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TRAVELING WAVE SOLUTIONS FOR A REACTION DIFFUSION EQUATION WITH DOUBLE DEGENERATE NONLINEARITIES

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Abstract. This paper studies the traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities. The existence, uniqueness, asymptotics as well as the stability of the wave solutions are investigated. The traveling wave solutions, existed for a continuance of wave speeds, do not approach the equilibria exponentially with speed larger than the critical one. While with the critical speed, the wave solutions approach to one equilibrium exponentially fast and to the other equilibrium algebraically. This is in sharp contrast with the asymptotic behaviors of the wave solutions of the classical KPP and $m-th$ order Fisher equations. A delicate construction of super- and sub-solution shows that the wave solution with critical speed is globally asymptotically stable. A simpler alternative existence proof by LaSalle's Wazewski principle is also provided in the last section.

1. **Introduction.** We study the asymptotic behaviors and the stability of the traveling wave solutions of the reaction-diffusion equation

$$\begin{cases} u_t = u_{xx} + f(u), \\ u(x,0) = \psi(x), \end{cases} \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+ \tag{1}$$

with the nonlinear term f satisfying the following conditions :

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1. For some number $0 < \alpha < 1$, $f(s)$ is $C^{1,\alpha}$ on the interval $[0, 1]$,
2. $f(0) = f(1) = 0$,
3. $f'(0) = f'(1) = 0$,
4. $f'(s) > 0$, $f'(1-s) < 0$ for small $s > 0$,
5. $f > 0$ on $(0, 1)$.

The initial condition ψ will be specified later. The above conditions imply that $u = 0$ and $u = 1$ are two equilibria of equation (1), and that both equilibria are double degenerated. An example of (1) with the nonlinear term f satisfying conditions $(1) - (5)$ may be found, for example, in [20].

If we disregard the initial condition for a moment, a traveling wave solution of (1) has the form $u(x,t) := u(x + ct) \equiv u(\xi)$, $x + ct = \xi \in R$, and connects the equilibrium $u = 0$ and $u = 1$ as ξ goes from $-\infty$ to ∞. The constant c is the wave speed, and such solution also satisfies the following boundary value problem,

$$\begin{cases} u'' - cu' + f(u) = 0, \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{cases} \tag{2}$$

where prime denotes the differentiation with respect to ξ. Integrating (2) from $-\infty$ to $+\infty$, one immediately sees that $c > 0$ is a necessary condition for the existence of solutions. We will therefore assume $c > 0$ throughout this paper.

The proofs of the existence, uniqueness, as well as the stability properties of the traveling wave solutions are based on the *a priori* estimates of the wave solutions at infinities, namely, the asymptotic behaviors. Once one has control of the traveling wave solution at infinities, the comparison principle, the compactness argument can be applied to derive the above mentioned properties of the wave solutions. As is well known, the asymptotic behaviors of the wave solution depend on the properties of the nonlinear source term f. In the KPP-Fisher equation where $f(u) = u(1-u)$, the wave solution approaches the equilibria exponentially fast for any wave speed $c \geq 2$; however in the $m - th$ order Fisher equation or Zadovich equation where $f(u) = u^m(1-u)$, $m > 1$, the wave solution approaches to both equilibria exponentially with the critical/minimum speed, and algebraically to equilibrium $u = 0$ and exponentially to $u = 1$ with other wave speed. The criticality of the minimum speed has already been observed by KPP in their seminal paper [16] where the solution of the corresponding initial value problem (1) evolves naturally to the traveling wave solution with step initial values 0 at one end and 1 at the other end. It was proved again in [3] that such criticality extends to the more general Zodovich equation for any $m > 1$ which resolved a long time open problem on stability of the wave solutions. The traveling wave solutions have been shown in ([25], [26], [15], [14], [33], [22]) to be asymptotically stable in the exponentially weighted Banach spaces. Similar results have also been derived for equations in higher dimensions ([3], [5]) and monostable reaction diffusion systems ([28], [5], [31]). (Please see [5], [3], [24], [28], [29], [23] and the references therein for more developments and applications in Biology as well as in Physics and Chemistry). For the equation with double degenerated nonlinear source, less results are known, in particular, the asymptotic behaviors of the wave solutions and their stabilities are still remained unanswered ([4]). We should point out that in [23], Liang and Zhao estabished, among other things, that a more general existence result of the traveling wave solutions of system (1) with the wave speed c larger than or equal to the asymptotic spreading speed. Note that in [23] the requirement for system (1) to support such traveling wave solutions is $0 < f \in C^1([0,1],\mathbb{R})$ and $f(0) = f(1) = 0$. Please see [23] Theorem 2.17, Theorem 4.3 and Theorem 4.4 there for more details. We remark here the ideas and methods in this paper are different from those [23].

We are trying to study those questions in this paper. It is easy to see that one can approximate system (2) with the following system:

$$\begin{cases} u_\theta'' - c_\theta u_\theta' + f_\theta(u) = 0, \\ u_\theta(-\infty) = 0, \quad u_\theta(\infty) = 1 \end{cases} \tag{3}$$

where $0 < \theta < 1/4$, $f_\theta(s) = f \cdot \xi_\theta(s)$, and $\xi_\theta(s)$ is a cut-off function having the form:

$$\xi_\theta(s) = \begin{cases} 0, & s \in [0,\ \theta); \\ \text{a smooth positive function connecting 0 and 1}, & s \in (\theta,\ 2\theta); \\ 1, & s \in [2\theta,\ 1]. \end{cases} \tag{4}$$

The existence of traveling wave solutions of (3) is shown in ([6]). Noticing that as $\theta \to 0$, $f_\theta \to f$. Therefore, as long as the $u_\theta(\xi)$ is uniformly bounded in local $W^{2,p}$ $(p > 1)$ norm, non-vanishing, and that c_θ is bounded from below by zero and above by a positive number, the existence of the traveling wave solutions for (2) will follow by letting $\theta \to 0$ and a compactness argument. The key point here is the asymptotic estimate of the solutions of (3).

We then proceed to study the stability of the traveling wave solution. A spectral analysis reveals that the essential spectrum ([13]) of the linearized operator around the traveling wave solutions touches origin, which implies that in the exponentially weighted Banach spaces, the traveling wave solution of (1) is unstable or at most marginally stable. Following the ideas of [3], [10], by suitably constructing super- and sub-solutions of equation (1), we are able to show that for certain range of initial functions, the traveling wave solution with the critical wave speed is asymptotically stable. We remark that the stability of the traveling wave solution with non-critical speed of (1) has recently been studied in [22], by means of Evans function and linear spectral analysis in the exponential-polynomially weighted Banach spaces.

We now state the main results of this paper.

Theorem 1.1. *Suppose that f satisfies conditions (1)-(5), then system (1) has a unique traveling wave solution $u^*(\xi)$ (up to a translation of origin) with speed $c^* > 0$, and the wave speed is described by $\lim_{\theta\to 0} c_\theta = c^*$. The traveling wave solution is monotonically increasing in $\mathbb{R}$, and has the following asymptotic behaviors,*

$$u(\xi) = \begin{cases} H_1(\frac{1}{c^*}\xi(1 + o(1))), & \text{as } \xi \to +\infty, \\ Ae^{c^*\xi} + o(e^{c^*\xi}), & \text{as } \xi \to -\infty, \end{cases} \tag{5}$$

where $A > 0$ is a constant, $H_1 = F_1^{-1}$, $F_1 = \int_{u_0}^{u} \frac{ds}{f(s)}$, $u_0 < u < 1$ and $1 - u_0$ is small enough.

The next theorem indicates that the wave speed c^* obtained in Theorem 1.1 is minimal or critical.

Theorem 1.2. *Let c^* be as in Theorem 1.1, then for every $c > c^*$ (2) has a unique (up to a translation of the origin) solution. The solution is monotonically increasing on $\mathbb{R}$, and has the following asymptotic behaviors,*

$$u(\xi) = \begin{cases} H_1(\frac{1}{c}\xi(1+o(1))), & \text{as } \xi \to +\infty, \\ H_0(\frac{1}{c}\xi(1+o(1))), & \text{as } \xi \to -\infty, \end{cases} \tag{6}$$

where $H_0 = F_0^{-1}$, $F_0 = \int_{u_0}^{u} \frac{ds}{f(s)}$, $0 < u < u_0$ for some small u_0 , $H_1 = F_1^{-1}$, $F_1 = \int_{u_1}^{u} \frac{ds}{f(s)}$, $u_1 < u < 1$ and u_1 is close to 1.

Remark 1. More detailed asymptotic behaviors can be obtained via iterations, please refer to the proof of Lemma 2.3.

On the stability of the traveling wave solution with the critical speed we have,

Theorem 1.3. *There exist constants ξ_0, ζ_0, $q_0 \in \mathbb{R}$ and a constant $\bar{M} > 0$ sufficiently large, such that if initial value $\psi(x)$ satisfies*

$$0 \le \psi(x) \le Ae^{c^*(x-\xi_0)} + q_0 e^{\frac{c^*}{2}(x-\xi_0)}, \quad \text{as } x < -\bar{M}, \tag{7}$$

and

$$H_1(x+\zeta_0) - q_0 e^{-\frac{c^*}{2}(x+\xi_0)} \le \psi(x) \le H_1(x-\zeta_0) + q_0 e^{-\frac{c^*}{2}(x+\xi_0)}, \quad \text{as } x > \bar{M}, \tag{8}$$

then the solution $u(x,t)$ of the initial value problem (1) has the following property

$$u^*(\xi + \zeta(t) - \xi_1, t) \le u(x,t) \le u^*(\xi - \zeta(t) + \xi_2, t), \tag{9}$$

for every $t > 0$, where $\xi = x + c^* t$, $\zeta(t) = \zeta_0 e^{-\beta t}$, $\beta = \frac{c^{*2}}{8}$ and ξ_1, ξ_2 are two positive constants.

Letting $t \to +\infty$ in (9), we can see that $u(x,t) \to u^*(x,t)$ exponentially.

Remark 2. Noting the method used in the proof of this theorem in section 4 is different from that of [14], where the local stability of the traveling wave solution with the critical wave speed was studied. This is mainly due to the fact that the traveling wave solutions of (1) no longer have exponential decay at the positive infinity (please see section 2).

