BIFURCATIONS OF PERIODIC SOLUTIONS OF HAMITONIAN SYSTEMS USING FIXED POINT METHODS

KENNETH R. MEYER

ABSTRACT. A variation of a fixed point theorem of G. D. Birkhoff is used to study the bifurcations of periodic solutions of a periodic Hamiltonian system of two degrees of freedom depending on two parameters.

I. Introduction. The use of normal forms has become standard in the study of the bifurcation of periodic solutions in Hamiltonian systems of equations (or equivalently of the bifurcation of periodic points of a symplectic mapping.) If a system of equations is in normal form it is quite easy in general to establish the existence and uniqueness of bifurcating periodic solutions by using the implicit function theorem. In practice putting a system into normal form is not always easy even with the powerful algebraic processors available today. No matter how powerful the computer is it is easy to construct simple examples which defy that power.

The reason is simple. In order to establish the existence and uniqueness of a bifurcation from a periodic solution which has a characteristic multiplier which is an n^{th} root of unity typically the equations must be put into normal form through terms of order n. Consider for example the conservative Duffing's equation

$$\ddot{x} + \omega^2 x = \epsilon(\gamma x^3 + A\cos t)$$

which can be written as a Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}$$

with Hamiltonian

$$H = \frac{1}{2}(y^2 + \omega^2 x^2) - \epsilon(\gamma x^4 + Ax\cos t).$$

For $\epsilon = 0$ the origin is a 2π periodic solution with characteristic multipliers $\exp(\pm i2\pi\omega)$ so when ω is not an integer there is a small (order ϵ) 2π periodic solution with characteristic multipliers close to $\exp(\pm i2\pi\omega)$. This solution is called the *harmonic*. When ω is a rational number $m/n, n$ and m relatively prime, one expects that an $n2\pi$ periodic solution would bifurcate (*a subharmonic bifurcation*). The usual normal form procedure for handling this problem would be to put the equation in action-angle coordinates (or complex conjugate coordinates) and normalize the Hamiltonian through terms of order n. If at order n an angle term appears in the normal form then it is easy to apply to the implicit function theorem

This research partially supported by grants from the National Science Foundation and the Taft Foundation.

Appeared in: *Differential Equations and Dynamical Systems*, 3(1), 1995, 1-6.

to establish the existence and uniqueness of a pair of subharmonic solutions which bifurcate from the harmonic – one elliptic and one hyperbolic subharmonic.

The amount of work it takes to normalize an equation to order n seems to grows exponentially with n and so this approach must fail for large n. However, a variation on an argument of Poincaré(1912) which is partly analytic and partly topological establishes the existence of the subharmonic solutions for all $m/n \notin \mathbf{Z}$. This argument requires that the Hamiltonian be normalized only through the fourth order – in this case an easy task by hand. Like most fixed point arguments there are no uniqueness, continuity or stability conclusions. See Meyer and Hall (1992) pp. 214 ff. for the details of this argument.

In a short note Birkhoff(1931) gave an n-dimensional generalization of Poincaré's argument which lead to the classic Birkhoff-Lewis theorem. In this note I will show how Birkhoff's idea can be used to establish bifurcation results in higher dimensions with a fixed amount of computation – i.e. the amount of computation does not depend in an essential way on the particular characteristic multiplier. The method will be applied to a model two parameter bifurcation problem.

In general terms the main bifurcation result is as follows. Start with a 2 degree of freedom 2π-periodic Hamiltonian system with two parameters, say μ and ν. Generically with two parameters one encounters a 2π-periodic solution with two pairs of multipliers where one pair are p^{th} roots of unity and the other pair are q^{th} roots of unity where p and q are relatively prime integers. For simplicity assume p and q are large, say $p, q > 4$. (The case when p and q are small can be attacked by traditional normal form methods.) Assume there is such a periodic solution when $\mu = \nu = 0$. By the implicit function theorem there exists a nearby 2π-periodic solution for all small μ and ν.

The basic assumptions are two. First, assume that the multipliers vary in a nondegenerate manner as the two parameters are varied (i.e. assume that the map from the parameter space to the space of multipliers is injective when $\mu = \nu = 0$.) Second, assume that when the Hamiltonian is normalized through terms of order four when $\mu = \nu = 0$ that a type of twist condition is satisfied (i.e. a certain determinant defined by the coefficients in the normal form is non-zero.)

Under these assumptions there is a region S in the μ,ν-parameter plane bounded by two smooth c_1, c_2 curves as shown in Figure 1. For values of the parameters on one boundary curve the system has at least two $2p\pi$-periodic solution and on the other boundary curve the system has at least two $2q\pi$-periodic solution. In the sector S between the two boundary curves the system has at least three $2pq\pi$-periodic solutions.

This result is purely an existence result. There is no information about how these solutions vary with the parameters nor what the stability properties of these periodic solutions are. That would require assumptions on the higher order terms in the normal form and theorems of this type are abundant now.

