}{\rho^3}(\Phi^2 + \frac{\Theta^2}{\sin^2\phi}) - \frac{1}{\rho^2} = 0.$$

That is, $G^2/\rho^3 - 1/\rho^2 = 0$ so $G^2 = \rho = const$. Since $0 = \dot{\rho} = R$, the Hamiltonian evaluated on a circular solution becomes

$$\frac{-1}{2L^2} = \frac{1}{2}\{\frac{G^2}{\rho^2}\} - \frac{1}{\rho}$$

which implies $L^2 = \rho = G^2$. Thus, as in the two dimensional Kepler problem, $L = \pm G$ corresponds to the circular orbits.

To investigate three dimensional circular solutions then, we need to introduce Poincaré elements of some kind. Begin by considering Delaunay coordinates which are defined by the generating function

$$W(\rho, \theta, \phi, L, G, K) = \theta K + \int_{\pi/2}^{\phi} \left\{ G^2 - \frac{K^2}{\sin^2 z} \right\}^{1/2} dz +$$

$$\int_a^{\rho} \left\{ -\frac{G^2}{z^2} + \frac{2}{z} - \frac{1}{L^2} \right\}^{1/2} dz$$

where $a = L[L = (L^2 - G^2)^{1/2}]$.

The meaning of L is shown above, and if i is the inclination of the invariant plane in the Kepler problem, then $G^2 \cos i = \Theta^2$, so G is the magnitude of total angular momentum (see above). Near circular orbits, the variables L, G, and K present no problems, so we consider the angular variables $l = \partial W/\partial L$, $g = \partial W/\partial G$, $k = \partial W/\partial K$. Let

$$I_0 = \frac{1}{L^3} \int_a^{\rho} \left\{ -\frac{G^2}{z^2} + \frac{2}{z} - \frac{1}{L^2} \right\}^{-1/2} dz$$

$$I_1 = -G \int_a^{\rho} \frac{1}{z^2} \left\{ -\frac{G^2}{z^2} + \frac{2}{z} - \frac{1}{L^2} \right\}^{-1/2} dz$$

From Schimdt [7] (via a clever substitution to evaluate the integrals), we find that

$$I_0 = \arccos \frac{1 - \rho/L^2}{\sqrt{1 - (G/L)^2}} - \frac{\rho R}{L}$$

$$I_1 = -\arccos \frac{G^2/\rho - 1}{\sqrt{1 - (G/L)^2}}$$

Since $l = I_0$, it is clear that this variable is undefined when $L = \pm G$. Now, let

$$I_2 = \int_{\pi/2}^{\phi} \left\{ G^2 - \tfrac{K^2}{\sin^2 z} \right\}^{-1/2} G dz$$

$$I_3 = \int_{\pi/2}^{\phi} \left\{ G^2 - \tfrac{K^2}{\sin^2 z} \right\}^{-1/2} \tfrac{K}{\sin^2 z} dz$$

These last two integrals can be integrated to obtain (since $K = \Theta$)

$$I_2 = -\arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}$$

$$-\arctan \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi - (\Theta/G)^2}} \qquad \text{if } \sin^2\phi \neq (\frac{\Theta}{G})^2$$

or

$$I_3 = -\pi/2 \qquad \text{if } \sin^2\phi = (\frac{\Theta}{G})^2$$

And I_2 is defined since $\Theta^2 = G^2 \cos i$ therefore $0 < 1 - (\Theta^2/G^2) < 1$ as we assume that $0 < i < \pi/2$. Now, the Delaunay coordinates are as follows:

$$\begin{aligned} l = I_0 \quad &= \arccos A - R\rho/L \\ g = I_1 - I_2 \quad &= -\arccos B + \arcsin S \\ k = \theta - I_3 \quad &= \theta + \arctan T \end{aligned}$$

Where

$$A = \frac{1-\rho/L^2}{\sqrt{1-(G/L)^2}} \qquad B = \frac{G^2/\rho - 1}{\sqrt{1-(G/L)^2}}$$

$$S = \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}} \qquad T = \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi - (\Theta/G)^2}}$$