We will prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in Section 2 to Section 4 respectively, in the Appendix we will provide an alternative simpler and more direct existence proof of the traveling wave solutions. The proof works for KPP, $m-th$ order Fisher as well as the current nonlinearities. However, the proof does not lead to the asymptotic estimates of the traveling wave solutions.

2. **Existence of traveling wave solutions, proof of Theorem 1.1.**

In this section, we will show the existence of critical wave speed and the corresponding traveling wave solution. We first study the asymptotic behaviors of the traveling wave solutions of system (3), and based on which we will derive several comparison results. The existence of the traveling wave solutions of (1) will follow from those results and a compactness argument.

Lemma 2.1. *Let $u_\theta(\xi)$ be a traveling wave solution of system (3), and c_θ be its speed. Writing*

$$\phi(\xi) = \frac{u_\theta''(\xi)}{u_\theta'(\xi)}, \tag{10}$$

then

$$\lim_{\xi\to+\infty}\phi(\xi)=\lambda, \tag{11}$$

where λ *is a solution of the equation*

$$\lambda^2-c_\theta\lambda=0. \tag{12}$$

Proof. By [6], $u_\theta'(\xi)>0$ for $\xi\in\mathbb{R}$.

It is easy to see that $u_\theta'(\xi)$ satisfies the following equation

$$(u_\theta')''-c_\theta(u_\theta')'+f_\theta'(u_\theta)u_\theta'=0, \tag{13}$$

then it follows that

$$\frac{u_\theta'''}{u_\theta'}-c_\theta\frac{u_\theta''}{u_\theta'}+f_\theta'(u_\theta)=0.$$

By condition (3), we have

$$\frac{u_\theta'''}{u_\theta'}-c_\theta\frac{u_\theta''}{u_\theta'}=-f_\theta'(u_\theta)\to 0,\quad \text{as}\ \xi\to+\infty.$$

From (11),

$$\phi'(\xi)=\frac{u_\theta'''u_\theta'-(u_\theta'')^2}{(u_\theta')^2}=\frac{u_\theta'''}{u_\theta'}-\phi^2, \tag{14}$$

then one has

$$\begin{aligned}\phi'-c_\theta\phi+\phi^2 &= \frac{u_\theta'''}{u_\theta'}-\phi^2-c_\theta\phi+\phi^2\\ &= \frac{u_\theta'''}{u_\theta'}-c_\theta\frac{u_\theta''}{u_\theta'}\\ &= -f_\theta'(u_\theta)\to 0\end{aligned} \tag{15}$$

as $\xi\to+\infty$.

We claim that $\phi(\xi)$ is bounded as $\xi\to+\infty$.

Suppose that the claim is not true, then ϕ is unbounded, it is either monotone or oscillating as $\xi\to+\infty$. First we suppose that ϕ is monotone on the interval $(M,+\infty)$ for some M large, then one must have

$$\lim_{\xi\to+\infty}|\phi(\xi)|=+\infty,$$

consequently,

$$\lim_{\xi\to+\infty}\frac{1}{\phi(\xi)}=0. \tag{16}$$

On the other hand, by (15) one has

$$\begin{aligned}-\frac{f_\theta'(u_\theta)}{\phi^2} &= \frac{\phi'-c_\theta\phi+\phi^2}{\phi^2}\\ &= \frac{\phi'}{\phi^2}-c_\theta\frac{1}{\phi}+1,\end{aligned} \tag{17}$$

so (15) and (17) imply that as $\xi \to +\infty$,

$$(\frac{1}{\phi})' = -\frac{\phi'}{\phi^2} \to 1,$$

which contradicts (16).

We then assume $\phi(\xi)$ oscillating as $\xi \to +\infty$. Since ϕ is unbounded, we can choose a sequence ξ_i, $i \in N$ such that $\xi_i \to +\infty$ as $i \to +\infty$, and ϕ takes extreme values at ξ_i . Then the unboundedness of ϕ implies that $\lim_{i\to\infty} |\phi(\xi_i)| = \infty$. Since $\{\phi(\xi_i)\}$ are local extrema, $\phi'(\xi_i) = 0$ for $i \in N$. Then it follows that one has

$$\begin{aligned} \phi'(\xi_i) - c_\theta\phi(\xi_i) + \phi^2(\xi_i) &= -c_\theta\phi(\xi_i) + \phi^2(\xi_i) \\ &= \phi(\xi_i)(-c_\theta + \phi(\xi_i)) \to +\infty \end{aligned} \tag{18}$$

which is in contradiction with (15). Hence, $\phi(\xi)$ is bounded as $\xi \to +\infty$.

We next show that $\lim_{\xi\to+\infty} \phi(\xi)$ exists.

Let $\alpha = \limsup_{\xi\to+\infty} \phi(\xi)$, $\beta = \liminf_{\xi\to+\infty} \phi(\xi)$. Since $\phi(\xi)$ is bounded, α and β are well defined and $\alpha \geq \beta$. Choosing sequences M_i and N_i, $i \in N$ such that $\phi(M_i)$ and $\phi(N_i)$ are the local maximum and minimum of $\phi(\xi)$ respectively, and M_i, $N_i \to +\infty$ as $i \to +\infty$, then one has

$$\begin{aligned} &\phi'(M_i) - c_\theta\phi(M_i) + \phi^2(M_i) \\ &\quad = -c_\theta\phi(M_i) + \phi^2(M_i) \to -c_\theta\alpha + \alpha^2 = 0, \end{aligned}$$

and

$$\phi'(N_i) - c_\theta\phi(N_i) + \phi^2(N_i) = -c_\theta\beta + \beta^2 = 0,$$

then it follows that both α and β are solutions of (12). We claim that $\alpha = \beta$. In fact, if the claim is not true, then one must have $\alpha = c_\theta$ and $\beta = 0$. Choosing ϵ small enough and a sequence r_i with $r_i \to +\infty$, such that $\phi(r_i) = \alpha - \epsilon/2$ and $\phi'(r_i) \leq 0$, then

$$\phi'(r_i) - c_\theta\phi(r_i) + \phi^2(r_i) \leq (\alpha - \frac{\epsilon}{2})(\alpha - \frac{\epsilon}{2} - c_\theta) < 0$$

as $r_i \to +\infty$. This contradicts (15). Therefore, $\alpha = \beta$ and $\lim_{\xi\to+\infty} \phi(\xi)$ exists. □

Lemma 2.2. *Let λ be defined as in (11), then $\lambda = 0$.*

Proof. We first prove that for any constants A, $d > 0$, if ξ is large enough then $u_\theta(\xi) \leq 1 - Ae^{-d\xi}$.

By conditions (2) and (3), there exists ξ_0 large enough, such that for $s \in [u_\theta(\xi_0), 1]$ and a constant $d > 0$ (to be determined later),

$$f_\theta(s) \leq (d^2 + c_\theta d)(1 - s)$$

Now consider equation

$$u'' - c_\theta u' + (d^2 + c_\theta d)(1 - u) = 0 \tag{19}$$

Clearly, for any $A > 0$, $1 - Ae^{-d\xi}$ is a solution of (19) while $u_\theta(\xi)$ is a subsolution. Choosing $A > 0$ small enough, we have

$$u_\theta(\xi_0) \leq 1 - Ae^{-d\xi_0}.$$

By Maximum Principle, we have

$$u_\theta(\xi) \le 1 - Ae^{-d\xi} \tag{20}$$

for $\xi \in (\xi_0, +\infty)$.

However, (11) and (12) imply that if $\lambda = c_\theta$, there exists a constant $\bar{A} > 0$, such that

$$u_\theta(\xi) = 1 - \bar{A}e^{-c_\theta \xi} + o(e^{-c_\theta \xi})$$

as ξ is large enough.

Choosing $d = c_\theta/2$ in (20), then one would have

$$1 - \bar{A}e^{-c_\theta \xi} + o(e^{-c_\theta \xi}) = u_\theta(\xi) \le 1 - \bar{A}e^{-\frac{c_\theta}{2}\xi},$$

for ξ large enough, which is impossible. This implies that $\lambda = 0$. □

We next derive the asymptotic behaviors of $u_\theta(\xi)$ at $\xi = +\infty$.

Lemma 2.3. *The traveling wave solution u_θ with speed c_θ of (3), has the following asymptotic behavior as ξ is close to $+\infty$,*

$$u_\theta(\xi) = H_1(\frac{1}{c_\theta}\xi(1 + o(1))).$$

Proof. Since $u'_\theta(\xi) > 0$ for $\xi \in R$, then one has

$$\frac{u''_\theta}{u'_\theta} - c_\theta + \frac{f_\theta(u_\theta)}{u'_\theta} = 0,$$

or equivalently,

$$\frac{u'_\theta}{f_\theta(u_\theta)} = \frac{1}{c_\theta - \dfrac{u''_\theta}{u'_\theta}}. \tag{21}$$

By Lemma 2.2, one has as $\xi \to +\infty$, $\frac{u''_\theta}{u'_\theta} \to 0$, then (21) implies by simple integration that

$$u_\theta(\xi) = H_\theta(\frac{1}{c_\theta + o(1)}\xi), \text{ and } u'_\theta = f_\theta(H_\theta(\frac{1}{c_\theta + o(1)}\xi)). \tag{22}$$

To gain further insight into (22), we rewrite (21) as

$$\begin{aligned}\frac{u'_\theta}{f_\theta(u_\theta)} &= \frac{1}{c_\theta}\frac{1}{1 - \dfrac{1}{c_\theta}\dfrac{u''_\theta}{u'_\theta}} \\ &= \frac{1}{c_\theta}(1 + \frac{1}{c_\theta}\frac{u''_\theta}{u'_\theta} + o(\frac{u''_\theta}{u'_\theta})).\end{aligned}$$