II Main Result. Specifically assume that the Hamiltonian H is defined in a neighborhood of the origin in $\mathbf{R}^4$ for small values of the parameters μ and ν and is normalized through terms of order 4. Use action-angle variable $(I_1, I_2, \phi_1, \phi_2)$ in $\mathbf{R}^4$. The Hamiltonian is of the form

$$H = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2}(A I_1^2 + 2\mathrm{B} \mathrm{I}_1 I_2 + C I_2^2) + K \tag{1}$$

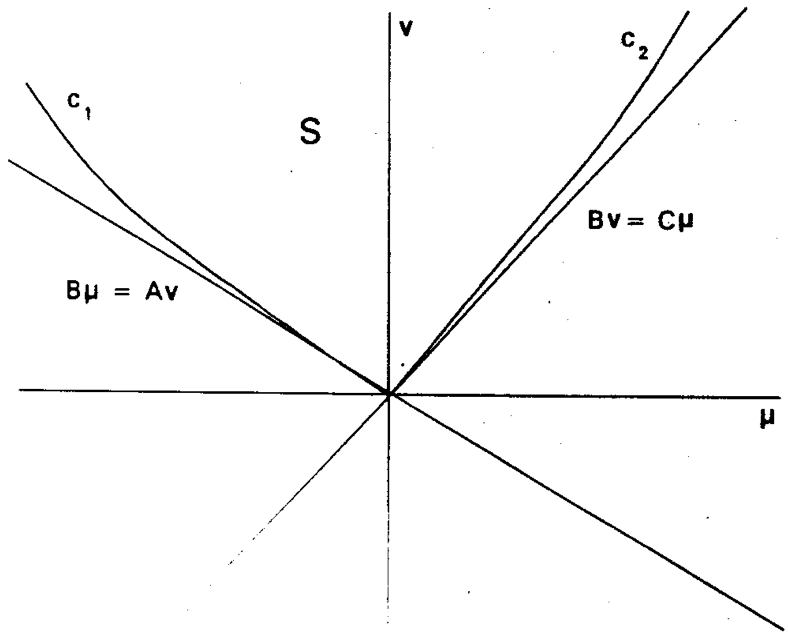

FIGURE 1. The sector S in parameter space.

where $H = H(I_1, I_2, \phi_1, \phi_2, \mu, \nu, t), K = K(I_1, I_2, \phi_1, \phi_2, \mu, \nu, t)$ are smooth functions for I_1, I_2, μ, ν small, are 2π-periodic in ϕ_1, ϕ_2, t and that K is of higher order (say at least cubic in I_1, I_2). Assume that $\omega_1 = 1/p + \mu$ and $\omega_2 = 1/q + \nu$ where p and q are relatively prime integers. Assume that A, B and C are constants with

$$D = B^2 - AC \neq 0. \tag{2}$$

Thus I have assumed that the system has undergone a considerable amount of preliminary changes of variables, but this is consistent with the general statement of the main result given above. First of all the 2π-periodic solution which exists for small μ, ν has been shifted to the origin. Second the general assumption about the frequencies is $\omega_1 = 1/p, \omega_2 = 1/q$ and $\partial(\omega_1, \omega_2)/\partial(\mu, \nu) \neq 0$ when $\mu = \nu = 0$, but clearly a change of parameters makes the frequencies of the above form. Thus the origin is a 2π-periodic solution with multipliers $\exp(\pm 2\pi\omega_1 i), \exp(\pm 2\pi i \omega_2)$. Of course it is assumed that the system has been normalized through terms of order four. In general the coefficients A, B, C depend on the parameters and so assumption (2) need only be checked when $\mu = \nu = 0$. I have dropped the dependence of the coefficients on the parameters to simplify the discussion.

Introduce a small parameter ϵ and scale by $I_i \to \epsilon I_i$, $H \to \epsilon^{-1} H$, $\mu \to \epsilon\mu$, $\nu \to \epsilon\nu$. The Hamiltonian becomes

$$H = p^{-1} I_1 + q^{-1} I_2 + \frac{\epsilon}{2}(\mu I_1 + \nu I_2 + A I_1^2 + 2B I_1 I_2 + C I_2^2) + O(\epsilon^2) \tag{3}$$