The Poincaré elements, which we will show *are* defined on circular solutions, are given by $Q_1 = l+g+k, \quad Q_2 = -\sqrt{2(L-G)}\sin(k+g), \quad Q_3 = l+g$. With the Delaunay coordinates, we can write (Q_3 is the simplest case, so we begin here): $Q_3 = \arccos A + \arccos B + \arcsin S - R\rho/L$. Combining the first two terms gives $Q_3 = \arccos(1 + \frac{R^2\rho}{1+G/L}) + \arcsin S - R\rho/L$ which now has *no* singularity at $L = \pm G$ and is defined as long as long as $0 < i < \pi/2$. Next, observe that

$$\begin{aligned} Q_1 \quad &= l + g + k = Q_3 + k \\ &= \arccos\left(1 + \frac{R^2\rho}{1+G/L}\right) + \arcsin S - R\rho/L + \theta + \arctan T \end{aligned}$$

which has the same property.

Q_2 is more complicated. Start by looking at

$$k + g = \theta + \arctan T + \arccos B + \arcsin S.$$

So

$$
\begin{aligned}
\sin(g+k) &= \sin\left[\arccos B\right]\cos\left[\theta + \arctan T + \arcsin S\right] + \\
&\quad \cos\left[\arccos B\right]\sin\left[\theta + \arctan T + \arcsin S\right] \\
&= \sin\left[\arccos \frac{G^2/\rho - 1}{\sqrt{1-(G/L)^2}}\right]\cos\left[\theta + \arctan T + \arcsin S\right] + \\
&\quad \cos\left[\arccos \frac{G^2/\rho - 1}{\sqrt{1-(G/L)^2}}\right]\sin\left[\theta + \arctan T + \arcsin S\right] \\
&= \frac{GR}{\sqrt{1-(G/L)^2}}\cos\left[\theta + \arctan T + \arcsin S\right] + \\
&\quad \frac{G^2/\rho - 1}{\sqrt{1-(G/L)^2}}\sin\left[\theta + \arctan T + \arcsin S\right].
\end{aligned}
$$

Thus

$$
\sqrt{2(L-G)}\sin(g+k) = \frac{\sqrt{2}LGR}{\sqrt{L+G}}\cos\left[\theta + \arctan T + \arcsin S\right] +
$$

$$
\frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}}\sin\left[\theta + \arctan T + \arcsin S\right].
$$

Where, again, the singularity at $L = \pm G$ is eliminated. *On the circular solutions,* since $R = 0$ and $G^2 = \rho$ we have $Q_2 \equiv 0$.

7. Initial Conditions for Doubly Symmetric Periodic Orbits in Poincaré Elements

Let position $= (x, y, z)$ and let momentum $= (X, Y, Z)$. To guarantee that the third mass moving according to the motions given in the restricted problem have a doubly symmetric, three dimensional periodic orbit it is sufficient that at $t = 0$ we have $X = y = z = 0$, and at time $t = \frac{1}{4}T$ we have $X = y = Z = 0$. We want to determine what these sufficient conditions must be in the Poincaré elements discussed above, so that such an orbit could be nearly circular for $0 < t < \frac{1}{4}T$. To do this, we simply determine what the conditions must be in (symplectic) spherical coordinates and then use the representations developed above to translate these conditions into Poincaré elements.

In spherical coordinates we have at $t = 0$ when $X = y = z = 0$:

$$0 = z = \rho \cos\phi \Rightarrow \phi = \pi/2,$$

$$0 = y = \rho \sin\phi \sin\theta \text{ and } \phi = \pi/2 \Rightarrow \theta = n\pi,$$

$$R = xX + yY + zZ \Rightarrow R = 0,$$

$$\Theta = -X\rho \sin\phi \sin\theta + Y\rho \sin\phi \cos\theta \text{ and } \theta = n\pi \Rightarrow \Theta = Y\rho$$

$$\Phi = X\rho \cos\phi \cos\theta + Y\rho \cos\phi \sin\theta - Z\rho \sin\phi \Rightarrow \Phi = -Z\rho$$

Thus it is sufficient that at $t = 0$; $\phi = \pi/2$, $\theta = n\pi$, $R = 0$.