Consequently, there exists a ξ_0 large enough (we note that ξ_0 may depend on θ but it causes no problem in the convergence proof below), such that for $\xi > \xi_0$,

$$\int_{\xi_0}^{\xi} \frac{u'_\theta}{f(u_\theta)} ds = \int_{\xi_0}^{\xi} \frac{1}{c_\theta}(1 + \frac{1}{c_\theta}\frac{u''_\theta}{u'_\theta} + o(\frac{u''_\theta}{u'_\theta}))ds,$$

then

$$\begin{aligned}\int_{\xi_0}^{\xi}\frac{u_\theta'}{f(u_\theta)}ds &= \int_{\xi_0}^{\xi}\frac{1}{c_\theta}(1+\frac{1}{c_\theta}\frac{u_\theta''}{u_\theta'}+o(\frac{u_\theta''}{u_\theta'}))ds\\ &= \frac{1}{c_\theta}((\xi-\xi_0)+\frac{1}{c_\theta}\ln\frac{u_\theta'(\xi)}{u_\theta'(\xi_0)}+o((\frac{u_\theta''}{u_\theta'})(\xi-\xi_0)))\\ &= \frac{1}{c_\theta}(\xi-\xi_0)(1+O(\frac{1}{\xi-\xi_0}\ln f_\theta(H_\theta(\frac{1}{c_\theta+o(1)}\xi)))).\end{aligned}$$

Writing $F_\theta(u_\theta)=\int_{u_\theta(\xi_0)}^{u_\theta(\xi)}\frac{ds}{f(s)}$, since $F_\theta'(u_\theta)=\frac{1}{f_\theta(u_\theta)}>0$, we see that F_θ is invertible, let $F_\theta^{-1}=H_\theta$. In addition since $\frac{1-s}{f(s)}\to\infty$ as $s\to 1$ we have

$$\frac{1}{\xi-\xi_0}\ln f_\theta(H_\theta(\frac{1}{c_\theta+o(1)}\xi))\to 0 \text{ as } \xi\to+\infty,$$

and hence

$$u_\theta(\xi)=H_\theta(\frac{1}{c_\theta}(\xi-\xi_0)(1+o(1))).$$

After a shifting of the origin, we can obtain

$$u_\theta(\xi)=H_\theta(\frac{1}{c_\theta}\xi(1+o(1))),$$

for ξ large enough.

It is easy to see that with iteration we could show after a shifting of the origin that, for example:

$$u_\theta(\xi)=H_\theta(\frac{1}{c_\theta}\xi(1+\frac{1}{c_\theta}\frac{1}{\xi}\ln f_\theta(H_\theta(\frac{1}{c_\theta}\xi))(1+o(1)))). \tag{23}$$

Note that the Lemma is proved since $f_\theta\equiv f$ near 1. □

Remark 3. The traveling wave solution $u_\theta(\xi)$ of (3) has the following asymptotic behavior at $-\infty$,

$$u_\theta(\xi)=Be^{c_\theta\xi}$$

where $B>0$ is a constant, see ([6]).

Next we show that as θ decreases to 0, u_θ approaches a solution $u(\xi)$ of (2). To this end, we will need the following comparison lemmas.

Lemma 2.4. *Let g be any $C^{1,\alpha}$ function and $a>0$ be a constant. Assume that functions v and z satisfy*

$$\begin{cases} v''-cv'+g(v) \ge 0,\\ z''-cz'+g(z) \le 0,\end{cases} \quad on\ [-a,a],$$

respectively, and that

$$\begin{cases} v(-a) < z(\xi), & \xi\in(-a,a];\\ v(\xi) < z(a), & \xi\in[-a,a),\end{cases}$$

then

$$v(\xi)\le z(\xi),\qquad \xi\in(-a,a).$$

Furthermore, if

$$z(a)>v(a),\quad z(-a)>v(-a),$$

then

$$v(\xi) < z(\xi), \qquad \xi \in [-a, a].$$

Proof. The lemma was proved in [4] by Sliding domain method. □

Lemma 2.5. *Suppose that f satisfies conditions (1)-(5), let $\theta_1 < \theta_2$ and u_{θ_i} be the solutions of (3) with speed c_{θ_i}, $i = 1, 2$, then $c_{\theta_1} \geq c_{\theta_2}$.*

Proof. Suppose that there exist θ_1 and θ_2, with $0 < \theta_1 < \theta_2$, but $c_{\theta_1} < c_{\theta_2}$.

From Lemma 2.3, for ξ large enough, u_{θ_i}, $i = 1$, 2 can be written as

$$\begin{aligned} u_{\theta_1}(\xi) &= H(\frac{1}{c_{\theta_1}}\xi(1 + o(1))), \\ u_{\theta_2}(\xi) &= H(\frac{1}{c_{\theta_2}}\xi(1 + o(1))), \end{aligned} \tag{24}$$

noting that the same H is used since $f_{\theta_1}(u)$ and $f_{\theta_2}(u)$ agree for u near 1.

By Remark 3, we have as ξ is close to $-\infty$ that there exist constants A, $B > 0$ such that

$$\begin{aligned} u_{\theta_1}(\xi) &= Ae^{c_{\theta_1}\xi}, \\ u_{\theta_2}(\xi) &= Be^{c_{\theta_2}\xi}. \end{aligned} \tag{25}$$

We further normalize $u_{\theta_1}(\xi)$ and $u_{\theta_2}(\xi)$ such that

$$u_{\theta_1}(0) = u_{\theta_2}(0) = \frac{1}{2}. \tag{26}$$

Given $c_{\theta_1} < c_{\theta_2}$, the dependence of H on c_θ and (22) imply that for ξ large enough,

$$u_{\theta_1}(\xi) > u_{\theta_2}(\xi). \tag{27}$$

Then from (23) and (25), there exists a $T > 0$ large enough, such that

$$u_{\theta_1}(-T) > u_{\theta_2}(-T),$$

and

$$u_{\theta_1}(T) > u_{\theta_2}(T).$$

Noting that $f_{\theta_1} > f_{\theta_2}$, one has

$$\begin{cases} u''_{\theta_1} - c_{\theta_1}u'_{\theta_1} + f_{\theta_2}(u_{\theta_1}) \leq 0, \\ u''_{\theta_2} - c_{\theta_2}u'_{\theta_2} + f_{\theta_2}(u_{\theta_2}) = 0, \end{cases} \quad \text{on} \quad [-T, T].$$

The monotonicity of $u_{\theta_i}(\xi)$, $i = 1, 2$ and Lemma 2.4 imply

$$u_{\theta_2}(\xi) < u_{\theta_1}(\xi) \text{ on } (-T, T),$$

which contradicts with the normalization condition (26). □

Lemma 2.6. *The wave speed c_θ has an upper bound independent on θ as $\theta \to 0$.*

Proof. The idea of the proof is similar to section 8 of [4]. Let θ be chosen such that $0 < \theta < 1/4$. Normalizing $u_\theta(\xi)$ such that $u_\theta(0) = 1/2$. We will compare the solution of (3) and c_θ with those of a KPP-Fisher equation

$$\begin{cases} w'' - cw' + kw(1-w) = 0, \\ w(-\infty) = 0,\ w(+\infty) = 1, \end{cases} \tag{28}$$

where in equation (28), we choose the positive constant k such that $ks(1-s) > f_\theta(s)$, for $s \in (0,\ 1)$. We can normalize $w(\xi)$ and $u_\theta(\xi)$ such that

$$w(0) = u_\theta(0) = \frac{1}{2} \tag{29}$$

Recalling that ([25], [11], [16]) system (28) has a solution $w(\xi)$, $\xi \in R$ for every $c \geq 2\sqrt{k}$, and the solution satisfies that $w'(\xi) > 0$. The solution $w(\xi)$ has the following asymptotic behaviors:

$$w(\xi) = A_w e^{\frac{c-\sqrt{c^2-4k}}{2}\xi} + o(e^{\frac{c-\sqrt{c^2-4k}}{2}\xi}), \quad \text{as } \xi \leq -N; \tag{30}$$

and

$$w(\xi) = 1 - B_w e^{\frac{c-\sqrt{c^2+4k}}{2}\xi} + o(e^{\frac{c-\sqrt{c^2+4k}}{2}\xi}), \quad \text{as } \xi \geq N, \tag{31}$$

where in (30), (31), A_w and B_w are two positive numbers.

Since $\frac{c-\sqrt{c^2-4k}}{2} \to 0$ as $c \to \infty$, we can choose a sufficiently large c, such that $0 < \frac{c-\sqrt{c^2-4k}}{2} < c_{\frac{1}{4}} \leq c_\theta$. Then Remark 3 implies

$$u_\theta(\xi) < w(\xi), \text{ as } \xi \leq -N, \tag{32}$$

and by Lemma 2.3, we also have

$$u_\theta(\xi) < w(\xi), \text{ as } \xi \geq N. \tag{33}$$

We then consider (3) and (26) on the interval $(-N,\ N)$. We claim that $c_\theta \leq c$. Suppose on the contrary $c_\theta > c$, then one would have

$$w'' - c_\theta w' + f_\theta(w) < 0. \tag{34}$$

Inequality (32) along with (30) and (31) as well as Lemma 2.6 imply that $w(\xi) > u_\theta(\xi)$ on $[-N,\ N]$, which contradicts (29).

Since c is fixed and does not depend on θ, the Lemma is then proved. ∎

Lemma 2.7. *Suppose that f satisfies conditions (1)-(5), then there exist a $c^* > 0$ and a function $u^*(\xi) > 0$, $\xi \in \mathbb{R}$ such that*

$$\begin{cases} (u^*)'' - c^*(u^*)' + f(u^*) = 0, \\ u^*(-\infty) = 0,\ u^*(+\infty) = 1. \end{cases} \tag{35}$$

Proof. Choosing a decreasing sequence $\{\theta_i\}$, $i = 1, 2, 3, \cdots$, such that $\theta_i \to 0$ as $i \to +\infty$ and normalizing $u_{\theta_i}(\xi)$ such that $u_{\theta_i}(0) = \frac{1}{2}$. As $i \to \infty$, we have, by Lemma 2.3 and Lemma 2.5, that

$$c^* = \lim_{i \to +\infty} c_{\theta_i}.$$

For each $i = 1, 2, 3...$, u_{θ_i} solves boundary value problem

$$\begin{cases} u''_{\theta_i} - c_{\theta_i} u'_{\theta_i} + f_{\theta_i}(u_{\theta_i}) = 0, \\ \\ u_{\theta_i}(-\infty) = 0, \ u_{\theta_i}(+\infty) = 1. \end{cases}$$

Normalizing u_{θ_i} such that $u_{\theta_i}(0) = 1/2$, we then consider sequence $\{u_{\theta_i}\}_{i=1}^{\infty}$ on the interval $[-a,\ a]$ with $a > 0$ large. From Schauder estimates, on the interval $[-a,\ a]$,$\{u_{\theta_i}\}_{i=1}^{\infty}$ has a subsequence $\{u_{\theta_{i_1}}\}_{i_1=1}^{\infty}$ convergent to a solution of the initial value problem

$$\begin{cases} u'' - c^* u' + f(u) = 0, \ \text{on } [-a, a], \\ \\ u(\frac{1}{2}) = 0, \end{cases}$$

We further consider sequence $\{u_{\theta_{i_1}}\}_{i_1=1}^{\infty}$ on the interval $[-2a,\ 2a]$, the Schauder estimates again imply that $\{u_{\theta_{i_1}}\}_{i_1=1}^{\infty}$ has a convergent subsequence $\{u_{\theta_{i_2}}\}_{i_2=1}^{\infty}$ on $[-2a,\ 2a]$ and the limiting function solves the initial value problem

$$\begin{cases} u'' - c^* u' + f(u) = 0, \ \text{on } [-2a,\ 2a], \\ \\ u(\frac{1}{2}) = 0, \end{cases}$$

Iterating the above process, we will have for each $n \in \mathbb{Z}$, there is a subsequence $\{u_{\theta_{i_n}}\}$ of $\{u_{\theta_i}\}$ which is uniformly convergent to a solution of the corresponding initial value problem on the interval $[-na,\ na]$.