The equations of motion are

$$\begin{array}{l}\dot{I}_1 = \partial H/\partial \phi_1 = \cdots \ , \\ \dot{I}_2 = \partial H/\partial \phi_2 = \cdots \ , \\ \dot{\phi}_1 = -\partial H/\partial I_1 = -p^{-1} + \epsilon(\mu + AI_1 + BI_2) + \cdots \ , \\ \dot{\phi}_2 = -\partial H/\partial I_2 = -q^{-1} + \epsilon(\nu + BI_1 + CI_2) + \cdots \ ,\end{array} \tag{3}$$

where here and below the dots stand for higher order terms (e.g. $O(\epsilon^2)$). Denote the period map by P and the pq^{th} iterate by $P' : (I_1, I_2, \phi_1, \phi_2) \to (I_1', I_2', \phi_1', \phi_2')$ where

$$\begin{array}{l}I_1' = I_1 + \cdots \ , \\ I_2' = I_2 + \cdots \ , \\ \phi_1' = \phi_1 + 2\pi q + 2\pi pq\epsilon(-\mu + AI_1 + BI_2) + \cdots \\ \phi_2' = \phi_2 + 2\pi p + 2\pi pq\epsilon(-\nu + BI_1 + CI_2) + \cdots \ .\end{array} \tag{4}$$

Seek to solve for the set where the angle variables do not change, namely solve

$$\begin{array}{c}(\phi_1' - \phi_1 - 2\pi q)/2\pi pq\epsilon = (-\mu + AI_1 + BI_2) + \cdots = 0 \\ \\ (\phi_2' - \phi_2 + 2\pi p)/2\pi pq\epsilon = (-\nu + \ \mathrm{BI}_1 + CI_2) + \cdots = 0.\end{array} \tag{5}$$

When $\epsilon = 0$ the right hand sides reduce to the linear equations

$$\begin{array}{c}AI_1 + BI_2 = \mu, \\ \\ BI_1 + CI_2 = \nu,\end{array} \tag{6}$$

so the assumption that $D = B^2 - AC \neq 0$ assures that these equations have a unique solution for I_1 and I_2, namely $I_1 = (B\nu - C\mu)/D, I_2 = (B\mu - A\nu)/\mathrm{D}$. The two line in the μ, ν-plane where $I_1 = 0$ and $I_2 = 0 (B\nu = C\mu, B\mu = A\nu)$ are lines through the origin which divides the parameter plane into four rectilinear sectors. In one of these sectors both I_1 and I_2 are positive. Let this sector be denoted by S'. For each point (μ, ν) in the interior of S' the solution set of (6) in $\mathbf{R}^4$ is a two torus and for each point on the boundary of S' the solution set of (6) in $\mathbf{R}^4$ is a circle. This is the analysis when $\epsilon = 0$, but the implicit function theorem guarantees that the essentially the same conclusions hold when ϵ is non-zero but small.

In particular by applying the implicit function theorem there are two smooth curves c_1 and c_2 in the μ, ν-parameter space which are tangent to the lines $B\nu = C\mu (I_1 = 0)$ and $B\mu = A\nu (I_2 = 0)$ at the origin. They divide a small disk about the origin into four sectors in the interior of one (denoted by S) the solutions to (5) have both I_1 and I_2 positive. On one boundary curve, $c_1, I_1 = 0$ and $I_2 > 0$ and on the other boundary curve, c_2, $I_1 > 0$ and $I_2 = 0$. Fix values of the parameters μ, ν in the interior or S so the solution of (5) is a two-torus in $\mathbf{R}^4$(denote by T^2). The mapping P' is symplectic and so the form

$$\omega = \sum_{i=1}^{2} (I_i' d\phi_i' - I_i d\phi_i) \tag{7}$$

is closed and in a disk about the origin in $\mathbf{R}^4$ it is exact. Thus there is a smooth real-valued function f defined on a disk about the origin in $\mathbf{R}^4$ such that $df = \omega$. On T, $d\phi' = d\phi$, and so on T^2 one has

$$d(f') = \sum_{i=1}^{2} (I'_i - I_i) d\phi_i, \tag{8}$$

where f' is the restriction of f to T^2).

At a critical of f' where $d(f') = 0$ it follows from (8) that $I'_i = I_i$. But by the definition of $T^2 : \phi'_i = \phi_i$. Therefore a critical point of f' is a fixed point of P' which of course is the initial condition for $a2pq\pi$-periodic solution of (3). Since T^2 is a torus f' must have at least three critical points (see Milnor(1963)) and so for μ, ν in the interior of S equations (3) have at least three $2pq\pi$-periodic solutions.

For μ, ν on the boundary of c_1 of S the solution set of (5) is a circle (denoted by T^1) and let $f' = f$ restricted to T^1. The function f' must have at least two critical points which correspond to periodic solutions of (3) of period $2pq\pi$. But since $I_1 = 0$ along these solutions their period must be $2q\pi$-periodic.

References.

Birkhoff, G. D. 1931:: Une généralizsation àn dimensions du dernier théorème de géométrie de Poincaré, **Comptes Rendus Acad. Sci. 192**, 196-198.

Meyer, K. R. and Hall, G. R. 1992:: Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, Springer-Verlag, New York.

Milnor, J. 1963:: Morse Theory, Princeton University Press, Princeton, NJ.

Poincaré, H. 1912:: Sur un théorème de Géométrie, **Rendiconti del Circolo de Matematico di Palermo 33**, 1912, 375.

Institute for Dynamics, Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221-0025