Now at $t = \frac{1}{4}T$ when $X = y = Z = 0$:

$$\begin{aligned}
\phi &\neq 0 \text{ so } 0 = y = \rho \sin\phi \sin\theta \Rightarrow \theta = m\pi \\
R &= xX + yY + zZ \Rightarrow R = 0 \text{ as before} \\
\Theta &= Y\rho \text{ as before, but} \\
\Phi &= X\rho \cos\phi \cos\theta + Y\rho \cos\phi \sin\theta - Z\rho \sin\phi \\
&\quad \text{and } X = Z = 0,\ \theta = m\pi \Rightarrow \Phi = 0
\end{aligned}$$

Thus it is sufficient that at $t = \frac{1}{4}T$; $\theta = m\pi$, $R = 0$, $\Phi = 0$.

Now since the expressions for Poincaré elements obtained above are nearly all in spherical coordinates, we need only evaluate according to the above conditions. The expressions from above are:

$$\begin{aligned}
Q_1 = &\ \arccos(1 + \frac{R^2\rho}{1 + G/L}) + \arcsin S - R\rho/L + \theta + \arctan T \\
Q_2 = &\ \frac{\sqrt{2}LGR}{\sqrt{L+G}} \cos[\theta + \arctan T + \arcsin S] + \\
&\quad \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin[\theta + \arctan T + \arcsin S] \\
Q_3 = &\ \arccos(1 + \frac{R^2\rho}{1 + G/L}) + \arcsin S - R\rho/L
\end{aligned}$$

Since $R = 0$ at both $t = 0$ and at $t = \frac{1}{4}T$ we can reduce the expressions to:

$$\begin{aligned}
Q_1 = &\ 0 + \arcsin S - 0 + \theta + \arctan T \\
Q_2 = &\ 0 + \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin[\theta + \arctan T + \arcsin S] \\
Q_3 = &\ 0 + \arcsin S + 0
\end{aligned}$$

Which is equal to

$$
\begin{aligned}
Q_1 = &\quad \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}} + \theta + \arctan \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi - (\Theta/G)^2}} \\
Q_2 = &\quad \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin\left[\theta + \arctan \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi-(\Theta/G)^2}} + \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}\right] \\
Q_3 = &\quad \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}
\end{aligned}
$$

Now at $t = 0$, $\phi = \pi/2$ and $\theta = n\pi$ so this becomes

$$
\begin{aligned}
Q_1 = &\quad \arcsin 0 + n\pi + \arctan 0 = n\pi \\
Q_2 = &\quad \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin[n\pi] = 0 \\
Q_3 = &\quad \arcsin 0 = j\pi
\end{aligned}
$$

To see the initial conditions at $t = \frac{1}{4}T$, since here ϕ is not determined, we need to expand the expressions $\frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi-(\Theta/G)^2}}$ and $\frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}$ in terms of spherical coordinates. To do this, recall that $G^2 = \Phi^2 + \Theta^2/\sin^2\phi$. But at $t = \frac{1}{4}T$, this becomes $G^2 = \Theta^2/\sin^2\phi$ since $\Phi = 0$. So $\sin^2\phi = (\Phi/G)^2$ and the integral I_3 becomes $-\pi/2$. In addition, $1-(\Phi/G)^2$ becomes $\cos\phi$. Thus at $t = \frac{1}{4}T$ when $\theta = m\pi$, $\quad R = 0$, $\quad \Phi = 0$ the coordinates are

$$
\begin{aligned}
Q_1 = &\quad \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}} + m\pi + \arctan \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi - (\Theta/G)^2}} \\
Q_2 = &\quad \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin\left[m\pi + \arctan \frac{(\Theta/G)\cos\phi}{\sqrt{\sin^2\phi-(\Theta/G)^2}} + \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}\right] \\
Q_3 = &\quad \arcsin \frac{\cos\phi}{\sqrt{1-(\Theta/G)^2}}
\end{aligned}
$$

or

$$
\begin{aligned}
Q_1 = &\quad \arcsin 1 + m\pi - \pi/2 = m\pi \\
Q_2 = &\quad \frac{\sqrt{2}L(G^2/\rho - 1)}{\sqrt{L+G}} \sin[m\pi - \pi/2 + \arcsin 1] = 0 \\
Q_3 = &\quad \arcsin 1 = k\pi/2
\end{aligned}
$$

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