Extracting the diagonal sequence $\{u_{\theta_{ii}}\}_{i=1}^{\infty}$ and Letting $n \to \infty$, there exists a function $u^*(\xi)$ that solves the equation

$$\begin{cases} (u^*)'' - c^*(u^*)' + f(u^*) = 0, \quad \xi \in \mathbb{R}, \\ \\ (u^*)(0) = \frac{1}{2}. \end{cases} \tag{36}$$

Furthermore since $u'_{\theta_i} > 0$, we then have $(u^*)' \geq 0$. Also for each i, $0 \leq u_{\theta_i} \leq 1$, we have $0 \leq u^*(\xi) \leq 1$, $\xi \in R$.

By condition (1), $u(\xi) \equiv \frac{1}{2}$ is not a constant solution of (2), and $u^*(-\infty) = \lim_{\xi \to -\infty} u^*(\xi) \leq \frac{1}{2} \leq \lim_{\xi \to \infty} u^*(\xi) = u^*(\infty)$. Since $u^*(-\infty)$ and $u^*(+\infty)$ must be equilibrium points of f , then $u^*(-\infty) = 0$, $u^*(\infty) = 1$. We can integrate (36) to conclude that $c^* > 0$. □

Lemma 2.8. *Corresponding to the wave speed c^*, the traveling wave solution u^* has the following asymptotic behaviors,*

$$u^*(\xi) = \begin{cases} Ae^{c^*\xi} + o(e^{c^*\xi}); \quad as \quad \xi \to -\infty, \\ \\ H_1(\frac{1}{c^*}\xi(1 + o(1))), \quad as \quad \xi \to +\infty, \end{cases} \tag{37}$$

where $A > 0$ is some constant.

Proof. One can repeat the same proof as in Lemma 2.1 and Lemma 2.2 to conclude that as $\xi \to +\infty$,

$$u^*(\xi) = H(\frac{1}{c^*}\xi(1 + o(1))).$$

To derive the asymptotic behaviors of $u^*(\xi)$ at $\xi = -\infty$, we first show

$$\lim_{\xi\to-\infty}\frac{(u^*)''(\xi)}{(u^*)'(\xi)} = c^*. \tag{38}$$

The same proofs of Lemma 2.1 and Lemma 2.2 can be carried over to show

$$\lim_{\xi\to-\infty}\frac{(u^*)''(\xi)}{(u^*)'(\xi)} = \lambda,$$

where λ is a solution of $\lambda^2 - c^*\lambda = 0$.

We next show $\lambda \neq 0$. Suppose on the contrary, one has $\lim_{\xi\to-\infty}\frac{(u^*)''(\xi)}{(u^*)'(\xi)} = 0$, then for any ϵ with $0 < \epsilon < c_{\frac{1}{4}}/2$, there exists a $M > 0$ such that as $\xi \in (-\infty, -M]$,

$$-\epsilon < \frac{(u^*)''(\xi)}{(u^*)'(\xi)} < \epsilon.$$

Choosing a sequence $\{\theta_i\}_{i=1}^{\infty}$ with $0 < \theta_i < \frac{1}{4}$ and $\lim_{i\to+\infty}\theta_i = 0$ as in the proof of Lemma 2.7, then the convergence of $(u_{\theta_i},\, c_{\theta_i}) \to (u^*,\, c^*)$ as $i \to \infty$ implies that

$$-2\epsilon < c_{\theta_i} < 2\epsilon < c_{\frac{1}{4}},$$

a contradiction to Lemma 2.7.

Then $\lambda = c^*$, integrating (38), we have the desired conclusion. □

Next, we show the uniqueness of the wave speed c^* derived in Lemma 2.7. The uniqueness is understood in the sense that c^* is the only wave speed such that the corresponding wave solution decays exponentially at $-\infty$; and there is only one (modulo translation) traveling wave solution $u^*(\xi)$ associated with such wave speed.

Lemma 2.9. *The wave speed c^* is uniquely determined by (2).*

Proof. We first show that c^* is the only wave speed such that the corresponding traveling wave solution decays exponentially at $-\infty$.

Suppose on the contrary that u_1, u_2 are two traveling wave solutions of (2) with speed $c_1 > c_2$, and that both u_1 and u_2 decay exponentially at $\xi = -\infty$. Then there are two positive constants B_1 and B_2, such that

$$\begin{aligned} u_1(\xi) &= B_1 e^{c_1\xi} + o(e^{c_1\xi}),\\ u_2(\xi) &= B_2 e^{c_2\xi} + o(e^{c_2\xi}), \end{aligned} \tag{39}$$

We normalize u_1 and u_2 such that $u_1(0) = u_2(0) = \frac{1}{2}$

At $\xi = +\infty$, u_1 and u_2 have the following asymptotic behaviors

$$\begin{aligned} u_1(\xi) &= H_1(\frac{1}{c_1}\xi(1+o(1))),\\ u_2(\xi) &= H_1(\frac{1}{c_2}\xi(1+o(1))). \end{aligned} \tag{40}$$

It can be shown that both u_1 and u_2 are monotonically increasing.

It is clear that

$$u_1(\xi) < u_2(\xi)$$

for $|\xi| \to +\infty$. Then there exists a constant $M > 0$ such that for $\xi \in (-\infty, -M] \cup [M, +\infty)$,

$$u_1(\xi) < u_2(\xi).$$

Now we shift u_1 to the left. By the monotonicity of u_1 and u_2 with respect to ξ, we only need to shift u_1 to the left at most $2M$ units such that $u_1(\xi) < u_2(\xi)$ on $(-M, M)$. We note that the inequality holds even for $\xi \in (-\infty, \infty)$.

We shift u_1 back until it first touches u_2 at some finite point ξ_1 (because $u_1 < u_2$ at $\pm\infty$ no matter how much shift by (39), (40)). At this time, we have

$$u_1(\xi) \leq u_2(\xi)$$

and also we have by $u_2' > 0$,

$$u_1'' - c_1 u_1' + f(u_1) = 0,$$

$$u_2'' - c_1 u_2' + f(u_2) < 0.$$

Let $w(\xi) = u_1(\xi) - u_2(\xi)$, then w satisfies

$$\begin{cases} w'' - c_2 w' + f(u_1) - f(u_2) > 0, \\ w(-\infty) = 0, \ \ w(+\infty) = 0, \\ w \geq 0 \text{ for all } \xi \in R \text{ and } w(\xi_1) = 0. \end{cases}$$

Notice that

$$f(u_1) - f(u_2) \quad = \quad f'(\eta(\xi))(u_1 - u_2) = f'(\eta(\xi))w.$$

By the strong Maximal Principle for non-negative solutions, we have

$$w \equiv 0, \quad \text{for} \quad \xi \in \mathbb{R}.$$

This implies

$$u_1 \equiv u_2, \quad \text{for} \quad \xi \in \mathbb{R}.$$

A contradiction. □

Proof of Theorem 1.1: See Lemma 2.7 and Lemma 2.8.

3. **The range of wave speed, proof of Theorem 1.2.** In this section, we show that c^* is the minimal wave speed. To this end, we will show for every $c \geq c^*$ that system (2) has a wave solution, while for every $0 < c < c^*$, system (2) does not have any positive wave solution.

Lemma 3.1. *For any $0 < c < c^*$, (2) does not have any positive solution.*

Proof. We will use Lemma 2.4 to prove the conclusion. In order to verify the boundary conditions of Lemma 2.4, we first show that if $u(\xi)$ is any traveling wave solution of (2) with speed $c > 0$, then $u'(\xi) > 0$, note here we do not put any restriction on c. We have on the one hand that any positive wave solution of (2) with speed $c > 0$ must be monotone, on the other hand that it is impossible to have a monotone wave for any $0 < c < c^*$. This contradiction will lead us to the conclusion of this Lemma.

Since (2) is translation invariant, then for any $r > 0$, $u^r(\xi) = u(\xi + r)$ is also a solution of (2) with boundary conditions $u^r(-\infty) = 0$, $u^r(+\infty) = 1$. Similar to the proofs of Lemma 2.1 and Lemma 2.2, we have as $\xi \to -\infty$, either there exists a $\tilde{A} > 0$ such that

$$u(\xi) = \tilde{A}e^{c\xi} + o(e^{c\xi}), \tag{41}$$

for a constant $\tilde{A} > 0$, or

$$u(\xi) = H_0(\frac{1}{c}\xi(1 + o(1))) \ . \tag{42}$$

As $\xi \to +\infty$, $u(\xi)$ has the similar asymptotics, i.e., either there exists a $\tilde{B} > 0$, such that

$$u(\xi) = 1 - \tilde{B}e^{-c\xi} + o(e^{-c\xi}), \tag{43}$$

or

$$u(\xi) = H_1(\frac{1}{c}\xi(1 + o(1))). \tag{44}$$

In case of either (43) or (44), we have as in (21)

$$\frac{u'}{f(u)} \to \frac{1}{c}, \text{ which implies that } u' > 0 \text{ there.} \tag{45}$$

It then follows from (41)-(44) that, for any $r > 0$, there exists a $M > 0$, such that

$$u^r(\xi) > u(\xi), \quad \xi \in (-\infty, -M] \cup [M, +\infty). \tag{46}$$

We next show

$$u^r(\xi) > u(\xi), \quad \xi \in (-M, M). \tag{47}$$

Suppose that (47) is not true, then there exists a $\xi_0 \in (-M, M)$ such that

$$u^r(\xi_0) = u(\xi_0),$$

we then shift $u^r(\xi)$ to the left, that is, we increase r. There exists a $\bar{r} \geq r$, such that

$$u^{\bar{r}}(\xi) > u(\xi), \ \xi \in [-M, \ M - \bar{r}]. \tag{48}$$

By (46), $\bar{r}$ is at most $2M$, so (47) holds.

We next shift $u^{\bar{r}}(\xi)$ back, that is we decrease $\bar{r}$, until one of the following situation happens first:

Case (a). there exists a $\bar{r}_1$, such that $u^{\bar{r}_1}(\xi) \equiv u(\xi)$ on the interval $[-M, \ M]$, then we can apply Maximum Principle to $w(\xi) = u^{\bar{r}_1}(\xi) - u(\xi)$ on R to conclude that $u(\xi) \equiv u^{\bar{r}}(\xi)$, for any $\xi \in R$, which is in contradiction with (46).

Case (b). there exists a $\bar{r}_2$ and a $\xi_1 \in (-M, \ M)$, such that $u^{\bar{r}_2}(\xi_1) \equiv u(\xi_1)$, while at the same time $u^{\bar{r}_2}(\xi) > u(\xi)$, for $\xi \in (-M, M)$, $\xi \neq \xi_1$. The same proof as in Case (a) implies that this case is also impossible. We can then decrease $\bar{r}$ further.

In either case, since we can decrease $\bar{r}$ at most to 0, then necessarily, one has $u^r(\xi) > u(\xi)$ for $\xi \in (-M, \ M)$. This implies $u'(\xi) > 0$.

We then show that for any $0 < c < c^*$, (1.2) does not have monotone solutions.

Suppose on the contrary, for some $0 < c < c^*$, $u(\xi)$ is a solution of (2), then similar to the proof of Lemma 2.1 and Lemma 2.2, one gets to

$$\frac{u''(\xi)}{u'(\xi)} \to \lambda, \quad as \ |\xi| \to \infty$$

then either $\lambda = 0$ or $\lambda = c$, so there exists a $M > 0$, such that for $\xi \in (-\infty, -M] \cup [M, +\infty)$, one has

$$u(\xi) > u^*(\xi). \tag{49}$$

We next show (49) is true for all $\xi \in R$.

We consider $u(\xi)$ and $u^*(\xi)$ on the interval $[-M,\ M]$, suppose there is a $\bar{\xi} \in (-M,\ M)$ such that $u(\bar{\xi}) = u^*(\bar{\xi})$ then we can shift $u^*(\xi)$ to the right until we have $u(\xi) > u^*(\xi)$, $\xi \in (-M, M)$. Then, $u(\xi) > u^*(\xi)$,$\xi \in R$.

Now we shift $u^*(\xi)$ back until it first touches $u(\xi)$ at some ξ_0 and we have $u^*(\xi) \leq u(\xi)$, by maximum principle, we have $u(\xi) \equiv u^*(\xi)$, $\xi \in R$, a contradiction. □

Lemma 3.2. *Assuming the conditions of Theorem 1.1, then for every $c > c^*$ with c^* given by Theorem 1.1, system*

$$\begin{cases} u'' - cu' + f(u) = 0, \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{cases}$$

has a solution.

Proof. Step 1, we denote the solution obtained in Theorem 1.1 as $(u^*,\ c^*)$ and we have, by Theorem 1.1, $(u^*)' > 0$ for $\xi \in R$. Hence, for any $c > c^*$,

$$(u^*)'' - c(u^*)' + f(u^*) < 0.$$

This shows for $c > c^*$ that u^* is a super-solution of

$$u'' - cu' + f(u) = 0. \tag{50}$$

Also, for any $0 < h < 1$, we have $f(h) > 0$, then it follows that h is a sub-solution of (49). Now fix a constant $a \geq 1$, we choose $h \leq u^*(-a)$, therefore, there exists a function v, such that

$$\begin{cases} v'' - cv' + f(v) = 0, \\ v(-a) = h, \quad v(a) = u^*(a), \end{cases}$$

and v may be obtained by monotone iteration and we have $h \leq v \leq u^*$ on $[-a,\ a]$ and $v' > 0$ on $(-a,\ a)$.

Step 2. Now we shift u^* and let $u^{*r} = u^*(\xi + r)$ and $h^r = u^*(-a + r)$, then by step 1, there exists v^r, such that

$$\begin{cases} (v^r)'' - c(v^r)' + f(v^r) = 0, \\ v^r(-a) = h, \quad v^r(a) = u^*(a). \end{cases}$$

Uniqueness of the solution of the above equations implies v^r depending continuously on r. Since $(u^*)' > 0$, then v^r is a sub-solution corresponding to any $r' > r$ and then we have $v^{r'} > v^r$, let $r \to \infty$, we have $v^r \to 1$ by $u^* \to 1$ and $v^r \to 0$ as $r \to -\infty$. Then there exists some $r = \bar{r}$ such that $v^{\bar{r}}(0) = \frac{1}{2}$.

Fix $r = \bar{r}$ and let u^a be the solution of

$$\begin{cases} (v^{\bar{r}})'' - c(v^{\bar{r}})' + f(v^{\bar{r}}) = 0, & \xi \in [-a,\ a], \\ 0 < v^{\bar{r}} < 1,\ (v^r)' > 0, \\ v^{\bar{r}}(0) = \frac{1}{2}. \end{cases}$$

By $L^p\,(p > 1)$ estimates, there exists a sequence $a_j \to +\infty$, such that $u^{a_j} \to u$ on any compact set of $[-a, a]$ uniformly and u satisfies

$$\begin{cases} u'' - cu' + f(u) = 0, \\ u' \geq 0, \qquad u(0) = \frac{1}{2}. \end{cases}$$

Then, $u(\xi) \neq 0$, $u(\xi) \neq 1$, so $u(\xi)$ is not a constant and by taking limit as $\xi \to \pm\infty$,we have $u(-\infty) = 0$, $u(+\infty) = 1$. ☐

Lemma 3.3. *Suppose that $(u,\ c)$ is a traveling wave solution of (2) with speed $c > c^*$, then*

$$\begin{cases} u(\xi) = H_1(\frac{1}{c}\xi(1 + o(1))) & \text{as } \xi \to +\infty, \\ u(\xi) = H_0(\frac{1}{c}\xi(1 + o(1))) & \text{as } \xi \to -\infty. \end{cases}$$

where H_0, H_1 are defined by Theorem 1.2.

Proof. See Lemma 2.2 for the asymptotic behavior of $u(\xi)$ when $\xi \to +\infty$, Lemma 2.8 implies that $u(\xi)$ does not decay exponentially at $\xi \to -\infty$. ☐

Proof of Theorem 1.2: For the existence of the wave solution, see Lemma 2.8 and Lemma 2.9. For the uniqueness of the traveling wave solutions for each wave speed $c > c^*$, one can adapt the proof of Lemma *2.9* to the current situation.

4. **Stability, proof of Theorem 1.3.** In this section, we prove Theorem 1.3, the stability of the traveling wave solution with the critical speed c^*. Though the construction of the super- and sub-solution appears to be similar to that of [10], the technical details are markedly different.

Let

$$q(\xi, t) = q_0 e^{-\beta t} \min\{e^{\eta(\xi - \xi_0)}, e^{-\eta(\xi + \xi_0)}\}, \tag{51}$$

and

$$\zeta(t) = \zeta_0 e^{-\beta t}, \tag{52}$$

where $\eta = c^*/2$, $\beta = (c^*)^2/8$, and the constants q_0, ξ_0 and ζ_0 will be determined in the following.

It follows from conditions (2) and (3) that there exists a $\tau_0 > 0$ sufficiently small, such that

$$\sup_{0<\tau<\tau_0 \text{ or } 1-\tau_0<\tau<1} |f(\tau)| \leq c^*\eta - \eta^2 - \beta = \frac{(c^*)^2}{8}, \tag{53}$$

and also there exists $M_0 > 0$ large enough, such that $u^*(-M_0) = \tau_0/2$. We fix $q_0 = u^*(-M_0) = \tau_0/2$, $\zeta_0 = M_0/2$. It is not hard to show that

$$e^{\eta\xi}(u^*(\xi))' \to \infty, \qquad \text{as } \xi \to +\infty. \tag{54}$$

We then let

$$\min_{-M_0-\zeta_0\le\xi\le M_0+\zeta_0}(e^{\eta\xi}, e^{-\eta\xi}) = \bar{M},$$

and

$$\min_{-M_0-\zeta_0\le\xi\le M_0+\zeta_0}(u^*(\xi))' = \bar{u}.$$

Now let ξ_0 be defined as

$$\xi_0 = \frac{1}{\eta}\ln\frac{2(\beta + c^*\theta + \theta^2 + |f|_{C^{1,\alpha}})q_0\bar{M}}{\beta\zeta_0\bar{u}}.$$

Finally, letting

$$\bar{\theta}(\xi,t) = u^*(\xi-\zeta(t)) + q(\xi,t),$$

$$\underline{\theta}(\xi,t) = u^*(\xi+\zeta(t)) - q(\xi,t),$$

and

$$\bar{\alpha} = \min_{\xi\in R}(1,\ \bar{\theta}),$$

$$\underline{\alpha} = \max_{\xi\in R}(0,\ \underline{\theta}).$$

We have,

Lemma 4.1. *$\bar{\alpha}$ is a super-solution of (1).*

Proof. Since $u = 1$ is already a solution of (1), we only consider $\bar{\theta}$, then

$$\begin{aligned}
&\bar{\theta}_t - \bar{\theta}_{xx} - f(\bar{\theta})\\
=\ &(u^*(\xi-\zeta(t)) + q(\xi,t))_t - (u^*(\xi-\zeta(t)) + q(\xi,t))_{xx}\\
&-f(u^*(\xi-\zeta(t)) + q(\xi,t)) \qquad\qquad (55)\\
=\ &-\zeta'(t)u^*_{\xi\xi}(\xi-\zeta(t)) + (q(\xi,t))_t - q_{\xi\xi} + f(u^*(\xi-\zeta(t)))\\
&-f(u^* + q(\xi,t)),
\end{aligned}$$

where $u^*_{\xi\xi}$ denote the second partial derivative of u^* with respect to ξ.

We discuss (55) by cases.

Case 1, $\xi \le -M_0 + \zeta_0$.

Now (55) reads

$$-u^*_\xi(\xi-\zeta(t))\zeta'(t) + (-\beta + c^*\eta - \eta^2 - f'(u^* + \sigma q))q$$

for some constant σ. Because of (53), we have

$$-\beta + c^*\eta - \eta^2 - f'(u^* + \sigma q) \ge 0.$$

Case 2, $\xi \ge M_0 + \zeta_0$.

Now (55) turns into

$$\beta e^{-\beta t}u^*_\xi(\xi-\zeta(t))[1 + \frac{-\beta - c^*\eta - \eta^2}{\zeta_0\beta}\frac{q}{u_\xi e^{-\beta t}} - \frac{f'(u^*+\sigma q)}{\zeta_0\beta}\frac{q}{u^*_\xi e^{-\beta t}}].$$

We only need to show that

$$1+\frac{-\beta-c^*\eta-\eta^2}{\zeta_0\beta}\frac{q}{u_\xi e^{-\beta t}}-\frac{f'(u^*+\sigma q)}{\zeta_0\beta}\frac{q}{u_\xi^* e^{-\beta t}}\geq 0. \tag{56}$$

In fact, by (54), we have

$$|\frac{(\beta+c^*\eta+\eta^2)q_0}{\zeta_0\beta}\frac{1}{u_\xi^* e^{\eta(\xi+\xi_0)}}|\ll\frac{1}{2}, \tag{57}$$

and

$$|\frac{f'(u^*+\sigma q)q_0}{\zeta_0\beta}\frac{1}{u_\xi^* e^{\eta(\xi+\xi_0)}}|\ll\frac{1}{2}. \tag{58}$$

Therefore, by (57) and (58), we have shown that

$$\beta\zeta(t)u_\xi^*(\xi-\zeta(t))[1+\frac{-\beta-c^*\eta-\eta^2}{\zeta_0\beta}\frac{q}{u_\xi^* e^{-\beta t}}-\frac{f'(u+\sigma q)}{\zeta_0\beta}\frac{q}{u_\xi^* e^{-\beta t}}]\geq 0.$$

Case 3, $-M+\zeta_0\leq\xi\leq M_0+\zeta_0$. We consider two subcases,

Subcase 1, $-M_0+\zeta_0\leq\xi\leq 0$.

$$\begin{aligned}&-\zeta'(t)u_\xi(\xi-\zeta(t))+(-\beta-c^*\eta-\eta^2-f'(u+\sigma q))q\\ =\;&\beta\zeta_0 e^{-\beta t}u_\xi(\xi-\zeta(t))+(-\beta-c^*\eta-\eta^2-f'(u+\sigma q))q.\end{aligned} \tag{59}$$

Subcase 2, $0\leq\xi\leq M_0+\zeta_0$.

$$\begin{aligned}&-\zeta'(t)u_\xi(\xi-\zeta(t))+(-\beta+c^*\eta-\eta^2-f'(u+\sigma q))q\\ =\;&\beta\zeta_0 e^{-\beta t}u_\xi(\xi-\zeta(t))+(-\beta+c^*\eta-\eta^2-f'(u+\sigma q))q.\end{aligned} \tag{60}$$

from (53), we see that (59) and (60) are both nonnegative. □

Lemma 4.2. $\underline{\alpha}$ *is a subsolution of (1).*

Proof. Since $u=0$ is already a solution of (1), we need only to consider $\underline{\theta}$. We have

$$\begin{aligned}&\underline{\theta}_t-\underline{\theta}_{xx}-f(\underline{\theta})\\ =\;&(u^*(\xi+\zeta(t))-q(\xi,t))_t-(u^*(\xi+\zeta(t))-q(\xi,t))_{xx}\\ &-f(u^*(\xi+\zeta(t))-q(\xi,t))\\ =\;&\zeta'(t)u_\xi^*(\xi+\zeta(t))-(q(\xi,t))_t-q_{\xi\xi}\\ &+f(u^*(\xi+\zeta(t)))-f(u^*(\xi+\zeta(t))-q(\xi,t)).\end{aligned} \tag{61}$$

Similar to the proof of Lemma 4.1, we consider the following cases.

Case 1. $\xi\leq -M_0-\zeta_0$.

For some constant σ_1, (61) changes into

$$\zeta'(t)u_\xi^*(\xi+\zeta(t))+(\beta-c^*\eta+\eta^2+f'(u^*-\sigma_1 q))q. \tag{62}$$

We need to show (62) is negative, actually, by (53), we have

$$\begin{aligned} & \beta - c^*\eta + \eta^2 + f'(u^* - \sigma_1 q) \\ = \quad & -\frac{c^{*2}}{8} + f'(u^* - \sigma_1 q) < 0 \end{aligned}$$

Case 2. $\xi \geq M_0 - \zeta_0$.
Now (62) is

$$-\beta\zeta_0 e^{-\beta t}u^*_\xi(1 + \frac{-\beta - c^*\eta - \eta^2}{\zeta_0\beta}\frac{q}{u^*_\xi e^{-\beta t}} - \frac{f'(u+\sigma_1 q)}{\zeta_0\beta}\frac{q}{u^*_\xi e^{-\beta t}}), \tag{63}$$

we need to show (63) is positive. The same proof as in Lemma 4.1 can be applied here to get the conclusion.
Case 3. $-M_0 - \zeta_0 \leq \xi \leq M_0 - \zeta_0$.
we have two subcases to be considered:
Subcase 1. $0 \leq \xi \leq M_0 - \zeta_0$.
In this case, (61) changes to

$$-\beta\zeta_0 e^{-\beta t}u^*_\xi + (\beta + c^*\eta + \eta^2)q + f'(u^* - \sigma_1 q). \tag{64}$$

Subcase 2. $-M_0 - \zeta_0 \leq \xi \leq 0$.
In this case, (61) changes to

$$-\beta\zeta_0 e^{-\beta t}u_\xi + (\beta - c^*\eta + \eta^2)q + f'(u - \sigma_1 q). \tag{65}$$

In either case, we can argue as in Lemma 4.1 that (64) and (65) are negative. □

Lemma 4.3. *There exist $\xi_1 > 0$ and $\xi_2 > 0$ such that*

$$\bar{\theta}(\xi + \xi_2, t) > \underline{\theta}(\xi - \xi_1, t). \tag{66}$$

Proof. By Theorem 1.1, for $\xi + \zeta_0 \leq -M_0$, we have

$$\begin{aligned} \bar{\theta}(\xi, t) &= u^*(\xi - \zeta(t)) + q(\xi, t) \\ &= u^*(\xi - \zeta_0 e^{-\beta t}) + q_0 e^{-\beta t}e^{\eta(\xi - \xi_0)} \\ &= Ae^{c^*(\xi - \zeta_0 e^{-\beta t})} + q_0 e^{-\beta t}e^{\eta(\xi - \xi_0)}, \end{aligned} \tag{67}$$

and

$$\begin{aligned} \underline{\theta}(\xi, t) &= u^*(\xi + \zeta(t)) - q(\xi, t) \\ &= u^*(\xi + \zeta_0 e^{-\beta t}) - q_0 e^{-\beta t}e^{\eta(\xi - \xi_0)} \\ &= Ae^{c^*(\xi + \zeta_0 e^{-\beta t})} - q_0 e^{-\beta t}e^{\eta(\xi - \xi_0)}, \end{aligned} \tag{68}$$

while for $\xi - \zeta_0 \geq M_0$, we have

$$\bar{\theta}(\xi, t) = H_1(\frac{1}{c^*}(\xi - \zeta(t))(1 + o(1))) + q_0 e^{-\beta t}e^{-\eta(\xi + \xi_0)}, \tag{69}$$

and

$$\underline{\theta}(\xi, t) = H_1(\frac{1}{c^*}(\xi + \zeta(t))(1 + o(1))) - q_0 e^{-\beta t}e^{-\eta(\xi + \xi_0)}. \tag{70}$$

Since H is increasing with respect to ξ, then there exists $\xi_1 > 0$ such that

$$\bar{\theta}(\xi, t) > \underline{\theta}(\xi - \xi_1, t). \tag{71}$$

We then consider $\xi \pm \zeta_0 \in (-M_0,\, M_0)$, by Theorem 1.1 and because $(-M_0,\, M_0)$ is finite, there exists a $0 < \xi_2 \le M_0$ such that

$$\bar{\theta}(\xi - \xi_2, t) > \underline{\theta}(\xi + \xi_1, t)$$

for all $t > 0$. □

Lemma 4.4. *Let u be the solution of (1) with initial value ψ satisfying $\underline{\alpha}(x, 0) \le \psi \le \bar{\alpha}(x, 0)$, then for all $t > 0, u(t, x) \ge \underline{\alpha}$ and $u(t, x) \le \bar{\alpha}$.*

Proof. By Maximum principle and comparing ψ with $\underline{\alpha}(x, 0)$ and $\bar{\alpha}(x, 0)$. □

Proof of Theorem 1.3. *Lemma 4.1-4.4 amount to the conclusion of the theorem.*

5. **Application.** As a special case of system (1), we consider the following system

$$\begin{cases} u_t = u_{xx} + 1 + \sin u, \\ u(x, 0) = \psi(x) \end{cases} \tag{72}$$

and its counterpart in the moving coordinates $\xi = x + ct$ with boundary conditions

$$\begin{cases} u'' - cu' + 1 + \sin u = 0, \\ u(-\infty) = -\dfrac{\pi}{2}, \quad u(+\infty) = \dfrac{3\pi}{2}. \end{cases} \tag{73}$$

System (73) is the classical pendulum or single point Josephson Junction equation ([20]), and satisfies conditions (1)-(5) after rescaling. According to Theorem 1.1 to Theorem 1.3, there is a $c^* > 0$ such that for any $c \ge c^*$ system (73) has a unique (modulo transformation) traveling wave solution. Also, the traveling wave solution u^* with speed c^* is asymptotically stable for suitably chosen initial condition $\psi(x)$.

We interpret the dynamics of (73) in the phase plane $(u,\, u')$. System (73) has a unique heteroclinic orbit connecting the equilibrium $(u,\, u') = (-\pi/2,\, 0)$ with $(u,\, u') = (3\pi/2,\; 0)$ for every $c \ge c^*$ (see also [20]). An equilibrium analysis reveals that $(-\pi/2,\, 0)$ and $(3\pi/2,\, 0)$ are both saddle-nodes. Also from the asymptotic analysis in section 2, when $c = c^*$ the heteroclinic orbit is formed by the connection between a strongly unstable manifold of the saddle-node $(-\pi/2,\, 0)$ and a center stable manifold of $(3\pi/2,\, 0)$ on the upper half $(u,\, u')$ plane; while for $c > c^*$, the heteroclinic orbit is formed by connecting the center unstable manifold of $(-\pi/2,\, 0)$ with a center stable manifold of $(3\pi/2,\, 0)$. Hence as the wave speed changes from c^* to $c > c^*$ the heteroclinic orbit breaks and then immediately re-forms between the two equilibria. In this point of view, our results are in sharp contrast with those of KPP equation, since in the later equation the heteroclinic connection persists as c changes from $c^*$to $c > c^*$. The driving force for the traveling wave solution of KPP equation is reaction and diffusion process. This is more evident from the following corollary, in which we allow the nonlinear reaction term f to be slightly more general than $1 + \sin u$.

Corollary 1. *Suppose in addition to conditions (1)-(5), the nonlinear term f in (2) satisfies $f''(1) > 0$ and $f''(0) > 0$, and f is of $C^{3,\alpha}$, then corresponding to c^*, then the traveling wave solution of (2) has the following asymptotic behaviors,*

$$\begin{cases} u^*(\xi) = 1 - \dfrac{2c^*}{f''(1)}\dfrac{1}{\xi} + o(\dfrac{1}{\xi}) & \text{as } \xi \to \infty, \\ u^*(\xi) = Ae^{c^*\xi} + o(e^{c^*\xi}) & \text{as } \xi \to -\infty, \end{cases} \tag{74}$$

while for $c > c^*$, the traveling wave solutions has the following asymptotic behaviors

$$\begin{cases} u(\xi) = 1 - \dfrac{2c}{f''(1)}\dfrac{1}{\xi} + o(\dfrac{1}{\xi}) & \text{as } \xi \to \infty, \\ u(\xi) = -\dfrac{2c}{f''(0)}\dfrac{1}{\xi} + o(\dfrac{1}{\xi}) & \text{as } \xi \to -\infty. \end{cases} \tag{75}$$

Proof. The asymptotic behavior for u^* at $-\infty$ has been derived by Theorem 1.1. We now derive the asymptotic expansion for u^* at $+\infty$.

Consider function

$$v(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B\ln\xi}{\xi^2}, \tag{76}$$

where the constant B is to be determined.

Differentiating $v(\xi)$, we have

$$\begin{aligned} v'(\xi) &= \frac{2c^*}{f''(1)}\frac{1}{\xi^2} + \frac{B - 2B\ln\xi}{\xi^3}, \\ v''(\xi) &= -\frac{4c^*}{f''(1)\xi^3} - \frac{5B - 6B\ln\xi}{\xi^4}, \end{aligned}$$

then

$$\begin{aligned} \hat{L}v &\doteq v'' - c^*v' + f(v) \\ &= \frac{1}{\xi^5}(-Bc^* - \frac{4c^*}{f''(1)\xi} - \frac{f'''(1)}{6}(\frac{2c^*}{f''(1)})^3 + o(\frac{1}{\xi})). \end{aligned} \tag{77}$$

Choosing $B = B_1 > 0$ sufficiently large and letting

$$v_1(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B_1\ln\xi}{\xi^2}, \tag{78}$$

we then have $\hat{L}v_1 < 0$ for ξ large enough, i.e, v_1 is a super solution for $\hat{L}v = 0$.

Similarly we can choose $-B_2 > 0$ large enough in (77) such that

$$v_2(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B_2\ln\xi}{\xi^2} \tag{79}$$

defines a sub-solution of $\hat{L}v = 0$ for $\xi > 0$ sufficiently large.

It follows that $v_2(\xi) < v_1(\xi)$ for ξ large enough. Consequently, there exists $T > 0$, such that

$$0 < v_2(T) < v_1(T) < 1. \tag{80}$$

Since system (2) is shifting invariant, we can make a shifting of origin of $u^*(\xi)$ such that

$$0 < v_2(T) < u^*(T) < v_1(T).$$

We then have

$$\begin{cases} \hat{L}v_2 > \hat{L}u^* > \hat{L}v_1, \\ v_2(T) < u^*(T) < v_1(T), \\ v_2(+\infty) = u^*(+\infty) = v_1(+\infty) = 1. \end{cases} \tag{81}$$

Consequently,

$$\begin{aligned} \hat{L}(v_2 - u^*) &= (v_2 - u^*)'' - c^*(v_2 - u^*)' + f(v_2) - f(u^*) \\ &\doteq w'' - c^* w' + f'(\mu v_2 + (1-\mu)u^*)w > 0, \end{aligned}$$

where we set $w = v_2 - u^*$ and for each ξ, $\mu(\xi)$ is a constant between 0 and 1. By Maximum Principle, we see $v_2(\xi) < u^*(\xi)$ for $\xi \geq T$. Similarly, we have $v_1 > u^*$ for $\xi > T$.

Therefore, we can write

$$u^*(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + o(\frac{1}{\xi}) \qquad \text{as } \xi \to \infty.$$

This shows (74). The verification for (75) is similar, so we skip it. □

We further remark that if the nonlinear term f as in (1) is only $C^{1,\alpha}$, the phase plane analysis method for the KPP equation as in [25] does not work. Even for $f = u^m(1-u)^n$, $1 < m\,, n \in \mathbb{N}$, the phase plane analysis as appears in [20] will be quite lengthy, see [12].

Appendix A. **An alternative existence proof.** In this section, we give an alternative existence proof of the traveling wave solutions, the proof is simpler than the previous one and covers KPP, $m-th$ Fisher as well as the current double degenerate case. As a cost, this proof does not yield the asymptotic results as in Theorems 1.1, 1.2.

Consider the equation

$$u'' - cu' + f(u) = 0 \tag{A-1}$$

where $f\,:\,[0,\,1] \to \mathbb{R}^+$ is smooth and $f(0) = f(1) = 0$. To simplify our arguments we will make the odd extension to f, that is we define $f(\xi) = -f(-\xi)$ for $\xi < 0$. A change of variable $\xi \to -\xi$ changes the sign of c so we will assume without loss of generality that $c \geq 0$.

We can write the equation as a system by introducing $v = u'$, so (A-1) becomes

$$u' = v, \qquad v' = cv - f(u) \tag{A-2}$$

The equilibrium points of (A-2) are of the form $(\gamma,\,0)$ where $f(\gamma) = 0$. A *heteroclinic solution of (A-2)* is a solution $(u(\xi), v(\xi))$ such that

$$\lim_{\xi\to-\infty}(u(\xi), v(\xi)) = (\gamma_1, 0), \quad \lim_{\xi\to+\infty}(u(\xi), v(\xi)) = (\gamma_2, 0) \tag{A-3}$$

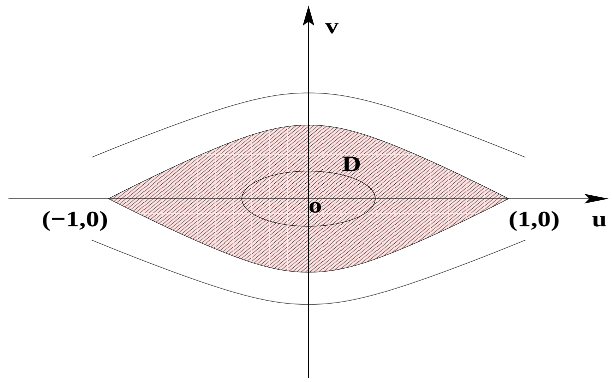


FIGURE 1. The Level Curve of V

where γ_1 and γ_2 are two distinct zeros of f. We sometimes say the solution is heteroclinic from $(\gamma_1, 0)$ to $(\gamma_2, 0)$.

Lemma A.1. *If $c > 0$ then any solution of (A-2) which is bounded for $\xi \leq 0$ approaches an equilibrium point as $\xi \to -\infty$.*

Proof. The classical Liapunov type theorems deal behavior as $\xi \to +\infty$. Therefore, there will be several sign shifts in the following argument.

Consider the (Liapunov) function

$$V = \frac{1}{2}v^2 + F(u), \quad F(u) = \int_0^u f(\tau)d\tau, \quad F_1 = \int_0^1 f(\tau)d\tau. \tag{A-4}$$

The derivative of V along a solution of (A-2) is

$$V' = \frac{\partial V}{\partial v}v' + \frac{\partial V}{\partial u}u' = v(cv - f(u)) + f(u)v = cv^2 \geq 0. \tag{A-5}$$

Since $V' \geq 0$ it follows from LaSalle's Theorem (Theorem 2, page 282 of [17]) that a solution which is bounded for $\xi < 0$ tends to the largest invariant set in $M = \{(u,v) : V'(u,v) = 0\}$ as $\xi \to -\infty$. (Also see ([18, 19])) If $c > 0$ then M is the set where $v = 0$. On M we have $v' = -f(u)$, so to remain in M we must have $f(u) = 0$. □

The level curves of V are shown in Figure 1. The shaded region is the set $D = \{(u,v)| - 1 \leq u \leq 1, V \leq F_1\}$.

Since $V' \geq 0$, a solution that starts in D at $\xi = 0$ remains D for $\xi \leq 0$. The set D is compact, so by Lemma A.1 all solutions that start in D tend as $\xi \to -\infty$ to one of the three equilibrium points $(-1,0)$, $(0,0)$, $(1,0)$.

Lemma A.2. *For all $c > 0$, at least one solution which starts in D tends to the equilibrium $(1,0)$ as $\xi \to +\infty$.*

Proof. Let $W = \{(u,v) : u \geq 0, v \geq 0, \frac{1}{2}F_1 \leq V \leq F_1\}$, see the shaded region in Figure 2. This region does not quite satisfy the hypothesis of Wazewski's theorem since there are both ingress and egress points on the boundary ([12], [32]). The region is not quite an isolating block in the sense of Conley and Easton ([8]), since there is an equilibrium points on the boundary. However, the ideas of these works readily apply.

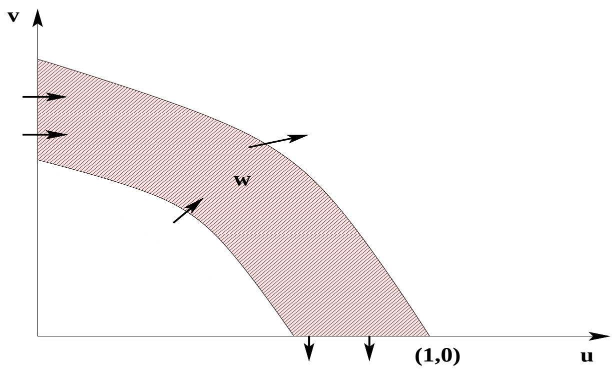


FIGURE 2. The Region W

The set of ingress points are the points on the boundary of W where the trajectories enter W. The set of ingress points is homeomorphic to an open interval. The set of egress points are the points on the boundary of W where the trajectories exit W. The set of egress points is homeomorhpic to the union of two open intervals. (The intervals are separated by the equilibrium point $(1,0)$.)

Assume that no trajectory approaches $(1,0)$. Then all trajectories enter the ingress set and exit the egress set. The ingress and egress sets are cross sections to the flow and cross section maps define a homeomorphism. But this is a contradiction since the ingress set is connected and the egress set is disconnected. □

Proposition 1. *There is at least one heteroclinic orbit from* $(0,0)$ *to* $(1,0)$.

Proof. This follows from the two lemmas given above. □

Let $\psi(\xi)$ be such a heteroclinic solution. We note that $\psi(\xi)$ is unique (up to a translation in ξ) if $f'(1) < 0$ since in that case the critical point at $(1,0)$ is a saddle point.

A positive heteroclinic solution $(u(\xi), v(\xi))$ satisfies (A-3) and $u(t) > 0$ for all $-\infty < \xi < \infty$.

Proposition 2. *Let* $f(u) \leq \alpha u$ *for* $0 \leq u \leq 1$. *If* $0 \leq 4\alpha \leq c^2$ *then* $\psi(\xi)$ *is a positive heteroclinic solution.*

If $d = f'(0) > 0$, *then the heteroclinic solution* $\psi(\xi)$ *from* $(0,0)$ *to* $(1,0)$ *is not positive when* $c^2 - 4d < 0$.

Proof. We remark that the first part of this proposition does not require that $f'(0) > 0$ and/or $f'(1) < 0$.

If $d = f'(0) > 0$ and $c^2 - 4d < 0$, then the critical point at the origin is a unstable spiral (focus). Since $\psi(\xi)$ approaches 0 as $\xi \to -\infty$ in the phase plane $(\psi(\xi), \psi'(\xi))$ spirals around the origin with $\psi(\xi)$ taking positive and negative values. See [7].

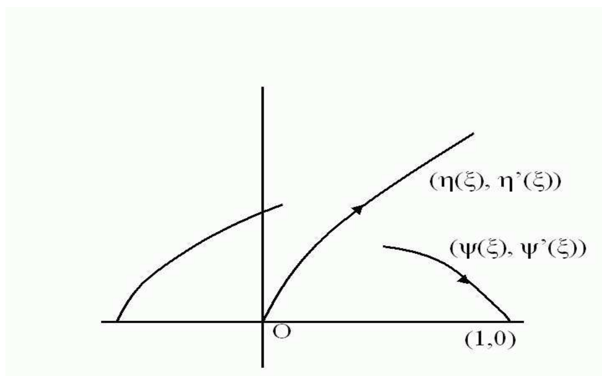
Now let $f(u) \leq \alpha u$ for $0 \leq u \leq 1$ with $0 \leq 4\alpha \leq c^2$. We compare equation (A-2) and the linear equation

$$u' = v, \quad v' = cv - \alpha u \tag{A-6}$$

The equilibrium of equation (A-6) is a linear unstable node with particular solutions along the eigenvectors of the coefficient matrix, that is along the vectors $(c \pm \sqrt{c^2 - 4\alpha}, 2\alpha)$. These vectors have positive slope. Let $(u(\xi), v(\xi))$ be a solution of (A-2) and $(u_l(\xi), v_l(\xi))$ be a solution of (A-6). Then at any point in the first quadrant

$$\frac{dv}{du} = c - \frac{f(u)}{v} \geq c - \frac{\alpha u}{u} = \frac{dv_l}{du_l}.$$

The solutions of equation (A-2) grow faster than the solutions of (A-6) so there are solutions of (A-2) which remain in the first quadrant and leave D. Call such a solution $(\eta(\xi), \eta'(\xi))$. The solution $(\eta(\xi), \eta'(\xi))$ defines a curve in the first quadrant that starts at the origin as $\xi \to -\infty$ and leaves D.



If the solution $\psi(\xi)$ became negative then $(\psi(\xi), \psi'(\xi))$ would have loop around the origin. It must ultimately tend to $(1, 0)$ through the first quadrant, but the curve $(\eta(\xi), \eta'(\xi))$ acts as a barrier since two trajectories cannot cross in the phase plane. □

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REFERENCES

[1] D. G. Aronson and H. Weinberger, "Nonlinear Diffusion in Population Genetics, Combustion and Nerve Pulse Propagation," Partial Differential Equations and Related Topics, Lecture notes in mathematics, 446, Springer, Berlin, 1975.

[2] D. G. Aronson and H. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math., **30** (1978), 33–76.

[3] J. W. Bebernes, C. Li and Y. Li, *Travelling fronts in cylinders and their stability*, Rocky Mountain J. Math., **27** (1997), 123–150.

[4] H. Berestycki and L. Nirenberg, *Travelling fronts in cylinders*, Annales de l'IHP (C) Analyse non linéaire, **9** (1992), 497–572.

[5] H. Berestycki and L. Nirenberg, *Some qualitative properties of solutions of semi-linear elliptic equations in cylindrical domains*, Analysis et. Cetera, ed. P. Rabinowitz et al., Academic Pr (1990), 115–164.

[6] H. Berestycki, B. Nicolaenko and B. Scheurer, *Travelling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal., **16** (1985), 1207–1242.

[7] E. Coddington and N. Levinson, "The Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

[8] C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Tans. Amer. Math. Soc., **158**, (1971), 137–143.

[9] P. C. Fife, "Mathematical Aspects of Reaction and Diffusing Systems," Lecture notes in Biomathematics, 28. Springer, Berlin-New Yorker, 1979.

[10] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling frony solutions*, Archive for Rational Mechanics and Analysis, **65** (1977), 335–361.
[11] R. A. Fisher, *The wave of advance of advantages genes*, Ann. Eugenics, **7** (1937), 355–369.
[12] P. Hartman, "Ordinary Differential Equations," John Wiley, New York, 1964.
[13] D. Henry, "Geometric Theory of Semi-Linear Parabolic Equations," Lecture Notes in Mathematics, Vol. **840**, Springer, New York, 1980.
[14] X. J. Hou and Y. Li, *Local stability of traveling wave solutions of nonlinear reaction diffusion equations*, Discrete and Continuous Dynamical Systems, **15** (2006), 557–594.
[15] K. Kirchgassner, *On the nonlinear dynamics of travelling fronts*, Journal of Differential Equations, **96** (1992), 256–278.
[16] A. N. Kolmogrov, I. G. Petrovskii and N. S. Piskunov, *Study of the diffusion equation with growth of quantity of matter and its application to a biological problem*, Bull. Univ. Moscou, Ser. Int. Sec. A, **1** (1937), 1–25.
[17] J. P. LaSalle, *An invariance principle in the theory of stability*, Differential equations and dynamical Syatems (Eds. J. K. Hale & J.P. LaSalle), Academic Press, New York, 1967, 227–286.
[18] J. P. LaSalle, *Stability theory for ordinary differential equations*, Journal of Differential Equations, **4** (1969), 57–65.
[19] J. P. LaSalle and Lefschetz, "Stability by Liapunov's Direct Method with Applications," Academic Press, New York, 1961.
[20] M. Levi, F. C. Hoppensteadt and W. L. Miranker, *Dynamics of the Josephson junction*, Quarterly of Applied Math., July (1978), 167–198.
[21] Y. Li, *Asymptotic behavior of positive solutions of equation* $\Delta u + K(x)u^p = 0$ *in* R^n, Journal of Differential Equations, **95** (1992), 304–330.
[22] Y. Li and Y.-P. Wu, *Stability of travelling waves with noncritical speeds for double degenerate Fisher-type equations*, Discrete Contin. Dyn. Syst. Ser. B, **10** (2008), 149-170.
[23] X. Liang and X. Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Communications on Pure and Applied Math., **60** (2007), 1–40.
[24] D. J. Needham and A. N. Barnes, *Reaction diffusion and phase waves occurring in a class of scalar reaction diffusion equations*, Nonlinearity, **12** (1999), 41–58.
[25] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Advances in Math., **22** (1976), 312–355.
[26] D. H. Sattinger, *Weighted norms for the stability of travelling waves*, Journal of Differential Equations, **25** (1977), 130–144.
[27] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, New York, 1983.
[28] A. I. Volpert, V. A. Volpert and V. A. Volpert, *Travelling wave solutions of parabolic systems*, Translations of Mathematical Monographs, **140**. A.M.S. Providence, RI, 1994.
[29] J. Xin, *Front propagation in heterogeneous media*, SIAM Rev., **42** (2000), 161–230.
[30] I. C. Gohberg and M. G. Krein, *The Basic propositions on defect numbers, root numbers and indices of linear operators*, A.M.S. Translation Series 2, Vol. **13** (1960), 185–265.
[31] M. Wang and Q. X. Ye, *Traveling wave solutions for some degenerate parabolic equations II*, Acta Math. Appl. Sinica (English Ser.), **9** (1993), 396–382.
[32] T. Waszewski, *Sur l'unicitet la limitation des int égrales des équations aux dérivées partielles du premier ordre*, Atti R. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., **18** (1933), 372–376.
[33] Y. P. Wu, X. Xing and Q. X. Ye, *Stability of travelling waves with algebraic decay for n-degree Fisher-type equations*, Discrete Contin. Dyn. Syst., **16** (2006), 47–66.

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